

$\sigma(*)$ -RINGS AND THEIR EXTENSIONS AS 2-PRIMAL RINGS

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Abstract

In this article, we discuss the prime radical of skew polynomial rings over Noetherian rings. We recall $\sigma(*)$ property on a ring R (i.e. $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of R), where σ is an endomorphism of R . Also recall that a ring R is 2-primal if and only if $P(R)$ and the set of nilpotent elements of R are same, if and only if the prime radical is a completely semiprime ideal. It can be seen that a $\sigma(*)$ is a 2-primal ring. In this article we show that if R is a Noetherian $\sigma(*)$ -ring, which is also an algebra over \mathbb{Q} ; σ is an automorphism of R and δ a σ -derivation of R , then the Ore extension $R[x; \sigma, \delta]$ is 2-primal Noetherian.

1 Introduction

A ring R always means an associative ring with identity. The field of rational numbers and the set of natural numbers are denoted by \mathbb{Q} and \mathbb{N} respectively unless otherwise stated. The set of prime ideals of R is denoted by $Spec(R)$. The sets of minimal prime ideals of R is denoted by $Min.Spec(R)$. Prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. Let R be a ring and σ an automorphism of R . Let I be an ideal of R such that $\sigma^m(I) = I$ for some $m \in \mathbb{N}$. We denote $\bigcap_{i=1}^m \sigma^i(I)$ by I^0 . For any two ideals I, J of R ; $I \subset J$ means that I is strictly contained in J .

This article concerns the study of skew polynomial rings (Ore extensions) in terms of 2-primal rings. Recall that a ring R is 2-primal if and only if

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$N(R) = P(R)$ if and only if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We note that a commutative ring is 2-primal. Also the ring

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} \text{ where } F \text{ is a field, is 2-primal.}$$

For further details on 2-primal rings, we refer the reader to [1, 2, 3, 7, 10].

Recall that $R[x; \sigma, \delta]$ is the usual polynomial ring with coefficients in R , in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x; \sigma, \delta]$ to be of the form $f(x) = \sum_{i=0}^n x^i a_i$. We denote $R[x; \sigma, \delta]$ by $O(R)$. In case δ is the zero map, we denote $R[x; \sigma]$ by $S(R)$ and in case σ is the identity map, we denote $R[x; \delta]$ by $D(R)$. The study of Ore-extension $O(R) = R[x; \sigma, \delta]$ and its special cases $S(R)$ and $D(R)$ have been of interest to many authors. For example [4, 5, 6, 9, 10, 11].

2-primal rings have been studied in recent years and are being treated by authors for different structures. In [10], Greg Marks discusses the 2-primal property of $R[x; \sigma, \delta]$, where R is a local ring, σ an automorphism of R and δ a σ -derivation of R . In Greg Marks [10], it has been investigated that when R is a local ring with a nilpotent maximal ideal, the Ore extension $R[x; \sigma, \delta]$ will or will not be 2-primal depending on the δ -stability of the maximal ideal of R . In the case where $R[x; \delta]$ is 2-primal, it will satisfy an even stronger condition; in the case where $R[x; \sigma, \delta]$ is not 2-primal, it will fail to satisfy an even weaker condition.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [7]. 2-primal near rings have been discussed by Argac and Groenewald in [1].

Recall that in Krempa [8], a ring R is called σ -rigid if there exists an endomorphism of R with the property that $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. In [9], Kwak defines a $\sigma(*)$ -ring R to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and establishes a relation between a 2-primal ring and a $\sigma(*)$ -ring. The property is also extended to the skew-polynomial ring $R[x; \sigma]$.

It is known that if R is a 2-primal Noetherian \mathbb{Q} -algebra, and δ is a derivation of R , then $R[x; \delta]$ is 2-primal Noetherian. (Theorem (2.4) of Bhat [3]).

In this paper we generalize the above result for $R[x; \sigma, \delta]$. But before that we note that a $\sigma(*)$ -ring is a 2-primal ring [Proposition (2.3)]. We also note that if σ is an automorphism of R , then it can be extended to an automorphism of $R[x; \sigma]$ such that $\sigma(x) = x$; i.e. $\sigma(\sum_{i=0}^n x^i a_i) = \sum_{i=0}^n x^i \sigma(a_i)$, and prove that if R is a $\sigma(*)$ -ring, then $R[x; \sigma]$ is also a $\sigma(*)$ -ring [Theorem (2.9)].

