East-West J. of Mathematics: Vol. 10, No 1 (2008) pp. 73-81

$\sigma(*)$-RINGS AND THEIR EXTENSIONS AS 2-PRIMAL RINGS

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Abstract

In this article, we discuss the prime radical of skew polynomial rings over Noetherian rings. We recall $\sigma(*)$ property on a ring R (i.e. $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of R), where σ is an endomorphism of R. Also recall that a ring R is 2-primal if and only if $P(R)$ and the set of nilpotent elements of R are same, if and only if the prime radical is a completely semiprime ideal. It can be seen that a $\sigma(*)$ is a 2-primal ring. In this article we show that if R is a Noetherian $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$; σ is an automorphism of R and δ a σ-derivation of R, then the Ore extension $R[x; \sigma, \delta]$ is 2-primal Noetherian.

1 Introduction

A ring R always means an associative ring with identity. The field of rational numbers and the set of natural numbers are denoted by $\mathbb{Q}$ and $\mathbb{N}$ respectively unless otherwise stated. The set of prime ideals of R is denoted by $Spec(R)$. The sets of minimal prime ideals of R is denoted by $Min.Spec(R)$. Prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. Let R be a ring and σ an automorphism of R. Let I be an ideal of R such that $\sigma^m(I) = I$ for some $m \in \mathbb{N}$. We denote $\cap_{i=1}^{m} \sigma^i(I)$ by I^0. For any two ideals I, J of R; $I \subset J$ means that I is strictly contained in J.

This article concerns the study of skew polynomial rings (Ore extensions) in terms of 2-primal rings. Recall that a ring R is 2-primal if and only if

Key words: Minimal prime, prime radical, 2-primal, automorphism, derivation, $\sigma(*)$-ring.
2000 AMS Mathematics Subject Classification: Primary 16-XX,; Secondary 16S36, 16N40, 16P40, 16S32, 16W20, 16W25.

$N(R) = P(R)$ if and only if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We note that a commutative ring is 2-primal. Also the ring

$$\text{R} = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} \text{ where F is a field, is 2-primal.}$$

For further details on 2-primal rings, we refer the reader to [1, 2, 3, 7, 10].

Recall that $R[x;\sigma,\delta]$ is the usual polynomial ring with coefficients in R, in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x;\sigma,\delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote $R[x;\sigma,\delta]$ by O(R). In case δ is the zero map, we denote $R[x;\sigma]$ by S(R) and in case σ is the identity map, we denote $R[x;\delta]$ by D(R). The study of Ore-extension $O(R) = R[x;\sigma,\delta]$ and its special cases S(R) and D(R) have been of interest to many authors. For example [4, 5, 6, 9, 10, 11].

2-primal rings have been studied in recent years and are being treated by authors for different structures. In [10], Greg Marks discusses the 2-primal property of $R[x;\sigma,\delta]$, where R is a local ring, σ an automorphism of R and δ a σ-derivation of R. In Greg Marks [10], it has been investigated that when R is a local ring with a nilpotent maximal ideal, the Ore extension $R[x;\sigma,\delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of R. In the case where $R[x;\sigma,\delta]$ is 2-primal, it will satisfy an even stronger condition; in the case where $R[x;\sigma,\delta]$ is not 2-primal, it will fail to satisfy an even weaker condition.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [7]. 2-primal near rings have been discussed by Argac and Groenewald in [1].

Recall that in Krempa [8], a ring R is called σ-rigid if there exists an endomorphism of R with the property that $a\sigma(a) = 0$ implies a = 0 for $a \in R$. In [9], Kwak defines a $\sigma(*)$-ring R to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to the skew-polynomial ring $R[x;\sigma]$.

It is known that if R is a 2-primal Noetherian $\mathbb{Q}$-algebra, and δ is a derivation of R, then $R[x;\delta]$ is 2-primal Noetherian. (Theorem (2.4) of Bhat [3]).

In this paper we generalize the above result for $R[x;\sigma,\delta]$. But before that we note that a $\sigma(*)$-ring is a 2-primal ring [Proposition (2.3)]. We also note that if σ is an automorphism of R, then it can be extended to an automorphism of $R[x;\sigma]$ such that $\sigma(x) = x$; i.e. $\sigma(\Sigma_{i=0}^{n} x^i a_i) = \Sigma_{i=0}^{n} x^i \sigma(a_i)$, and prove that if R is a $\sigma(*)$-ring, then $R[x;\sigma]$ is also a $\sigma(*)$-ring [Theorem (2.9)].

