

An Interior Proximal Method for Solving Monotone Generalized Variational Inequalities

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Abstract

We present a new method for solving generalized variational inequalities on polyhedra. The method is based on an interior-quadratic term which replaces the usual quadratic term. This leads to an interior proximal type algorithm. We first solve a monotone generalized variational inequalities satisfying a certain Lipschitz condition. Next, we combine this technique with line search technique to obtain a convergent algorithm for monotone generalized variational inequalities without Lipschitz condition. Finally some preliminary computational results are given.

1 Introduction

Let C be a polyhedral set on the real Euclidean space \mathbb{R}^n defined by

$$C := \{x \in \mathbb{R}^n : Ax \leq b\},$$

where A is an $p \times n$ matrix, $b \in \mathbb{R}^p$, $p \geq n$. We suppose that the matrix A is of maximal rank, i. e., $\text{rank}A = n$ and $\text{int}C = \{x : Ax < b\}$ is nonempty. Let F be a continuous mapping from D into \mathbb{R}^n , and φ be a lower semicontinuous convex function from C into \mathbb{R} . We say that a point x^* is a solution of the following generalized variational inequality if it satisfies

$$\langle F(x^*), x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0 \quad \forall x \in C, \quad (VIP)$$

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where $\langle \cdot, \cdot \rangle$ denotes the standard dot product in \mathbb{R}^n . Throughout the paper, we assume that the mapping F is monotone over C .

This generalized variational inequality problem have many important applications in economics, nonlinear analysis and have been studied by many researchers (see [9, 13, 15, 17, 19, 22, 24]).

It is well-known that the interior-quadratic technique is a powerfull tool for analyzing and solving optimization problems (see [6, 23]). Recently this technique has been used to develop proximal iterative algorithm for variational inequalities (see [5, 6, 8]).

In our recent paper [1] we have used the logrithmic quadratic function for pseudomonotone equilibrium on $\mathbb{R}^n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0 \ \forall i = 1, \dots, n\}$ and developed algorithms for solving them.

In this paper we extend our results in [1, 2, 3] to the generalized variational inequality problem (VIP). Namely, we first develop a convergent algorithm for (VIP) with F being monotone function satisfying a certain Lipschitz type condition on C by using the interior-quadratic function. Next, in order to avoid the Lipschitz condition we will combine the line search method and this function to obtain a convergent algorithm for solving the generalized variational inequality problem (VIP) with the monotone function F .

The remaining part of the paper is structured as follows. In Section 2, we present a convergent algorithm for monotone and Lipschitz generalized variational inequality problems. In Section 3, we modify the algorithm by combining a line search with the interior-quadratic function, which allows avoiding the Lipschitz condition. Section 4 deals with some preliminary results of the proposed method.

2 Preliminaries on the interior-quadratic function

First, let us recall the well known concepts of monotonicity that will be used in the sequel.

Definition 2.1 *Let C be a convex set in \mathbb{R}^n , and $F : C \rightarrow \mathbb{R}^n$. The function F is said to be*

(i) *monotone on C if for each $x, y \in C$, we have*

$$\langle F(x) - F(y), x - y \rangle \geq 0;$$

(ii) *strongly monotone on C with constant $\beta > 0$ if for each $x, y \in C$, we have*

$$\langle F(x) - F(y), x - y \rangle \geq \beta \|x - y\|^2;$$

(iii) Lipschitz with constant $L > 0$ on C (shortly L -Lipschitz), if we have

$$\|F(x) - F(y)\| \leq L\|x - y\| \quad \forall x, y \in C.$$

Remark 2.2 Let A be $p \times n$ matrix, $\text{rank}A = n$, $C := \{x \in \mathbb{R}^n : Ax \leq b\}$, and $F : C \rightarrow \mathbb{R}^n$ be L -Lipschitz on C . Then we have

$$\|F(x) - F(y)\| \leq \bar{L}\|A(x - y)\| \quad \forall x, y \in C,$$

where $\bar{A} := (a_{ij})_{n \times n}$ is a submatrix of A such that $\text{rank} \bar{A} = n$ and

$$\|\bar{A}^{-1}\| = \sup_{\|x\|=1} \|\bar{A}^{-1}x\|,$$

and $\bar{L} = L\|\bar{A}^{-1}\|$.

Indeed, from

$$\|F(x) - F(y)\| \leq L\|x - y\| \quad \forall x, y \in C,$$

and

$$\|x - y\| = \|\bar{A}^{-1}(\bar{A}(x - y))\| \leq \|\bar{A}^{-1}\| \|\bar{A}(x - y)\| \quad \forall x, y \in \mathbb{R}^n,$$

it follows that

$$\|F(x) - F(y)\| \leq L\|\bar{A}^{-1}\| \|A(x - y)\| \quad \forall x, y \in C.$$

Note that when φ is differentiable on some open set containing C , then, since φ is lower semicontinuous proper convex, the variational inequality (VIP) is equivalent to the following one (see [11, 12]):

Find $x^* \in C$ such that

$$\langle F(x^*) + \nabla\varphi(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C.$$

In special case $\varphi = 0$, problem (VIP) can be written by the following:

Find $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C. \quad (VI)$$

It is well known that the problem (VI) can be formulated as finding the zero point of the operator $T(x) = \varphi(x) + N_C(x)$ where

$$N_C(x) = \begin{cases} \{y \in C : \langle y, z - x \rangle \leq 0, \forall z \in C\} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases} \quad (2.1)$$

A classical method to solve this problem is the proximal point algorithm (see [2, 21]), which starting with any point $x^0 \in C$ and $\lambda_k \geq \lambda > 0$, iteratively updates x^{k+1} conforming the following problem:

$$0 \in \lambda_k T(x) + \nabla_x h(x, x^k), \quad (2.2)$$

where

$$h(x, x^k) = \frac{1}{2} \|x - x^k\|^2.$$

Recently, Auslender et al. [7] have proposed a new type of proximal interior method for solving problem (VI) on $C = \mathbb{R}_+^n$ through replacing function $h(x, x^k)$ by $d_\phi(x, x^k)$ which is defined as

$$d_\phi(x, y) = \sum_{i=1}^n y_i^2 \phi(y_i^{-1} x_i),$$

where

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu(t - \log t - 1) & \text{if } t > 0, \\ +\infty & \text{otherwise,} \end{cases} \quad (2.3)$$

with $\nu > \mu > 0$. The fundamental difference here is that the term d_ϕ is used to force the iteratives $\{x^{k+1}\}$ to stay in the interior of \mathbb{R}_+^n .