We also find a relation between the minimal prime ideals of R and those of the Ore extension $R[x; \sigma, \delta]$, where R is a Noetherian $\sigma(*)$ -ring, which is also an algebra over \mathbb{Q} ; σ is an automorphism of R and δ a σ -derivation of R . This is proved in Theorem (2.15).

We ultimately prove the following result [Theorem (2.18)]:

Theorem: If R is a Noetherian $\sigma(*)$ -ring, which is also an algebra over \mathbb{Q} , where σ is an automorphism of R and δ a σ -derivation of R , then the Ore extension $R[x; \sigma, \delta]$ is 2-primal Noetherian.

2 2-primal skew-Polynomial rings

We begin with the following definition:

Definition 2.1 (Kwak [9]) Let R be a ring and σ an endomorphism of R . Then R is said to be a $\sigma(*)$ -ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$.

Example 2.2 Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \rightarrow R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that R is a $\sigma(*)$ -ring.

Recall that an ideal I of a ring R is called σ -invariant if $\sigma(I) = I$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$. We also recall that an ideal J of a ring is called a σ -prime ideal of R if J is σ -invariant and for any σ -invariant ideals K and L with $KL \subseteq J$, we have $K \subseteq J$ or $L \subseteq J$.

Proposition 2.3 Let R be a ring and σ an automorphism of R . Then R is a $\sigma(*)$ -ring implies R is 2-primal.

Proof Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. \square

The following example shows that there exists an endomorphism σ of a ring R such that the converse of the above Proposition does not hold.

Example 2.4 Let $R = F[x]$, F a field. Then R is a commutative domain, and therefore is 2-primal with $P(R) = 0$. Let $\sigma : R \rightarrow R$ be defined by $\sigma(f(x)) = f(0)$. Let $f(x) = xa$, $0 \neq a \in F$. Then $f(x)\sigma(f(x)) \in P(R)$, but $f(x) \notin P(R)$. Therefore R is not a $\sigma(*)$ -ring.

Recall that an ideal P of a ring R is said to be completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$.

We now give a necessary and sufficient condition for a Noetherian ring to be a $\sigma(*)$ -ring in the following Theorem:

Theorem 2.5 Let R be a Noetherian ring, and σ an automorphism of R . Then R is a $\sigma(*)$ -ring if and only if for each minimal prime U of R , $\sigma(U) = U$ and U is completely prime ideal of R .

Proof Let R be a Noetherian ring such that for each minimal prime U of R , $\sigma(U) = U$ and U is completely prime ideal of R . Let $a \in R$ be such that $a\sigma(a) \in P(R) = \bigcap_{i=1}^n U_i$, where U_i are the minimal primes of R . Now for each i , $a \in U_i$ or $\sigma(a) \in U_i$ as U_i are completely prime. Now $\sigma(a) \in U_i = \sigma(U_i)$ implies that $a \in U_i$. Therefore $a \in P(R)$. Hence R is a $\sigma(*)$ -ring.

Conversely, suppose that R is a $\sigma(*)$ -ring and let $U = U_1$ be a minimal prime ideal of R . Now by Proposition (2.3), $P(R)$ is completely semiprime. Let U_2, U_3, \dots, U_n be the other minimal primes of R . Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R . Renumber so that $\sigma(U) = U_n$. Let $a \in \bigcap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \bigcap_{i=1}^n U_i = P(R)$. Therefore $a \in P(R)$, and thus $\bigcap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Now suppose that $U = U_1$ is not completely prime. Then there exist $a, b \in R \setminus U$ with $ab \in U$. Let c be any element of $b(U_2 \cap U_3 \cap \dots \cap U_n)a$. Then $c^2 \in \bigcap_{i=1}^n U_i = P(R)$. So $c \in P(R)$ and, thus $b(U_2 \cap U_3 \cap \dots \cap U_n)a \subseteq U$. Therefore $bR(U_2 \cap U_3 \cap \dots \cap U_n)Ra \subseteq U$ and, as U is prime, $a \in U$, $U_i \subseteq U$ for some $i \neq 1$ or $b \in U$. None of these can occur, so U is completely prime. \square