We also find a relation between the minimal prime ideals of R and those of the Ore extension $R[x;\sigma,\delta]$, where R is a Noetherian $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$; σ is an automorphism of R and δ a σ-derivation of R. This is proved in Theorem (2.15).

We ultimately prove the following result [Theorem (2.18]:

Theorem: If R is a Noetherian $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$, where σ is an automorphism of R and δ a σ-derivation of R, then the Ore extension $R[x;\sigma,\delta]$ is 2-primal Noetherian.

2 2-primal skew-Polynomial rings

We begin with the following definition:

Definition 2.1 *(Kwak [9]) Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$.*

Example 2.2 *Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then P(R) = $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ Let $\sigma : R \to R$ be defined by $\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that R is a $\sigma(*)$-ring.*

Recall that an ideal I of a ring R is called σ-invariant if $\sigma(I) = I$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$. We also recall that an ideal J of a ring is called a σ-prime ideal of R if J is σ-invariant and for any σ-invariant ideals K and L with $KL \subseteq J$, we have $K \subseteq J$ or $L \subseteq J$.

Proposition 2.3 *Let R be a ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring implies R is 2-primal.*

Proof Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. □

The following example shows that there exists an endomorphism σ of a ring R such that the converse of the above Proposition does not hold.

Example 2.4 *Let $R = F[x]$, F a field. Then R is a commutative domain, and therefore is 2-primal with $P(R) = 0$. Let $\sigma : R \to R$ be defined by $\sigma(f(x)) = f(0)$. Let $f(x) = xa$, $0 \neq a \in F$. Then $f(x)\sigma(f(x)) \in P(R)$, but $f(x) \notin P(R)$. Therefore R is not a $\sigma(*)$-ring.*

Recall that an ideal P of a ring R is said to be completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for a, $b \in R$.

We now give a necessary and sufficient condition for a Noetherian ring to be a $\sigma(*)$-ring in the following Theorem:

Theorem 2.5 *Let R be a Noetherian ring, and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.*

Proof Let R be a Noetherian ring such that for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R. Let $a \in R$ be such that $a\sigma(a) \in P(R) = \cap_{i=1}^{n} U_i$, where U_i are the minimal primes of R. Now for each i, $a \in U_i$ or $\sigma(a) \in U_i$ as U_i are completely prime. Now $\sigma(a) \in U_i = \sigma(U_i)$ implies that $a \in U_i$. Therefore $a \in P(R)$. Hence R is a $\sigma(*)$-ring.

Conversely, suppose that R is a $\sigma(*)$-ring and let $U = U_1$ be a minimal prime ideal of R. Now by Proposition (2.3), P(R) is completely semiprime. Let $U_2, U_3, ..., U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Now suppose that $U = U_1$ is not completely prime. Then there exist $a, b \in R \setminus U$ with $ab \in U$. Let c be any element of $b(U_2 \cap U_3 \cap ... \cap U_n)a$. Then $c^2 \in \cap_{i=1}^{n} U_i = P(R)$. So $c \in P(R)$ and, thus $b(U_2 \cap U_3 \cap ... \cap U_n)a \subseteq U$. Therefore $bR(U_2 \cap U_3 \cap ... \cap U_n)Ra \subseteq U$ and, as U is prime, $a \in U$, $U_i \subseteq U$ for some $i \neq 1$ or $b \in U$. None of these can occur, so U is completely prime. □

We also note that if R is a Noetherian ring, then Min.Spec(R) is finite (Theorem (2.4) of Goodearl and Warfield [6]) and for any automorphism σ of R and for any $U \in Min.Spec(R)$, we have $\sigma^i(U) \in Min.Spec(R)$ for all $i \in \mathbb{N}$, therefore, it follows that there exists some $m \in N$ such that $\sigma^m(U) = U$ for all $U \in Min.Spec(R)$. As mentioned earlier we denote $\cap_{i=0}^{m} \sigma^i(U)$ by U^0. With this we have the following Theorem:

Theorem 2.6 *Let R be a Noetherian ring and σ an automorphism of R. Let $S(R) = R[x; \sigma]$ be as usual. Then:*

(1) If $P \in Min.Spec(S(R))$, then $P = (P \cap R)S(R)$ and there exists $U \in Min.Spec(R)$ such that $P \cap R = U^0$.

(2) If $U \in Min.Spec(R)$, then $U^0 S(R) \in Min.Spec(S(R))$.