Applying this idea to problem (VIP), in this paper we use the following function

$$d(x, y) = \begin{cases} \frac{1}{2} \|x - y\|^2 + \mu \sum_{i=1}^n y_i^2 \left(\frac{x_i}{y_i} \log \frac{x_i}{y_i} - \frac{x_i}{y_i} + 1 \right) & \text{if } x > 0, \\ +\infty & \text{otherwise,} \end{cases} \quad (2.4)$$

with $\mu \in (0, 1)$ and $y \in C$. Let a_i denotes the rows of the matrix A , and define the following quantities:

$$\begin{aligned} l_i(x) &= b_i - \langle a_i, x \rangle, \\ l(x) &= (l_1(x), l_2(x), \dots, l_p(x)), \\ D(x, y) &= d(l(x), l(y)). \end{aligned}$$

We denote by $\nabla_1 D(x, y)$ the gradient of $f(\cdot, y)$ at x for every $y \in C$. It is easy to see that

$$\nabla_1 D(x, y) = -A^T (l(x) - l(y) + \mu X_y \log \frac{l(x)}{l(y)}), \quad (2.5)$$

where $X_y = \text{diag}(l_1(y), \dots, l_p(y))$ and $\log \frac{l(x)}{l(y)} = (\log \frac{l_1(x)}{l_1(y)}, \dots, \log \frac{l_p(x)}{l_p(y)})$.

Now we consider the following gap function:

$$g(x) = \min\{F(x), y - x\} + \varphi(y) - \varphi(x) + D(y, x) : y \in C\}. \quad (2.6)$$

Since C is closed convex and the objective function are strongly convex, the mathematical programming problem (2.6) is always solvable for any $x \in C$. Let $h(x)$ denote the unique solution to problem(2.6). h is a marginal mapping onto C . Observe that when φ is a constant function and $D(x, y) = \frac{1}{2} \|x - y\|^2$, h coincides and becomes the marginal mapping for the projection gap function introduced in Fukushima (1992) (see [16]). The following lemma characterizes the solutions to problem (VIP) by means of the mapping h .

Lemma 2.3 *Suppose that the generalized variational inequality problem (VIP) has a solution. Then a point x^* is a solution to problem (VIP) if and only if $x^* = h(x^*)$.*

Proof. Let x^* be a solution to problem (VIP) and $h(x^*)$ be the unique solution to problem (2.6). Then

$$\langle F(x^*), h(x^*) - x^* \rangle + \varphi(h(x^*)) - \varphi(x^*) \geq 0. \quad (2.7)$$

Since $h(x^*)$ is the solution to problem (2.6), there exists a $z^* \in \partial_\varphi(h(x^*))$ such that

$$\langle F(x^*) + \nabla_1 D(h(x^*), x^*) + z^*, y - h(x^*) \rangle \geq 0 \quad \forall y \in C. \quad (2.8)$$

Replacing $y = x^*$ in (2.8) we get

$$\langle F(x^*) + \nabla_1 D(h(x^*), x^*) + z^*, x^* - h(x^*) \rangle \geq 0 \quad \forall y \in C. \quad (2.9)$$

Adding two inequalities (2.7) and (2.9) we obtain

$$\langle \nabla_1 D(h(x^*), x^*), x^* - h(x^*) \rangle \geq \langle z^*, h(x^*) - x^* \rangle + \varphi(x^*) - \varphi(h(x^*)). \quad (2.10)$$

Since $z^* \in \partial_\varphi(h(x^*))$, we have

$$\langle z^*, x^* - h(x^*) \rangle \leq \varphi(x^*) - \varphi(h(x^*)).$$

Thus

$$\langle z^*, x^* - h(x^*) \rangle - \varphi(x^*) + \varphi(h(x^*)) \leq 0. \quad (2.11)$$

From inequalities (2.10) and (2.11), it follows that

$$\langle \nabla_1 D(h(x^*), x^*), x^* - h(x^*) \rangle \geq 0.$$

By strongly monotonicity of $\nabla_1 D(\cdot, x^*)$ and $\nabla_1 D(x^*, x^*) = 0$, we have $x^* = h(x^*)$.

Conversely, suppose now $h(x^*) = x^*$. Then, by (2.8) we have

$$\langle F(x^*) + z^*, y - h(x^*) \rangle \geq 0 \quad \forall y \in C.$$

Since $z^* \in \partial_\varphi(h(x^*))$,

$$\langle z^*, y - x^* \rangle \leq \varphi(y) - \varphi(x^*) \quad \forall y \in C.$$

Adding the last two inequalities we have

$$\langle F(x^*), y - x^* \rangle + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in C,$$

which means that x^* is solution to problem (VIP).

□

Lemma 2.3 shows that the solution of the generalized variational inequality (VIP) can be approximated by an iterative procedure $x^{k+1} = h(x^k)$, $k = 0, 1, \dots$ where $c > 0$, x^0 is any starting point in C and $h(x^k)$ is the unique solution of the strongly convex program

$$\min\{\langle F(x^k), y \rangle + \varphi(y) + D(y, x^k) : y \in C\}.$$

However, generally, the sequence $\{x^k\}_{k \geq 0}$ does not converge to a solution of problem (VIP) (see [13, 18]). Our goal now is to construct iteratively a sequence such that it converges to a solution to problem (VIP).

Algorithm 2.4 Step 0. Choose $x^0 \in C$, $k := 0$, a positive sequence $\{c_k\}$ such that $c_k \rightarrow c > 0$ as $k \rightarrow +\infty$.

Step 1. Solve the strongly convex program:

$$\min\{\langle F(x^k), y - x^k \rangle + \varphi(y) + \frac{1}{c_k}D(y, x^k) : y \in C\} \quad (2.12)$$

to obtain the unique solution y^k .

If $y^k = x^k$, then terminate: x^k is a solution to problem (VIP).

Otherwise go to Step 2.

Step 2. Find x^{k+1} which is the unique solution to the strongly convex program:

$$\min\{\langle F(y^k), y - y^k \rangle + \varphi(y) + \frac{1}{c_k}D(y, x^k) : y \in C\}.$$

Step 3. Set $k := k + 1$, and return to Step 1.

In the next proposition, we justify the stopping criterion.

Proposition 2.5 If $y^k = x^k$, then x^k is a solution to problem (VIP).