We also note that if R is a Noetherian ring, then $\text{Min.Spec}(R)$ is finite (Theorem (2.4) of Goodearl and Warfield [6]) and for any automorphism σ of R and for any $U \in \text{Min.Spec}(R)$, we have $\sigma^i(U) \in \text{Min.Spec}(R)$ for all $i \in \mathbb{N}$, therefore, it follows that there exists some $m \in \mathbb{N}$ such that $\sigma^m(U) = U$ for all $U \in \text{Min.Spec}(R)$. As mentioned earlier we denote $\bigcap_{i=0}^m \sigma^i(U)$ by U^0 . With this we have the following Theorem:

Theorem 2.6 *Let R be a Noetherian ring and σ an automorphism of R . Let $S(R) = R[x; \sigma]$ be as usual. Then:*

- (1) *If $P \in \text{Min.Spec}(S(R))$, then $P = (P \cap R)S(R)$ and there exists $U \in \text{Min.Spec}(R)$ such that $P \cap R = U^0$.*
- (2) *If $U \in \text{Min.Spec}(R)$, then $U^0 S(R) \in \text{Min.Spec}(S(R))$.*

Proof See Theorem (2.4) of Bhat [4]. \square

Corollary 2.7 *Let R be a Noetherian $\sigma(*)$ -ring, where σ is an automorphism of R . Then $P \in \text{Min.Spec}(S(R))$ if and only if there exists $Q \in \text{Min.Spec}(R)$ such that $S(Q) = P$ and $(P \cap R) = Q$.*

Proof R is a Noetherian $\sigma(*)$ -ring, therefore $U^0 = U$ for any $U \in \text{Min.Spec}(R)$ by Theorem (2.5). Now use Theorem (2.6). \square

Corollary 2.8 *Let R be a Noetherian $\sigma(*)$ -ring, where σ is an automorphism of R . Then $P(R)[x; \sigma] = P(R[x; \sigma])$.*

Theorem 2.9 *Let R be a Noetherian $\sigma(*)$ -ring, where σ is an automorphism of R . Then $R[x; \sigma]$ is also a Noetherian $\sigma(*)$ -ring.*

Proof $R[x; \sigma]$ is Noetherian by Hilbert Basis Theorem (Theorem (1.12) of Goodearl and Warfield [6]). Now we have $P(R)[x; \sigma] = P(R[x; \sigma])$ by Corollary (2.8). Let $f(x) = \sum_{i=0}^n x^i a_i \in R[x; \sigma]$ be such that $f(x)\sigma(f(x)) \in P(R[x; \sigma]) = P(R)[x; \sigma]$; i.e.

$$(x^n a_n + \dots + a_0)(x^n \sigma(a_n) + \dots + \sigma(a_0)) \in P(R)[x; \sigma],$$

or

$$x^{2n} \sigma^n(a_n) \sigma(a_n) + \dots + a_0 \sigma(a_0) \in P(R)[x; \sigma],$$

which implies that $a_0 \sigma(a_0) \in P(R)$, and therefore $a_0 \in P(R)$, as R is a $\sigma(*)$ -ring.

Therefore $g(x)\sigma(g(x)) \in P(R)[x; \sigma]$, where $g(x) = \sum_{i=1}^n x^i a_i$. With the same process as above, in a finite number of steps, we get that $a_i \in P(R)$ for all i , $1 \leq i \leq n$. Thus $f(x) \in P(R)[x; \sigma] = P(R[x; \sigma])$. Hence $R[x; \sigma]$ is also a Noetherian $\sigma(*)$ -ring. \square

We now give a relation between the minimal prime ideals of R and those of $R[x; \sigma, \delta]$, where R is a Noetherian \mathbb{Q} -algebra, σ an automorphism of R and δ a σ -derivation of R . This is proved in Theorem (2.15). Towards this we have the following:

Proposition 2.10 *Let R be a Noetherian \mathbb{Q} -algebra, σ an automorphism and δ a σ -derivation of R . Then $e^{t\delta}$ is an automorphism of $T = R[[t, \sigma]]$, the skew power series ring.*

Proof The proof is on the same lines as in Seidenberg [11] and in the non-commutative case on the same lines as provided by Blair and Small in [5]. \square

Hence forth we denote $R[[t, \sigma]]$ by T .

Lemma 2.11 *Let R be a Noetherian \mathbb{Q} -algebra, σ an automorphism and δ a σ -derivation of R . Then an ideal I of R is δ -invariant if and only if TI is $e^{t\delta}$ -invariant.*

Proof Let TI be $e^{t\delta}$ -invariant. Let $a \in I$. Then $a \in TI$. So $e^{t\delta}(a) \in TI$; i.e. $a + t\delta(a) + (t^2\delta^2/2!)(a) + \dots \in TI$. Therefore $\delta(a) \in I$.

