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Semi co–Hopfian and Semi Hopfian Modules

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Abstract

A module M is called *semi co–Hopfian* (resp. *semi Hopfian*) if any injective (resp. surjective) endomorphism of M has a direct summand image (resp. kernel). In this article, some properties of semi co–Hopfian and semi Hopfian modules are investigated with examples.

1 Introduction

Hopfian and co–Hopfian groups, rings and modules have been studied by many authors since 1960s. Recall that a module M is called *co–Hopfian* (resp. *Hopfian*) if any injective (resp. surjective) endomorphism of M is an isomorphism. Note that any Artinian module is co–Hopfian, and any Noetherian module is Hopfian. In this article, we concerned with semi co–Hopfian and semi Hopfian modules. A module M is called *semi co–Hopfian* (resp. *semi Hopfian*) if any injective (resp. surjective) endomorphism of M has a direct summand image (resp. kernel).

Semi co–Hopfian and semi Hopfian modules are used as a tool by many authors, for example, see [3, 17, 18]. In this paper, we deal with some properties of semi co-Hopfian and semi Hopfian modules and rings, among others direct sums and direct products of them are considered with many examples.

Recall from [13] that a module M has **(C2)** if for any submodule N of M which is isomorphic to a direct summand of M, is a direct summand of M; and

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(D2) if any submodule N such that M/N is isomorphic to a direct summand of M is a direct summand of M. If M has $(C2)$, then M is semi co–Hopfian. If M has $(D2)$, then M is semi Hopfian. Hence any self–injective module is semi co–Hopfian, and any self–projective module is semi Hopfian by [13]. But their converses are not true in general, examples are given in the article.

In the last section, we consider the ring of continuous functions. We prove that for a compact Hausdorff space X, if X is a semi co–Hopfian (resp. semi Hopfian) topological space, then $C(X)$ is a semi Hopfian (resp. semi co–Hopfian) $\mathbb{R}$–algebra.

A module M is called *Dedekind finite* if $M \cong M \oplus N$ for some module N, $N = 0$. For a ring R, ${}_RR$ is Dedekind finite if and only if $ab = 1$ implies that $ba = 1$ for any $a, b \in R$. It is well known that any co–Hopfian or Hopfian module is Dedekind finite.

Throughout this paper, R denotes an associative ring with identity and modules M are unitary left R–modules. For a module M, $Rad(M)$, $Soc(M)$ and $Z(M)$ are the Jacobson radical, the socle and the singular submodule of M, respectively. In the ring case we use the abbreviations $Z_r = Z(R_R)$ and $Z_l = Z({}_RR)$. For any $m \in M$, $l_R(m)$ will denote the left annihilator of m over R. $N \leq^{\oplus} M$ means that N is a direct summand of M.

2 Semi co–Hopfian Modules

A module M is called *semi co–Hopfian* if any injective endomorphism of M has a direct summand image, i.e. any injective endomorphism of M splits. A ring R is called *left semi co–Hopfian* if ${}_RR$ is a semi co–Hopfian module. In [17] and [18] semi co–Hopfian modules are named $GC2$. They generalized some results about injectivity via modules with GC2.

Clearly, any co–Hopfian module is semi co–Hopfian. The converse is not true in general, for example, let ${}_{\mathbb{Z}}M = \mathbb{Q}^{(\mathbb{N})}$. Since M is quasi–injective, it is semi co–Hopfian (by Lemma 2.1 and [13, Proposition 2.1]). But since $M \cong M \oplus \mathbb{Q}$, M is not Dedekind finite, hence not co–Hopfian. Also it is clear that if p is a prime and n is a positive integer , then any direct sum of copies of $\mathbb{Z}/\mathbb{Z}p^n$ is not a co–Hopfian $\mathbb{Z}$–module, but it is semi co–Hopfian because it is quasi–injective.

Lemma 2.1 *The following are equivalent for a module M.*

1) M is semi co–Hopfian.

2) Any submodule N of M which is isomorphic to M, is a direct summand of M.

Proof $(2 \Rightarrow 1)$ It is obvious. $(1 \Rightarrow 2)$ Let $N \leq M$ be such that $N \cong M$. Then we have an injective endomorphism α of M where $Im\alpha = N$. By (1), N is a direct summand of M. $\square$

By Lemma 2.1, if M has $C2$, then M is semi co–Hopfian. In particular any quasi–injective module is semi co–Hopfian by [13, Proposition 2.1]. Therefore, the concept of semi co–Hopfian modules is a generalization of co–Hopfian modules and modules with $C2$.

Example 2.2 *There exists a semi co–Hopfian module which has not $C2$.*

Proof Let R be the ring of 2×2 lower triangular matrices over a field F. Then R is Artinian and so co–Hopfian, but $_RR$ has not $C2$ (see [13, Example 2.9]). $\square$

Any semi co–Hopfian quasi continuous module has $C2$ (see [13, Lemma 3.14]).

For a ring R we have the following characterization.

Proposition 2.3 *The following are equivalent for a ring R.*
1) R is left semi co–Hopfian.
2) If $l_R(a) = 0$, $a \in R$, then aR is a direct summand of R.
3) If $l_R(a) = 0$, $a \in R$, then $aR = R$.
4) Every R–isomorphism $Ra \to R$, $a \in R$, extends to R.