Proof. If the algorithm terminates at Step1, then $y^k = x^k$. It means that x^k is the solution to problem (2.6). By Lemma 2.3 it is a solution to problem (VIP). □

In order to prove the convergence of Algorithm 2.4, we give the following key property of the sequence $\{x^k\}_{k \geq 0}$ generated by the algorithm.

Lemma 2.6 Suppose that the function $F : C \rightarrow \mathbb{R}^n \cup \{+\infty\}$ is monotone, L -Lipschitz on C , and φ is convex function on C . Then, if the algorithm does not terminate, then we have

$$\begin{aligned} \|A(x^{k+1} - x^*)\|^2 \leq & \|A(x^k - x^*)\|^2 - \frac{1 - 3\mu - c_k \|\bar{A}^{-1}\|^2}{1 + \mu} \|A(x^{k+1} - y^k)\|^2 \\ & - \frac{1 - 5\mu - c_k \bar{L}^2}{1 + \mu} \|A(x^k - y^k)\|^2, \end{aligned}$$

where x^* is any solution to problem (VIP).

Proof. Since y^k is the solution to problem (2.12), from an optimization results in convex programming [20], we have

$$0 = F(x^k) + w_1 + \frac{1}{c_k} \nabla_1 D(y^k, x^k),$$

where $w_1 \in \partial\varphi(y^k)$. It follows that

$$\frac{1}{c_k} \langle \nabla_1 D(y^k, x^k), y - y^k \rangle = -\langle F(x^k) + w_1, y - y^k \rangle \quad \forall y \in C. \quad (2.13)$$

Since $w_1 \in \partial\varphi(y^k)$, we have

$$\varphi(y) - \varphi(y^k) \geq \langle w_1, y - y^k \rangle \quad \forall y \in C.$$

From (2.13) and this inequality it follows that

$$\frac{1}{c_k} \langle \nabla_1 D(y^k, x^k), y - y^k \rangle \geq \langle F(x^k), y^k - y \rangle + \varphi(y^k) - \varphi(y) \quad \forall y \in C. \quad (2.14)$$

Replacing y by x^* , we obtain

$$\frac{1}{c_k} \langle \nabla_1 D(y^k, x^k), x^* - y^k \rangle \geq \langle F(x^k), y^k - x^* \rangle + \varphi(y^k) - \varphi(x^*). \quad (2.15)$$

Note that x^* is a solution to problem (VIP),

$$\langle F(x^*), x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0 \quad \forall x \in C.$$

By monotonicity of F , it follows that

$$\langle F(x^k), x^k - x^* \rangle + \varphi(x^k) - \varphi(x^*) \geq 0. \quad (2.16)$$

Combinating (2.15) and (2.16) we obtain that

$$\frac{1}{c_k} \langle \nabla_1 D(y^k, x^k), x^* - y^k \rangle \geq \langle F(y^k), y^k - x^k \rangle + \varphi(y^k) - \varphi(x^k). \quad (2.17)$$

On the other hand, since x^{k+1} is the solution to the strongly convex program

$$\min\{\langle F(y^k), y \rangle + \varphi(y) + \frac{1}{c_k} D(y, x^k) : y \in C\},$$

in the same way, we also have

$$\frac{1}{c_k} \langle \nabla_1 D(x^{k+1}, x^k), x^* - x^{k+1} \rangle \geq \langle F(y^k), x^{k+1} - y^k \rangle + \varphi(x^{k+1}) - \varphi(y^k). \quad (2.18)$$

Now, applying the Lipschitz condition of F and Remark 2.2 with $x = x^k, y = y^k$, we obtain

$$\begin{aligned} \langle F(x^k) - F(y^k), x^{k+1} - y^k \rangle &\leq \|F(x^k) - F(y^k)\| \|x^{k+1} - y^k\| \\ &\leq \frac{1}{2} \|F(y^k) - F(x^k)\|^2 + \frac{1}{2} \|x^{k+1} - y^k\|^2 \\ &\leq \frac{\bar{L}^2}{2} \|A(y^k - x^k)\|^2 + \frac{\|\bar{A}^{-1}\|^2}{2} \|A(x^{k+1} - y^k)\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \langle F(y^k), x^{k+1} - y^k \rangle &\geq \langle F(x^k), x^{k+1} - y^k \rangle - \frac{\bar{L}^2}{2} \|A(y^k - x^k)\|^2 \\ &\quad - \frac{\|\bar{A}^{-1}\|^2}{2} \|A(x^{k+1} - y^k)\|^2. \end{aligned} \quad (2.19)$$

From (2.5), (2.18) and (2.19), we have

$$\begin{aligned} \langle A(x^{k+1} - x^k), A(x^* - x^{k+1}) \rangle &\geq \\ &\mu \langle X_{x^k} \log \frac{l(x^{k+1})}{l(x^k)}, A(x^* - x^{k+1}) \rangle + c_k \langle F(x^k), x^{k+1} - y^k \rangle \\ &\quad + c_k \varphi(x^{k+1}) - c_k \varphi(y^k) - \frac{c_k \bar{L}^2}{2} \|A(y^k - x^k)\|^2 \\ &\quad - \frac{c_k \|\bar{A}^{-1}\|^2}{2} \|A(x^{k+1} - y^k)\|^2. \end{aligned} \quad (2.20)$$

If $y = x^{k+1}$, inequality (2.14) becomes

$$\begin{aligned} \langle F(x^k), x^{k+1} - y^k \rangle + \varphi(x^{k+1}) - \varphi(y^k) &\geq \frac{1}{c_k} \langle \nabla_1 D(y^k, x^k), y^k - x^{k+1} \rangle \\ &= \frac{1}{c_k} \langle A^T(l(x^k) - l(y^k)) - \mu X_{x^k} \log \frac{l(y^k)}{l(x^k)}, y^k - x^{k+1} \rangle \\ &= \frac{1}{c_k} \langle A(y^k - x^k), A(y^k - x^{k+1}) \rangle \\ &\quad - \frac{1}{c_k} \mu \langle X_{x^k} \log \frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1}) \rangle. \end{aligned} \quad (2.21)$$