Conversely suppose that $\delta(I) \subseteq I$ and let $f = \sum t^i a_i \in TI$. Then $e^{t\delta}(f) = f + t\delta(f) + (t^2\delta^2/2!)(f) + \dots \in TI$, as $\delta(a_i) \in I$. Therefore $e^{t\delta}(TI) \subseteq TI$. Replacing $e^{t\delta}$ by $e^{-t\delta}$, we get that $e^{t\delta}(TI) = TI$. \square

Let σ be an automorphism of a ring R , and I be an ideal of R such that $\sigma(I) = I$. Then it is easy to see that $TI \subseteq IT$ and $IT \subseteq TI$. Hence $TI = IT$ is an ideal of T .

Proposition 2.12 *Let R be a Noetherian $\sigma(*)$ -ring and T as usual. Then:*

- (1) $U \in \text{Min.Spec}(R)$ implies that $UT \in \text{Min.Spec}(T)$.
- (2) $P \in \text{Min.Spec}(T)$ implies that $P \cap R \in \text{Min.Spec}(R)$ and $P = (P \cap R)T$.

Proof (1) Let $U \in \text{Min.Spec}(R)$. Then $\sigma(U) = U$ by Theorem (2.5). Now $UT \in \text{Spec}(T)$. Suppose $UT \notin \text{Min.Spec}(T)$ and $J \subset UT$ is a minimal prime ideal of T . Then $(J \cap R) \subset UT \cap R = U$ which is a contradiction, as $(J \cap R) \in \text{Spec}(R)$. Therefore $UT \in \text{Min.Spec}(T)$.

(2) Let $P \in \text{Min.Spec}(T)$. Then $P \cap R \in \text{Spec}(R)$. Suppose $(P \cap R) \notin \text{Min.Spec}(R)$ and $M \subset P \cap R$ is a minimal prime ideal of R . Then $MT \subset (P \cap R)T \subseteq P$, which is a contradiction, as $MT \in \text{Spec}(R)$. Therefore $(P \cap R) \in \text{Min.Spec}(R)$. Now it is easy to see that $(P \cap R)T = P$. □

Proposition 2.13 Let R be a Noetherian $\sigma(\ast)$ -ring which is also an algebra over \mathbb{Q} , where σ is an automorphism of R and δ a σ -derivation of R . Then $P \in \text{Min.Spec}(R)$ implies $\delta(P) \subseteq P$.

Proof Let T be as usual. Now by Proposition (2.10) $e^{t\delta}$ is an automorphism of T . Let $P \in \text{Min.Spec}(R)$. Then by Proposition (2.12) $PT \in \text{Min.Spec}(T)$. Therefore there exists an integer $n \geq 1$ such that $(e^{t\delta})^n(PT) = PT$; i.e. $e^{nt\delta}(PT) = PT$. But R is a \mathbb{Q} -algebra, therefore $e^{t\delta}(PT) = PT$ and now Lemma (2.11) implies $\delta(P) \subseteq P$. □

Proposition 2.14 Let R be a $\sigma(\ast)$ -ring, which is also an algebra over \mathbb{Q} and σ is an automorphism of R . Let $U \in \text{Min.Spec}(R)$. Then $U(O(R)) = U[x; \sigma, \delta]$ is a completely prime ideal of $O(R) = R[x; \sigma, \delta]$, where δ is a σ -derivation of R .

Proof Let $U \in \text{Min.Spec}(R)$. Then $\sigma(U) = U$ by Theorem (2.5), and $\delta(U) \subseteq U$ by Proposition (2.13). Now R is 2-primal by Proposition (2.3) and further U is completely prime by Theorem (2.5). Now we note that σ can be extended to an automorphism $\bar{\sigma}$ of R/U and δ can be extended to a $\bar{\sigma}$ -derivation $\bar{\delta}$ of R/U . Now it is well known that $O(R)/U(O(R)) \simeq (R/U)[x; \bar{\sigma}, \bar{\delta}]$ and hence $U(O(R))$ is a completely prime ideal of $O(R)$. □

Theorem 2.15 Let R be a Noetherian $\sigma(\ast)$ -ring, which is also an algebra over \mathbb{Q} and σ is an automorphism of R . Let δ be a σ -derivation of R . Then $P \in \text{Min.Spec}(O(R))$ implies that $P \cap R \in \text{Min.Spec}(R)$, and conversely $P_1 \in \text{Min.Spec}(R)$ implies that $O(P_1) \in \text{Min.Spec}(O(R))$.