Proof See Theorem (2.4) of Bhat [4]. □

Corollary 2.7 *Let R be a Noetherian $\sigma(*)$-ring, where σ is an automorphism of R. Then $P \in Min.Spec(S(R))$ if and only if there exists $Q \in Min.Spec(R)$ such that $S(Q) = P$ and $(P \cap R) = Q$.*

Proof R is a Noetherian $\sigma(*)$-ring, therefore $U^0 = U$ for any $U \in Min.Spec(R)$ by Theorem (2.5). Now use Theorem (2.6). □

Corollary 2.8 *Let R be a Noetherian $\sigma(*)$-ring, where σ is an automorphism of R. Then $P(R)[x; \sigma] = P(R[x; \sigma])$.*

Theorem 2.9 *Let R be a Noetherian $\sigma(*)$-ring, where σ is an automorphism of R. Then $R[x; \sigma]$ is also a Noetherian $\sigma(*)$-ring.*

Proof $R[x;\sigma]$ is Noetherian by Hilbert Basis Theorem (Theorem (1.12) of Goodearl and Warfield [6]). Now we have $P(R)[x;\sigma] = P(R[x;\sigma])$ by Corollary (2.8). Let $f(x) = \Sigma_{i=0}^{n} x^i a_i \in R[x;\sigma]$ be such that $f(x)\sigma\big(f(x)\big) \in P(R[x;\sigma]) = P(R)[x;\sigma]$; i.e.

$$(x^n a_n + ... + a_0)(x^n \sigma(a_n) + ... + \sigma(a_0)) \in P(R)[x;\sigma],$$

or

$$x^{2n}\sigma^n(a_n)\sigma(a_n) + ... + a_0\sigma(a_0) \in P(R)[x;\sigma],$$

which implies that $a_0\sigma(a_0) \in P(R)$, and therefore $a_0 \in P(R)$, as R is a $\sigma(*)$-ring.

Therefore $g(x)\sigma\big(g(x)\big) \in P(R)[x;\sigma]$, where $g(x) = \Sigma_{i=1}^{n} x^i a_i$. With the same process as above, in a finite number of steps, we get that $a_i \in P(R)$ for all i, $1 \leq i \leq n$. Thus $f(x) \in P(R)[x;\sigma] = P(R[x;\sigma])$. Hence $R[x;\sigma]$ is also a Noetherian $\sigma(*)$-ring. □

We now give a relation between the minimal prime ideals of R and those of $R[x;\sigma,\delta]$, where R is a Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R. This is proved in Theorem (2.15). Towards this we have the following:

Proposition 2.10 *Let R be a Noetherian $\mathbb{Q}$-algebra, σ an automorphism and δ a σ-derivation of R. Then $e^{t\delta}$ is an automorphism of $T = R[[t,\sigma]]$, the skew power series ring.*

Proof The proof is on the same lines as in Seidenberg [11] and in the non-commutative case on the same lines as provided by Blair and Small in [5]. □

Hence forth we denote $R[[t,\sigma]]$ by T.

Lemma 2.11 *Let R be a Noetherian $\mathbb{Q}$-algebra, σ an automorphism and δ a σ-derivation of R. Then an ideal I of R is δ-invariant if and only if TI is $e^{t\delta}$-invariant.*

Proof Let TI be $e^{t\delta}$-invariant. Let $a \in I$. Then $a \in TI$. So $e^{t\delta}(a) \in TI$; i.e. $a + t\delta(a) + (t^2\delta^2/2!)(a) + ... \in TI$. Therefore $\delta(a) \in I$.

Conversely suppose that $\delta(I) \subseteq I$ and let $f = \sum t^i a_i \in TI$. Then $e^{t\delta}(f)$ $= f + t\delta(f) + (t^2\delta^2/2!)(f) + ... \in TI$, as $\delta(a_i) \in I$. Therefore $e^{t\delta}(TI) \subseteq TI$. Replacing $e^{t\delta}$ by $e^{-t\delta}$, we get that $e^{t\delta}(TI) = TI$. □

Let σ be an automorphism of a ring R, and I be an ideal of R such that $\sigma(I) = I$. Then it is easy to see that $TI \subseteq IT$ and $IT \subseteq TI$. Hence $TI = IT$ is an ideal of T.

Proposition 2.12 *Let R be a Noetherian $\sigma(*)$-ring and T as usual. Then:*

(1) $U \in Min.Spec(R)$ implies that $UT \in Min.Spec(T)$.

(2) $P \in Min.Spec(T)$ implies that $P \cap R \in Min.Spec(R)$ and $P = (P \cap R)T$.