From (2.20) and (2.21), it follows that

$$\begin{aligned} \langle A(x^{k+1} - x^k), A(x^* - x^{k+1}) \rangle &\geq \\ &\mu \langle X_{x^k} \log \frac{l(x^{k+1})}{l(x^k)}, A(x^* - x^{k+1}) \rangle - \frac{c_k \bar{L}^2}{2} \|A(y^k - x^k)\|^2 \\ &\quad + \langle A(y^k - x^k), A(y^k - x^{k+1}) \rangle - \frac{c_k \|\bar{A}^{-1}\|^2}{2} \|A(x^{k+1} - y^k)\|^2 \\ &\quad - \mu \langle X_{x^k} \log \frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1}) \rangle. \end{aligned} \quad (2.22)$$

Substituting

$\|A(x^k - x^*)\|^2 = \|A(x^k - x^{k+1})\|^2 + \|A(x^{k+1} - x^*)\|^2 + 2\langle A(x^{k+1} - x^k), A(x^* - x^{k+1}) \rangle$
into (2.22), we obtain the estimation

$$\begin{aligned} \|A(x^k - x^*)\|^2 &\geq \|A(x^k - x^{k+1})\|^2 + \|A(x^{k+1} - x^*)\|^2 - c_k \bar{L}^2 \|A(y^k - x^k)\|^2 \\ &\quad + 2\langle A(y^k - x^k), A(y^k - x^{k+1}) \rangle + 2\mu \langle X_{x^k} \log \frac{l(x^{k+1})}{l(x^k)}, A(x^* - x^{k+1}) \rangle \\ &\quad - 2\mu \langle X_{x^k} \log \frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1}) \rangle - c_k \|\bar{A}^{-1}\|^2 \|A(x^{k+1} - y^k)\|^2. \end{aligned}$$

Combining this inequality with the following equality

$\|A(x^{k+1} - x^k)\|^2 = \|A(x^{k+1} - y^k)\|^2 + \|A(x^k - y^k)\|^2 + 2\langle A(x^{k+1} - y^k), A(y^k - x^k) \rangle$,
we have

$$\begin{aligned} \|A(x^{k+1} - x^*)\|^2 &\leq \|A(x^k - x^*)\|^2 - \|A(x^{k+1} - y^k)\|^2 - \|A(x^k - y^k)\|^2 \\ &\quad + c_k \|\bar{A}^{-1}\|^2 \|A(x^{k+1} - y^k)\|^2 - 2\mu \langle X_{x^k} \log \frac{l(x^{k+1})}{l(x^k)}, A(x^* - x^{k+1}) \rangle \\ &\quad + 2\mu \langle X_{x^k} \log \frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1}) \rangle + c_k \bar{L}^2 \|A(y^k - x^k)\|^2. \end{aligned} \quad (2.23)$$

For each $t > 0$ we have $1 - \frac{1}{t} \leq \log t \leq t - 1$, then we obtain after multiplication by $l_i(x^*) \geq 0$ for each $i = 1, \dots, p$,

$$l_i(x^k) l_i(x^*) \log \frac{l_i(x^{k+1})}{l_i(x^k)} \leq l_i(x^*) (l_i(x^{k+1}) - l_i(x^k)), \quad (2.24)$$

and after multiplication by $-l_i(x^{k+1}) \leq 0$ for each $i = 1, \dots, p$,

$$\begin{aligned} -l_i(x^k) l_i(x^{k+1}) \log \frac{l_i(x^{k+1})}{l_i(x^k)} &\leq -l_i(x^k) l_i(x^{k+1}) \left(1 - \frac{l_i(x^k)}{l_i(x^{k+1})}\right) \\ &= l_i(x^k) (l_i(x^k) - l_i(x^{k+1})). \end{aligned} \quad (2.25)$$

Adding two inequalities (2.24) and (2.25), we obtain

$$\begin{aligned} &= |l_i(x^k) - l_i(x^*)|^2 + |l_i(x^k) - l_i(x^{k+1})|^2 \\ &\quad - |l_i(x^{k+1}) - l_i(x^*)|^2 \quad \forall i = 1, \dots, p. \end{aligned}$$

These inequalities deduce that

$$\begin{aligned} &2 \langle X_{x^k} \log \frac{l(x^{k+1})}{l(x^k)}, A(x^{k+1} - x^*) \rangle \leq \\ &\|A(x^k - x^*)\|^2 + \|A(x^k - x^{k+1})\|^2 - \|A(x^{k+1} - x^*)\|^2. \end{aligned} \quad (2.26)$$

In the same way, we also have

$$2\langle X_{x^k} \log \frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1}) \rangle \leq \|A(x^k - y^k)\|^2 + \|A(x^k - x^{k+1})\|^2 - \|A(y^k - x^{k+1})\|^2. \quad (2.27)$$

Adding the inequalities (2.23), (2.26) and (2.27), we get

$$\begin{aligned} \|A(x^{k+1} - x^*)\|^2 &\leq \|A(x^k - x^*)\|^2 - \|A(x^{k+1} - y^k)\|^2 - \|A(x^k - y^k)\|^2 \\ &\quad + c_k \bar{L}^2 \|A(y^k - x^k)\|^2 + c_k \|\bar{A}^{-1}\|^2 \|A(x^{k+1} - y^k)\|^2 + \mu (\|A(x^k - x^*)\|^2 \\ &\quad + \|A(x^k - x^{k+1})\|^2 - \|A(x^{k+1} - x^*)\|^2) + \mu (\|A(y^k - x^k)\|^2 \\ &\quad + \|A(x^{k+1} - x^k)\|^2 - \|A(x^{k+1} - y^k)\|^2), \end{aligned}$$

and consequently

$$\begin{aligned} (1 + \mu) \|A(x^{k+1} - x^*)\|^2 &\leq \\ &\quad (1 + \mu) \|A(x^k - x^*)\|^2 - (1 + \mu - c_k \|\bar{A}^{-1}\|^2) \|A(x^{k+1} - y^k)\|^2 \\ &\quad - (1 - \mu - c_k \bar{L}^2) \|A(x^k - y^k)\|^2 + 2\mu \|A(x^{k+1} - x^k)\|^2. \end{aligned} \quad (2.28)$$

Applying the following inequality

$$\|A(x^{k+1} - x^k)\|^2 \leq 2\|A(x^{k+1} - y^k)\|^2 + 2\|A(x^k - y^k)\|^2$$

to the last term in the right hand side of (2.28), we obtain

$$\begin{aligned} (1 + \mu) \|A(x^{k+1} - x^*)\|^2 &\leq \\ &\quad (1 + \mu) \|A(x^k - x^*)\|^2 - (1 - 3\mu - c_k \|\bar{A}^{-1}\|^2) \|A(x^{k+1} - y^k)\|^2 \\ &\quad - (1 - 5\mu - c_k \bar{L}^2) \|A(x^k - y^k)\|^2, \end{aligned}$$

which proves this lemma. \square

The following theorem establishes the convergence of the algorithm.