Proof Let $P_1 \in \text{Min.Spec}(R)$. Then $\sigma(P_1) = P_1$ by Theorem (2.5), and $\delta(P_1) \subseteq P_1$ by Proposition (2.13). Now it can be seen that that $O(P_1) \in \text{Spec}(O(R))$. Suppose $O(P_1) \notin \text{Min.Spec}(O(R))$ and $P_2 \subset O(P_1)$ be a minimal

prime ideal of $O(R)$. Then $P_2 = O(P_2 \cap R) \subset O(P_1) \subseteq \text{Min.Spec}(O(R))$. Therefore $(P_2 \cap R) \subset P_1$ which is a contradiction, as $(P_2 \cap R) \in \text{Spec}(R)$. Hence $O(P_1) \in \text{Min.Spec}(O(R))$.

Conversely suppose that $P \in \text{Min.Spec}(O(R))$, then it can be seen that $(P \cap R) \in \text{Spec}(R)$, and $O(P \cap R) \in \text{Spec}(O(R))$. Therefore $O(P \cap R) = P$. We now show that $(P \cap R) \in \text{Min.Spec}(R)$. Suppose $P_1 \subset (P \cap R)$ is a minimal prime ideal of R . Then $O(P_1) \subset O(P \cap R)$ and as in first paragraph $O(P_1) \in \text{Spec}(O(R))$ which is a contradiction. Hence $(P \cap R) \in \text{Min.Spec}(R)$. \square

Corollary 2.16 *Let R be a Noetherian $\sigma(*)$ -ring, which is also an algebra over \mathbb{Q} and σ is an automorphism of R . Let δ be a σ -derivation of R . Then $P(R[x; \sigma, \delta]) = P(R)[x; \sigma, \delta]$.*

We now prove the following Theorem, which is crucial in proving Theorem (2.18).

Theorem 2.17 *Let R be a Noetherian $\sigma(*)$ ring, which is also an algebra over \mathbb{Q} , σ an automorphism of R and δ a σ -derivation of R . Then $R[x; \sigma, \delta]$ is 2-primal if and only if $P(R)[x; \sigma, \delta] = P(R[x; \sigma, \delta])$.*

Proof Let $R[x; \sigma, \delta]$ be 2-primal. Now by Proposition (2.14) $P(R[x; \sigma, \delta]) \subseteq P(R)[x; \sigma, \delta]$. Let $f(x) = \sum_{j=0}^n x^j a_j \in P(R)[x; \sigma, \delta]$. Now R is a 2-primal subring of $R[x; \sigma, \delta]$ by Proposition (2.3), which implies that a_j is nilpotent and thus $a_j \in N(R[x; \sigma, \delta]) = P(R[x; \sigma, \delta])$, and so we have $x^j a_j \in P(R[x; \sigma, \delta])$ for each j , $0 \leq j \leq n$, which implies that $f(x) \in P(R[x; \sigma, \delta])$. Hence $P(R)[x; \sigma, \delta] = P(R[x; \sigma, \delta])$.

Conversely suppose that $P(R)[x; \sigma, \delta] = P(R[x; \sigma, \delta])$. We will show that $R[x; \sigma, \delta]$ is 2-primal. Let $g(x) = \sum_{i=0}^n x^i b_i \in R[x; \sigma, \delta]$, $b_n \neq 0$, be such that $(g(x))^2 \in P(R[x; \sigma, \delta]) = P(R)[x; \sigma, \delta]$. We will show that $g(x) \in P(R[x; \sigma, \delta])$. Now leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in \text{Min.Spec}(R)$. Now $\sigma(P) = P$ and P is completely prime by Theorem (2.5). Therefore we have $b_n \in P$, for all $P \in \text{Min.Spec}(R)$; i.e. $b_n \in P(R)$. Now $\delta(P) \subseteq P$ for all $P \in \text{Min.Spec}(R)$ by Proposition (2.13), we get $(\sum_{i=0}^{n-1} x^i b_i)^2 \in P(R[x; \sigma, \delta]) = P(R)[x; \sigma, \delta]$ and as above we get $b_{n-1} \in P(R)$. With the same process in a finite number of steps we get $b_i \in P(R)$ for all i , $0 \leq i \leq n$. Thus we have $g(x) \in P(R)[x; \sigma, \delta]$; i.e. $g(x) \in P(R[x; \sigma, \delta])$. Therefore $P(R[x; \sigma, \delta])$ is completely semiprime. Hence $R[x; \sigma, \delta]$ is 2-primal. \square