Proof **(1)** Let $U \in Min.Spec(R)$. Then $\sigma(U) = U$ by Theorem (2.5). Now $UT \in Spec(T)$. Suppose $UT \notin Min.Spec(T)$ and $J \subset UT$ is a minimal Prime ideal of T. Then $(J \cap R) \subset UT \cap R = U$ which is a contradiction, as $(J \cap R) \in Spec(R)$. Therefore $UT \in Min.Spec(T)$.

(2) Let $P \in Min.Spec(T)$. Then $P \cap R \in Spec(R)$. Suppose $(P \cap R) \notin Min.Spec(R)$ and $M \subset P \cap R$ is a minimal prime ideal of R. Then $MT \subset (P \cap R)T \subseteq P$, which is a contradiction, as $MT \in Spec(R)$. Therefore $(P \cap R) \in Min.Spec(R)$. Now it is easy to see that $(P \cap R)T = P$.

□

Proposition 2.13 *Let R be a Noetherian $\sigma(*)$-ring which is also an algebra over $\mathbb{Q}$, where σ is an automorphism of R and δ a σ-derivation of R. Then $P \in Min.Spec(R)$ implies $\delta(P) \subseteq P$.*

Proof Let T be as usual. Now by Proposition (2.10) $e^{t\delta}$ is an automorphism of T. Let $P \in Min.Spec(R))$. Then by Proposition (2.12) $PT \in Min.Spec(T)$. Therefore there exists an integer an integer $n \geq 1$ such that $(e^{t\delta})^n(PT) = PT$; i.e. $e^{nt\delta}(PT) = PT$. But R is a $\mathbb{Q}$-algebra, therefore $e^{t\delta}(PT) = PT$ and now Lemma (2.11) implies $\delta(P) \subseteq P$. □

Proposition 2.14 *Let R be a $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$ and σ is an automorphism of R. Let $U \in Min.Spec(R)$. Then $U(O(R)) = U[x;\sigma,\delta]$ is a completely prime ideal of $O(R) = R[x;\sigma,\delta]$, where δ is a σ-derivation of R.*

Proof Let $U \in Min.Spec(R)$. Then $\sigma(U) = U$ by Theorem (2.5), and $\delta(U) \subseteq U$ by Proposition (2.13). Now R is 2-primal by Proposition (2.3) and further more U is completely prime by Theorem (2.5). Now we note that σ can be extended to an automorphism $\overline{\sigma}$ of R/U and δ can be extended to a $\overline{\sigma}$-derivation $\overline{\delta}$ of R/U. Now it is well known that $O(R)/U(O(R)) \simeq (R/U)[x;\overline{\sigma},\overline{\delta}]$ and hence $U(O(R))$ is a completely prime ideal of O(R). □

Theorem 2.15 *Let R be a Noetherian $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$ and σ is an automorphism of R. Let δ be a σ-derivation of R. Then $P \in Min.Spec(O(R))$ implies that $P \cap R \in Min.Spec(R)$, and conversely $P_1 \in Min.Spec(R)$ implies that $O(P_1) \in Min.Spec(O(R)$.*

Proof Let $P_1 \in Min.Spec(R)$. Then $\sigma(P_1) = P_1$ by Theorem (2.5), and $\delta(P_1) \subseteq P_1$ by Proposition (2.13). Now it can be seen that that $O(P_1) \in Spec(O(R))$. Suppose $O(P_1) \notin Min.Spec(O(R))$ and $P_2 \subset O(P_1)$ be a minimal prime ideal of $O(R)$. Then $P_2 = O(P_2 \cap R) \subset O(P_1) \subseteq Min.Spec(O(R))$. Therefore $(P_2 \cap R) \subset P_1$ which is a contradiction, as $(P_2 \cap R) \in Spec(R)$. Hence $O(P_1) \in Min.Spec(O(R))$.

Conversely suppose that $P \in Min.Spec(O(R))$, then it can be seen that $(P \cap R) \in Spec(R)$, and $O(P \cap R) \in Spec(O(R))$. Therefore $O(P \cap R) = P$. We now show that $(P \cap R) \in Min.Spec(R)$. Suppose $P_1 \subset (P \cap R)$ is a minimal prime ideal of R. Then $O(P_1) \subset O(P \cap R)$ and as in first paragraph $O(P_1) \in Spec(O(R))$ which is a contradiction. Hence $(P \cap R) \in Min.Spec(R)$. □

Corollary 2.16 *Let R be a Noetherian $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$ and σ is an automorphism of R. Let δ be a σ-derivation of R. Then $P(R[x;\sigma,\delta]) = P(R)[x;\sigma,\delta]$.*

We now prove the following Theorem, which is crucial in proving Theorem (2.18).