Theorem 2.7 *Suppose that the function F is monotone and L -Lipschitz on C , that φ is convex and lower semicontinuous on C . Then, if the algorithm does not terminate and*

$$0 < \epsilon, 0 < \mu < \min\left\{\frac{1 - \epsilon - c_k \|\bar{A}^{-1}\|^2}{3}, \frac{1 - \epsilon - c_k \bar{L}^2}{5}\right\},$$

then the sequence $\{x^k\}_{k \geq 0}$ converges to a solution to problem (VIP).

Proof. From

$$0 < \mu < \min\left\{\frac{1 - \epsilon - c_k \|\bar{A}^{-1}\|^2}{3}, \frac{1 - \epsilon - c_k \bar{L}^2}{5}\right\}$$

and $\epsilon > 0$, we have

$$1 - 3\mu - c_k \|\bar{A}^{-1}\|^2 > 0 \quad \text{and} \quad 1 - 5\mu - c_k \bar{L}^2 > 0 \quad \forall k = 0, 1, \dots$$

Then, using Lemma 2.6 we obtain that

$$\|A(x^{k+1} - x^*)\|^2 \leq \|A(x^k - x^*)\|^2 \quad \forall k = 0, 1, \dots \quad (2.29)$$

It means that the sequence $\{\|A(x^k - x^*)\|\}_{k \geq 0}$ is nonincreasing. Since it is bounded below by 0, it must be convergent. Since A is of maximal rank the function $u \rightarrow \|u\|_A := \|Au\|$ is norm on \mathbb{R}^n and it follows that the sequence $\{\|x^k - x^*\|\}_{k \geq 0}$ converges. Then the sequence $\{x^k\}_{k \geq 0}$ is bounded and it has a subsequence $\{x^{k_i}\}_{i \geq 0}$ such that $x^{k_i} \rightarrow \bar{x}$ as $i \rightarrow +\infty$. From Lemma 2.6, we get

$$\frac{1 - 5\mu - c_k \bar{L}^2}{1 + \mu} \|A(x^k - y^k)\|^2 \leq \|A(x^k - x^*)\|^2 - \|A(x^{k+1} - x^*)\|^2 \quad \forall k = 0, 1, \dots$$

Applying these inequalities iteratively, we obtain

$$\sum_{k=0}^n \frac{1 - 5\mu - c_k \bar{L}^2}{1 + \mu} \|A(x^k - y^k)\|^2 \leq \|A(x^0 - x^*)\|^2 - \|A(x^{n+1} - x^*)\|^2 \quad \forall k \geq 0.$$

As the sequence $\{\|A(x^{n+1} - x^*)\|\}_{k \geq 0}$ is convergent, passing $n \rightarrow +\infty$ we have

$$\lim_{k \rightarrow +\infty} \frac{1 - 3\mu - c_k \|\bar{A}^{-1}\|^2}{1 + \mu} \|A(x^k - y^k)\|^2 = 0.$$

Using this with the assumption $1 - 5\mu - c_k \bar{L}^2 > \epsilon > 0$, we get

$$\lim_{k \rightarrow +\infty} \epsilon \|A(x^k - y^k)\| = 0,$$

which implies

$$\lim_{i \rightarrow +\infty} \|A(\bar{x} - y^{k_i})\| = 0.$$

It holds that

$$\lim_{i \rightarrow \infty} y^{k_i} = \bar{x}.$$

Recall that y^{k_i} is the solution of the problem

$$\min\{\langle F(x^{k_i}), y \rangle + \varphi(y) + \frac{1}{c_{k_i}} D(y, x^{k_i}) : y \in C\}.$$

Then

$$\langle F(x^{k_i}), y^{k_i} \rangle + \varphi(y^{k_i}) + \frac{1}{c_{k_i}} D(y^{k_i}, x^{k_i}) \leq \langle F(x^{k_i}), y \rangle + \varphi(y) + \frac{1}{c_{k_i}} D(y, x^{k_i}) \quad \forall y \in C.$$

Using the continuity of F , upper semicontinuity of $D(y, \cdot)$, passing to the limit as $i \rightarrow +\infty$ we obtain

$$\langle F(\bar{x}), y - \bar{x} \rangle + \varphi(y) - \varphi(\bar{x}) + \frac{1}{c}D(y, \bar{x}) \geq 0 \quad \forall y \in C.$$

Then, there exists a $\bar{w} \in \partial\varphi(\bar{x})$ such that

$$\langle F(\bar{x}) + \bar{w} + \frac{1}{c}\nabla_1 D(\bar{x}, \bar{x}), y - \bar{x} \rangle \geq 0 \quad \forall y \in C.$$

As $\nabla_1 D(\bar{x}, \bar{x}) = 0$, this reduces to

$$\langle F(\bar{x}) + \bar{w}, y - \bar{x} \rangle \geq 0 \quad \forall y \in C.$$

Combining this inequality with the convexity of φ ,

$$\varphi(y) - \varphi(\bar{x}) \geq \langle \bar{w}, y - \bar{x} \rangle \quad \forall y \in C,$$

we obtain that

$$\langle F(\bar{x}), y - \bar{x} \rangle + \varphi(y) - \varphi(\bar{x}) \geq 0 \quad \forall y \in C.$$

So \bar{x} is a solution to problem (VIP).

Replacing x^* by \bar{x} in (2.29) fields

$$\|A(x^{k+1} - \bar{x})\| \leq \|A(x^k - \bar{x})\| \quad \forall k = 0, 1, \dots$$

which implies that the sequence $\{\|A(x^k - \bar{x})\|\}_{k \geq 0}$ is convergent. We then have that the sequence $\{\|x^k - \bar{x}\|\}_{k \geq 0}$ is convergent. By the above proof, the sequence $\{x^k\}_{k \geq 0}$ has a subsequence converging to \bar{x} , we deduce that the whole sequence $\{x^k\}_{k \geq 0}$ converges to the solution \bar{x} of problem (VIP). □

3 The interior-quadratic proximal linesearch method

Convergence of Algorithm 2.4 requires that the function F satisfies the Lipschitz condition on C . This condition depends on positive constant L and in cases, it is unknown or difficult to approximate. So in this section, in order to avoid this assumption, we combine the interior-quadratic function with line search technique. This technique has been used widely in descent method for solving variational inequality problem (VI) on $C := \mathbb{R}_+^n$ (see [13, 23]).