Theorem 2.18 *Let R be a Noetherian $\sigma(*)$ -ring, which is also an algebra over \mathbb{Q} , σ an automorphism of R and δ a σ -derivation of R . Then $R[x; \sigma, \delta]$ is 2-primal Noetherian.*

Proof $R[x; \sigma, \delta]$ is Noetherian by Hilbert Basis Theorem (Theorem (1.12) of Goodearl and Warfield [6]). We now use Theorem (2.15) to get that $P(R)[x; \sigma, \delta] = P(R[x; \sigma, \delta])$, and the result now follows from Theorem (2.17). \square

The following example shows that if R is a Noetherian ring, then $R[x; \sigma, \delta]$ need not be 2-primal.

Example 2.19 Let $R = \mathbb{Q} \oplus \mathbb{Q}$ with $\sigma(a, b) = (b, a)$. Then the only σ -invariant ideals of R are 0 and R and, so R is σ -prime. Let $\delta : R \rightarrow R$ be defined by $\delta(r) = ra - a\sigma(r)$, where $a = (0, \alpha) \in R$. Then δ is a σ -derivation of R and $R[x; \sigma, \delta]$ is prime and $P(R[x; \sigma, \delta]) = 0$. But $(x(1, 0))^2 = 0$ as $\delta(1, 0) = -(0, \alpha)$. Therefore $R[x; \sigma, \delta]$ is not 2-primal. If δ is taken to be the zero map, then even $R[x; \sigma]$ is not 2-primal.

The following example shows that if R is a Noetherian ring, then even $R[x]$ need not be 2-primal.

Example 2.20 Let $R = M_2(\mathbb{Q})$, the set of 2×2 matrices over \mathbb{Q} . Then $R[x]$ is a prime ring with non-zero nilpotent elements and, so can not be 2-primal.

From these examples we conclude that if R is a Noetherian ring, then even $R[x]$ need not be two primal. But it is known that if R is 2-primal Noetherian \mathbb{Q} -algebra and δ is a derivation of R , then $R[x; \delta]$ is 2-primal Noetherian, and therefore there we have the following question:

Question 2.21 If R is a 2-primal Noetherian ring, is $R[x; \sigma, \delta]$ also a 2-primal Noetherian ring (even if R is commutative)?

References

- [1] N. Argac and N. J. Groenewald, A generalization of 2-primal near rings, *Questiones Mathematicae*, Vol. 27(4) (2004), 397-413.
- [2] V. K. Bhat, On 2-primal Ore extensions, *Ukrainian Math. Bull.*, Vol. 4(2) (2007), 173-179.
- [3] V. K. Bhat, Differential operator rings over 2-primal rings, *Ukrainian Math. Bull.*, Vol. 5(2), 153-158.
- [4] V. K. Bhat, Associated prime ideals of skew polynomial rings, *Beiträge zur Algebra und Geometrie*, Vol. 49/1 (2008), 277-283.
- [5] W. D. Blair and L. W. Small, Embedding differential and skew-polynomial rings into artinian rings, *Proc. Amer. Math. Soc.*, Vol. 109(4) 1990, 881-886.
- [6] K. R. Goodearl and R. B. Warfield Jr, *An introduction to non-commutative Noetherian rings*, Cambridge Uni. Press, 1989.
- [7] N. K. Kim and T. K. Kwak, Minimal prime ideals in 2-primal rings, *Math. Japonica*, Vol. 50(3) (1999), 415-420.
- [8] J. Krempa, Some examples of reduced rings, *Algebra Colloq.*, Vol. 3(4) (1996), 289-300.

- [9] T. K. Kwak, Prime radicals of skew-polynomial rings, *Int. J. of Mathematical Sciences*, Vol. 2(2) (2003), 219-227.
- [10] G. Marks, On 2-primal Ore extensions, *Comm. Algebra*, Vol. 29 (5) (2001), 2113-2123.
- [11] A. Seidenberg, Differential ideals in rings of finitely generated Type, *Amer. J. Math.* 89 (1967), 22-42.