Theorem 2.17 *Let R be a Noetherian $\sigma(*)$ ring, which is also an algebra over $\mathbb{Q}$, σ an automorphism of R and δ a σ-derivation of R. Then $R[x;\sigma,\delta]$ is 2-primal if and only if $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$.*

Proof Let $R[x;\sigma,\delta]$ be 2-primal. Now by Proposition (2.14) $P(R[x;\sigma,\delta]) \subseteq P(R)[x;\sigma,\delta]$. Let $f(x) = \sum_{j=0}^{n} x^j a_j \in P(R)[x;\sigma,\delta]$. Now R is a 2-primal subring of $R[x;\sigma,\delta]$ by Proposition (2.3), which implies that a_j is nilpotent and thus $a_j \in N(R[x;\sigma,\delta]) = P(R[x;\sigma,\delta])$, and so we have $x^j a_j \in P(R[x;\sigma,\delta])$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(R[x;\sigma,\delta])$. Hence $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$.

Conversely suppose that $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$. We will show that $R[x;\sigma,\delta]$ is 2-primal. Let $g(x) = \sum_{i=0}^{n} x^i b_i \in R[x;\sigma,\delta]$, $b_n \neq 0$, be such that $(g(x))^2 \in P(R[x;\sigma,\delta]) = P(R)[x;\sigma,\delta]$. We will show that $g(x) \in P(R[x;\sigma,\delta])$. Now leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in Min.Spec(R)$. Now $\sigma(P) = P$ and P is completely prime by Theorem (2.5). Therefore we have $b_n \in P$, for all $P \in Min.Spec(R)$; i.e. $b_n \in P(R)$. Now $\delta(P) \subseteq P$ for all $P \in Min.Spec(R)$ by Proposition (2.13), we get $(\sum_{i=0}^{n-1} x^i b_i)^2 \in P(R[x;\sigma,\delta]) = P(R)[x;\sigma,\delta]$ and as above we get $b_{n-1} \in P(R)$. With the same process in a finite number of steps we get $b_i \in P(R)$ for all i, $0 \leq i \leq n$. Thus we have $g(x) \in P(R)[x;\sigma,\delta]$; i.e. $g(x) \in P(R[x;\sigma,\delta])$. Therefore $P(R[x;\sigma,\delta])$ is completely semiprime. Hence $R[x;\sigma,\delta]$ is 2-primal. □

Theorem 2.18 *Let R be a Noetherian $\sigma(*)$-ring, which is also an algebra over $\mathbb{Q}$, σ an automorphism of R and δ a σ-derivation of R. Then $R[x;\sigma,\delta]$ is 2-primal Noetherian.*

Proof $R[x;\sigma,\delta]$ is Noetherian by Hilbert Basis Theorem (Theorem (1.12) of Goodearl and Warfield [6]). We now use Theorem (2.15) to get that $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$, and the result now follows from Theorem (2.17). □

The following example shows that if R is a Noetherian ring, then $R[x;\sigma,\delta]$ need not be 2-primal.

Example 2.19 *Let* $R = \mathbb{Q}\bigoplus\mathbb{Q}$ *with* $\sigma(a,b) = (b,a)$. *Then the only* σ-*invariant ideals of* R *are 0 and* R *and, so* R *is* σ-*prime. Let* $\delta : R \to R$ *be defined by* $\delta(r) = ra - a\sigma(r)$, *where* $a = (0,\alpha) \in R$. *Then* δ *is a* σ-*derivation of* R *and* $R[x;\sigma,\delta]$ *is prime and* $P(R[x;\sigma,\delta]) = 0$. *But* $(x(1,0))^2 = 0$ *as* $\delta(1,0) = -(0,\alpha)$. *Therefore* $R[x;\sigma,\delta]$ *is not 2-primal. If* δ *is taken to be the zero map, then even* $R[x;\sigma]$ *is not 2-primal.*

The following example shows that if R is a Noetherian ring , then even $R[x]$ need not be 2-primal.

Example 2.20 *Let* $R = M_2(\mathbb{Q})$, *the set of* 2×2 *matrices over* $\mathbb{Q}$. *Then* $R[x]$ *is a prime ring with non-zero nilpotent elements and, so can not be 2-primal.*

From these examples we conclude that if R is a Noetherian ring, then even $R[x]$ need not be two primal. But it is known that if R is 2-primal Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $R[x;\delta]$ is 2-primal Noetherian, and therefore there we have the following question:

Question 2.21 *If R is a 2-primal Noetherian ring, is* $R[x;\sigma,\delta]$ *also a 2-primal Noetherian ring (even if R is commutative)?*

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