The algorithm then can be described as follows.

Algorithm 3.1 Step 0. Take $x^0 \in C, k := 0$ and a sequence $\gamma_k \in (0; 2) \forall k \geq 0$.

Step 1. Find y^k which is the solution to the strongly convex program:

$$\min\{\langle F(x^k), y \rangle + \varphi(y) + \frac{1}{c_k}D(y, x^k) : y \in C\}. \quad (3.1)$$

If $y^k = x^k$, then stop.

Otherwise go to Step 2.

Step 2. Find $\lambda_k \in (0, 1)$ as the smallest number such that

$$\langle F((1-\lambda_k)x^k + \lambda_k y^k), y^k - x^k \rangle + \varphi(y^k) - \varphi((1-\lambda_k)x^k + \lambda_k y^k) + \frac{1}{2c_k}D(y^k, x^k) \leq 0. \quad (3.2)$$

Set $z^k := (1 - \lambda_k)x^k + \lambda_k y^k$, choose $g^k \in F(z^k) + \partial\varphi(z^k)$.

If $g^k = 0$, then stop.

Otherwise go to Step 3.

Step 3. Set $\delta_k := \gamma_k \frac{\lambda_k (\langle F(z^k), z^k - y^k \rangle + \varphi(z^k) - \varphi(y^k))}{(1-\lambda_k)\|g^k\|^2}$ and

$$x^{k+1} = P_C(x^k - \delta_k g^k),$$

$k := k + 1$ and return to Step 1.

Recall that $P_C(x)$ denotes the projection of x on C .

First we have to show that there always exists $\lambda_k \in (0, 1)$ as the smallest number satisfies (3.2). We suppose on the contrary that for every $\lambda \in (0, 1)$, we have

$$\langle F((1-\lambda)x^k + \lambda y^k), y^k \rangle + \varphi(y^k) - \varphi((1-\lambda)x^k + \lambda y^k) + \frac{1}{2c_k}D(y^k, x^k) > 0.$$

Passing to the limit in the above inequality (as $\lambda \rightarrow 0^+$), by the continuity of $F(y)$, we obtain

$$\langle F(x^k), y^k - x^k \rangle + \varphi(y^k) - \varphi(x^k) + \frac{1}{2c_k}D(y^k, x^k) \geq 0. \quad (3.3)$$

Since y^k is a solution to (3.1), it follows that

$$\langle F(x^k), y \rangle + \varphi(y) + \frac{1}{c_k}D(y, x^k) \geq \langle F(x^k), y^k \rangle + \varphi(y^k) + \frac{1}{c_k}D(y^k, x^k).$$

Replacing y by x^k in the above inequality, we have

$$0 \geq \langle F(x^k), y^k - x^k \rangle + \varphi(y^k) - \varphi(x^k) + \frac{1}{c_k}D(y^k, x^k). \quad (3.4)$$

Then from (3.3) and (3.4) it follows that $D(x^k, y^k) = 0$, i.e., $d(l(x^k), l(y^k)) = 0$. Since $l(x) = b - Ax$ and A is maximal rank, we obtain $x^k = y^k$. This contradicts to $x^k \neq y^k$ in Step 1.

Remark 3.2 The smallest number $\lambda_k \in (0, 1)$ in Step 2 of Algorithm 3.1 can be replaced by the following: With $\beta \in (0, 1)$, we find n as the smallest natural number such that

$$\langle F(\beta^n x^k + (1-\beta^n)y^k), y^k - x^k \rangle + \varphi(y^k) - \varphi(\beta^n x^k + (1-\beta^n)y^k) + \frac{1}{2c_k} D(y^k, x^k) \leq 0.$$

then set $\lambda_k := 1 - \beta^n$.

In the next proposition, we justify the stopping criterion.

Proposition 3.3 If $y^k = x^k$ or $g^k = 0$, then x^k is a solution to problem (VIP).

Proof. If the algorithm terminates at Step 1, then $y^k = x^k$. It means that x^k is the solution to problem (3.1). Then

$$\langle F(x^k), y \rangle + \varphi(y) + D(y, x^k) \geq \langle F(x^k), x^k \rangle + \varphi(x^k) + D(x^k, x^k) \quad \forall y \in C.$$

From $D(x^k, x^k) = 0$, this inequality follows that x^k is a solution to problem (VIP).

If the algorithm terminates at Step 2, then $g^k = 0$, that means $0 \in F(z^k) + \partial\varphi(z^k)$. Thus $0 = F(z^k) + w^k$, where $w^k \in \partial\varphi(z^k)$. Hence

$$\begin{aligned} \varphi(x) - \varphi(z^k) &\geq \langle w^k, x - z^k \rangle \\ &= -\langle F(z^k), x - z^k \rangle \quad \forall x \in C. \end{aligned}$$

So z^k is a solution to problem (VIP). \square

In order to prove the convergence of Algorithm 3.1, we give the following key property of the sequence $\{x^k\}_{k \geq 0}$ generated by the algorithm.

Lemma 3.4 Suppose that the function F is monotone on C and φ is convex on C . Then, if the algorithm does not terminate, then we have

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \frac{(2 - \gamma_k)\delta_k^2}{\gamma_k} \|g^k\|^2,$$

where x^* is any solution to problem (VIP).

Proof. We have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|P_k(x^k - \delta_k g^k) - x^*\|^2 \\ &\leq \|x^k - x^* - \delta_k g^k\|^2 \\ &= \|x^k - x^*\|^2 - 2\delta_k \langle g^k, x^k - x^* \rangle + (\delta_k \|g^k\|)^2. \end{aligned} \quad (3.5)$$

Note that, since x^* is a solution to problem (VIP),

$$\langle F(x^*), y - x^* \rangle + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in C.$$

Then by monotonicity, it follows that

$$\langle F(z^k), z^k - x^* \rangle + \varphi(z^k) - \varphi(x^*) \geq 0.$$

Combining this with

$$\begin{aligned} \langle g^k, x^k - x^* \rangle &= \langle g^k, x^k - z^k \rangle + \langle g^k, z^k - x^* \rangle \\ &\geq \langle g^k, x^k - z^k \rangle + \varphi(z^k) - \varphi(x^*) - \langle F(z^k), x^* - z^k \rangle, \end{aligned}$$

we obtain

$$\begin{aligned} \langle g^k, x^k - x^* \rangle &\geq \langle g^k, x^k - z^k \rangle \\ &= \frac{\lambda_k}{1 - \lambda_k} \langle g^k, z^k - y^k \rangle \\ &\geq \frac{\lambda_k}{1 - \lambda_k} (\langle F(z^k), z^k - y^k \rangle + \varphi(z^k) - \varphi(y^k)) \\ &= \frac{\delta_k}{\gamma_k} \|g^k\|^2. \end{aligned} \tag{3.6}$$

From (3.2) it follows that $\langle F(z^k), y^k - z^k \rangle + \varphi(y^k) - \varphi(z^k) < 0$. Hence

$$\delta_k := \gamma_k \frac{\lambda_k (\langle F(z^k), z^k - y^k \rangle + \varphi(z^k) - \varphi(y^k))}{(1 - \lambda_k) \|g^k\|^2} > 0. \tag{3.7}$$

Then from (3.5), (3.6) and (3.7), we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - 2 \frac{\delta_k^2}{\gamma_k} \|g^k\|^2 + (\delta_k \|g^k\|)^2 \\ &= \|x^k - x^*\|^2 - \frac{2 - \gamma_k}{\gamma_k} (\delta_k \|g^k\|)^2, \end{aligned}$$

which proves the above lemma. \square

Now we are in a position to consider a convergence of Algorithm 3.1 in a case which does not terminate.

Theorem 3.5 *Suppose that the sequences $\gamma_k \in (0, 2)$, $c_k \rightarrow \bar{c}$ as $k \rightarrow \infty$, and functions F, φ satisfies the following conditions:*

- (a) $\liminf \gamma_k (2 - \gamma_k) > 0$.
- (b) f is monotone on C .
- (c) φ is lower semicontinuous on C .

Then, if Algorithm 3.1 doesn't terminate at Step 1 or Step 2, then the sequence $\{x^k\}_{k \geq 0}$ converges to x^ which is a solution to problem (VIP).*

Proof. From Lemma 3.4, we have

$$\begin{aligned} \sum_{k=0}^n \frac{2-\gamma_k}{\gamma_k} (\delta_k \|g^k\|)^2 &\leq \sum_{k=0}^n (\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2) \\ &= \|x^0 - x^*\|^2 - \|x^{n+1} - x^*\|^2 \quad \forall n \geq 0. \end{aligned}$$

On the other hand, also since Lemma 3.4 deduces that $\{\|x^k - x^*\|\}$ is a decreasing sequence and is lower bounded by $\|x^0 - x^*\|$, then it must converge. It means that

$$\sum_{k=0}^{\infty} \frac{2-\gamma_k}{\gamma_k} (\delta_k \|g^k\|)^2 < +\infty.$$

Hence

$$\lim_{k \rightarrow \infty} \frac{2-\gamma_k}{\gamma_k} (\delta_k \|g^k\|)^2 = 0,$$

which together with $\liminf_{k \rightarrow \infty} (2-\gamma_k)\gamma_k > 0$ implies

$$\lim_{k \rightarrow \infty} \frac{\lambda_k (\langle F(z^k), z^k - y^k \rangle + \varphi(z^k) - \varphi(y^k))}{(1-\lambda_k)\|g^k\|} = 0.$$

From the convergence of $\{\|x^k - x^*\|\}_{k \geq 0}$, we have that the sequence $\{x^k\}_{k \geq 0}$ is bounded. Then by the maximum theorem [4], we can deduce that the sequence $\{g^k\}_{k \geq 0}$ is bounded too. Thus

$$\lim_{k \rightarrow \infty} \frac{\lambda_k (\langle F(z^k), z^k - y^k \rangle + \varphi(z^k) - \varphi(y^k))}{1-\lambda_k} = 0. \quad (3.8)$$

According to the rule (3.2), it is easy to see that

$$\frac{1}{2c_k} D(y^k, x^k) \leq -\langle F(z^k), y^k - x^k \rangle - \varphi(y^k) + \varphi(z^k). \quad (3.9)$$

We consider two cases:

Case 1: If $\limsup_{k \rightarrow \infty} \lambda_k > 0$, then there exists $\bar{\lambda} \in (0, 1]$ such that $\lambda_k \geq \bar{\lambda} \forall k \geq 0$.

From (3.8) and inequality (3.9), we have

$$\lim_{k \rightarrow \infty} D(y^k, x^k) = 0. \quad (3.10)$$

Since the sequence $\{x^k\}_{k \geq 0}$ is bounded, hence it has a subsequence $\{x^k : k \in M\}$ converging to a point \bar{x} . Using the limit (3.10) we see that the subsequence $\{y^k : k \in M\}$ also converges to \bar{x} . Note that y^k is a solution to problem (3.1), hence

$$\langle F(x^k), y \rangle + \varphi(y) + \frac{1}{c_k} D(y, x^k) \geq \langle F(x^k), y^k \rangle + \varphi(y^k) + \frac{1}{c_k} D(y^k, x^k) \quad \forall y \in C.$$

Passing to the limit as $k \rightarrow \infty$ and using the continuity of F , the lower semicontinuity of φ , we have

$$\langle F(\bar{x}), y \rangle + \varphi(y) + \frac{1}{c_k} D(y, \bar{x}) \geq \langle F(\bar{x}), \bar{x} \rangle + \varphi(\bar{x}) + \frac{1}{c_k} D(\bar{x}, \bar{x}) \quad \forall y \in C.$$

By Lemma 2.3, \bar{x} is a solution to problem (VIP), thus the proof of the theorem in this case is complete.

Case 2: If $\limsup_{k \rightarrow \infty} \lambda_k = 0$, then since $\{x^k\}$ is bounded, we have some subsequence $\{x^k : k \in M\}$ converging to some point \bar{x} as $k \rightarrow \infty$. From Step 1 of Algorithm 3.1, by lower semicontinuity of $\langle F(x^k), \cdot \rangle + \varphi(\cdot) + \frac{1}{c_k} D(\cdot, x^k)$, the sequence $\{y^k\}_{k \geq 0}$ is bounded too (see [4]). Thus, by taking a subsequence, if necessary, we may assume that the subsequence $\{y^k : k \in M\}$ also converges to some point \bar{y} . From

$$\langle F(x^k), y \rangle + \varphi(y) + \frac{1}{c_k} D(y^k, x^k) \geq \langle F(x^k), y^k \rangle + \varphi(y^k) + \frac{1}{c_k} D(y^k, x^k) \quad \forall y \in C,$$

by the lower semicontinuity of F , D and φ , taking the limit as $k \rightarrow \infty$, we can write

$$\langle F(\bar{x}), y \rangle + \varphi(y) + \frac{1}{\bar{c}} D(y, \bar{x}) \geq \langle F(\bar{x}), \bar{y} \rangle + \varphi(\bar{y}) + \frac{1}{\bar{c}} D(\bar{y}, \bar{x}) \quad \forall y \in C. \quad (3.11)$$

Substituting $y = \bar{x}$ we then have

$$0 \geq \langle F(\bar{x}), \bar{y} \rangle + \varphi(\bar{y}) - \varphi(\bar{x}) + \frac{1}{\bar{c}} D(\bar{y}, \bar{x}). \quad (3.12)$$

On the other hand, by Step 2 in Algorithm 3.1, since $\lambda_k \in (0, 1)$ is the smallest number satisfying

$$\langle F((1-\lambda_k)x^k + \lambda_k y^k), y^k - x^k \rangle + \varphi(y^k) - \varphi((1-\lambda_k)x^k + \lambda_k y^k) + \frac{1}{2c_k} D(y^k, x^k) \leq 0.$$

We deduce that

$$\langle F((1-\frac{1}{2}\lambda_k)x^k + \frac{1}{2}\lambda_k y^k), y^k - x^k \rangle + \varphi(y^k) - \varphi((1-\frac{1}{2}\lambda_k)x^k + \frac{1}{2}\lambda_k y^k) + \frac{1}{2c_k} D(y^k, x^k) > 0.$$

Passing $k \rightarrow \infty$, $k \in M$ the above inequality and using $\limsup_{k \rightarrow \infty} \lambda_k = 0$, we obtain

$$\langle F(\bar{x}), \bar{y} - \bar{x} \rangle + \varphi(\bar{y}) - \varphi(\bar{x}) + \frac{1}{2\bar{c}} D(\bar{y}, \bar{x}) \geq 0.$$

This together with (3.12) implies $D(\bar{x}, \bar{y}) = 0$, hence $\bar{x} = \bar{y}$. Then replacing \bar{y} in (3.11) by \bar{x} , we deduce that

$$\langle F(\bar{x}), y - \bar{x} \rangle + \varphi(y) - \varphi(\bar{x}) + \frac{1}{\bar{c}} D(y, \bar{x}) \geq 0 \quad \forall y \in C.$$

The proof is complete. \square

4 Numerical Results

The aim of this section is to illustrate the proposed algorithms on a class of generalized variational inequality (VIP) where

$$C := \mathbb{R}_+^n, \varphi = 0, \text{ and } F(x) = D(x) + Mx + q,$$

with the components of the $D(x)$ are $D_j(x) = d_j * \arctan(x_j) \forall j \geq 1$, d_j is chosen randomly in $(0, 1)$. The matrix $M = A^T A$ with A is $n \times n$ matrix whose entries are randomly generalized in the interval $(-1, 3)$. It is given by A. Bnouhachem (see [9]). Under these assumptions, it can be prove that F is continuous and monotone, that F is Lipschitz with constant $L \leq 1 + \|M\|$.

Note that in this case, the subproblem

$$y^k = \operatorname{argmin}\{\langle F(x^k), y - x^k \rangle + \frac{1}{c_k} d(y, x^k) : y \in C\}$$

where d is defined by (2.4). It is written as

$$y^k = \operatorname{argmin}\{\langle F(x^k), y - x^k \rangle + \frac{1}{2c_k} \|y - x^k\|^2 + \frac{\mu}{c_k} \sum_{i=1}^n (x_i^k)^2 \left(\frac{y_i}{x_i^k} \log \frac{y_i}{x_i^k} - \frac{y_i}{x_i^k} + 1 \right) : y \in C_+\},$$

where

$$C_+ := \{x \in \mathbb{R}^n : x_i > 0 \quad \forall i = 1, \dots, n\}.$$

It is not difficult to see that if we denote $y^k = (y_1^k, \dots, y_n^k)$ and $F(x) = (F_1(x), \dots, F_n(x))$ for all $x \in C$, then for every $i = 1, \dots, n$, we have y_i^k is the unique solution to the strongly convex problem

$$\min\{\frac{1}{2}t^2 + \eta_{ki} t + \xi_{ki} t \log t : t \in (0, +\infty)\},$$

where

$$\eta_{ki} := c_k F_i(x^k) - x_i^k - \mu x_i^k \log x_i^k - \mu x_i^k, \quad \xi_{ki} := \mu x_i^k \quad \forall i = 1, \dots, n.$$

In test we take the logarithmic parameter $\mu = 0.01$, $c_k = 0.01 \forall k \geq 1$ and the tolerance 10^{-7} . We obtained the following computational results.

Iter(k)	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k
0	1	1	1	1	1	1	1
1	1.0470	0.8957	1.0302	1.0030	0.9289	0.9641	0.9084
2	1.0886	0.8120	1.0654	1.0099	0.8549	0.9311	0.8199
3	1.1243	0.7436	1.1005	1.0155	0.7793	0.8975	0.7347
4	1.1528	0.6847	1.1296	1.0134	0.7022	0.8584	0.6540
5	1.1735	0.6296	1.1475	0.9976	0.6232	0.8095	0.5793
6	1.1864	0.5710	1.1496	0.9632	0.5422	0.7473	0.5112
7	1.1915	0.5067	1.1344	0.9088	0.4597	0.6706	0.4497
8	1.1887	0.4364	1.1014	0.8340	0.3763	0.5794	0.3946
9	1.1779	0.3601	1.0502	0.7383	0.2926	0.4733	0.3455
10	1.1590	0.2774	0.9801	0.6214	0.2095	0.3522	0.3020
11	1.1317	0.1882	0.8906	0.4829	0.1277	0.2159	0.2640

Table 1. Numerical results: Algorithm 2.4 with $n = 7$.

The approximate solution obtained after eleven iterations is

$$x^{10} = (1.1317, 0.1882, 0.8906, 0.4829, 0.1277, 0.2159, 0.2640)^T.$$

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