Proof (1) $\Rightarrow$ (4) If $Ra \cong R$, $a \in R$, then Ra is a direct summand of R by Lemma 2.1. So (4) holds.
(4) $\Rightarrow$ (3) Let $l_R(a) = 0$, $a \in R$. Then the isomorphism $f : Ra \to R$ defined by $f(ra) = r$, $r \in R$, extends to R by g. Then $1 = f(a) = g(a) = ag(1) \in aR$.
(3) $\Rightarrow$ (2) It is obvious.
(2) $\Rightarrow$ (1) Let $f : R \to R$ be a left R–monomorphism. Since $l_R(f(1)) = 0$, $f(1)R$ is a direct summand of R. Hence $Imf = Rf(1)$ is a direct summand of R. $\square$

It is well–known that if R is an integral domain, then $_RR$ has $C2$ if and only if R is a division ring. The following result is obvious by definitions.

Proposition 2.4 *If R is a ring with only idempotents 0 and 1, then the following are equivalent.*
1) $_RR$ has $C2$.
2) $_RR$ is co–Hopfian.

3) $_RR$ *is semi co–Hopfian.*

In particular, if R is an integral domain, $(1)-(3)$ *are equivalent to*

4) R is a division ring.

Another example of a semi co–Hopfian module is related with the summand sum property. A module M has the *summand sum property (SSP)* if the sum of any two direct summands of M is a direct summand. Note that M has SSP if and only if for every decomposition $M = A \oplus B$ and every R–homomorphism f from A to B, the image of f is a direct summand [1].

Hence if $M \oplus M$ has SSP, then M is semi co–Hopfian. But the converse is not true: Let the $\mathbb{Z}$–module $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$. Since M is Artinian, it is co–Hopfian. Denote the $\mathbb{Z}$–homomorphism $f : M \to M$ by $f(\overline{a}, \overline{b}) = (0, \overline{pb})$. Then $Imf = 0 \oplus p\mathbb{Z}_{p^2}$ is not a direct summand of M. Hence $M \oplus M$ has not SSP.

If M is co–Hopfian, then it is Dedekind finite. The converse is true if M is self–injective by [7, Proposition 1.4]. Also in [15, Proposition 3.7], it is proved that if M has finite uniform dimension and $C2$, then M is co–Hopfian. Since any self–injective module has $C2$ (see [13, Proposition 2.1]) and a module with a finite uniform dimension is Dedekind finite (see [11, Exercises 6(1)]), as a generalization of [7, Proposition 1.4] and [15, Proposition 3.7] we have the following proposition.

A module M is called *weakly co–Hopfian* (in [7]) if any injective endomorphism of M has an essential image.

Proposition 2.5 *The following are equivalent for a module M.*

1) M is co–Hopfian.

2) M is Dedekind finite and semi co–Hopfian.

3) M is weakly co–Hopfian and semi co–Hopfian.

Proof $(3) \Leftrightarrow (1) \Rightarrow (2)$ are obvious.

$(2) \Rightarrow (1)$ Let f be an injective endomorphism of M. Then $M = f(M) \oplus K$ for some $K \leq M$. Define a homomorphism $\varphi : M \oplus K \longrightarrow M$ by $\varphi(m, k) = f(m) + k$. Then φ is an isomorphism. Since M is Dedekind finite, $K = 0$. Hence $f(M) = M$ and so f is an isomorphism. □

Recall that a ring is *I–finite* if it contains no infinite set of orthogonal idempotents. R is I–finite if and only if R has ACC on right direct summands if and only if R has DCC on left direct summands (see [11, 6.59]). If a ring R is I–finite and has left $C2$, then $_RR$ is co–Hopfian by [14, Example 7.5]. By the same proof it can be seen that any I–finite and left semi co–Hopfian ring is left co–Hopfian.

From now on we investigate some properties of semi co–Hopfian modules.

Lemma 2.6 *Any direct summand of a semi co–Hopfian module is semi co–Hopfian.*

Proof Let N be a direct summand of M and $f : N \longrightarrow N$ a monomorphism. Write $M = N \oplus N'$. Then $g : M \to M$, $g(n + n') = f(n) + n'$ where $n \in N$, $n' \in N'$, is a monomorphism. Since $Img = Imf \oplus N'$ is a direct summand of M, we get that Imf is a direct summand of N. $\square$

We say that a submodule N of M is a *non–summand* of M if N is not a direct summand of M.

Lemma 2.7 *If for any non–summand submodule N of M, N is semi co–Hopfian, then M is semi co–Hopfian.*

Proof If M is not semi co–Hopfian, then there exists a non–summand submodule N of M such that $N \cong M$. By hypothesis, we have a contradiction. $\square$

Any finite direct sum of semi co–Hopfian modules need not be semi co–Hopfian.

Example 2.8 *There exists a simple module U and an injective module V such that $U \oplus V$ is not semi co–Hopfian.*

Proof Let R be a right Noetherian ring which is not a right V–ring (see [4]). Let U be a simple right R–module which is not injective and E denote the injective envelope of U. For each positive integer n, let $E_n = E$ and let $V = \oplus_{n \geq 1} E_n$. Then V is injective.

Let $M = U \oplus V$. Define $f : M \to M$ by $f((u, e_1, e_2, \ldots)) = (0, u, e_1, e_2, \ldots)$ where $u \in U$, $e_i \in E_i$. Then f is clearly a monomorphism. If $f(M)$ was a direct summand of M, then U would have to be a direct summand of E, a contradiction. $\square$

Proposition 2.9 *Let $M = \oplus_{i \in I} M_i$, where M_i is invariant under any injection of M for all $i \in I$. Then M is semi co–Hopfian if and only if M_i is semi co–Hopfian for all $i \in I$.*

Proof The necessity is by Lemma 2.6. For the sufficiency, let $f : M \to M$ be a monomorphism. Then restriction of f to M_i $(i \in I)$, is an injective endomorphism of M_i. By hypothesis, $f(M_i) \leq^{\oplus} M_i$ $(i \in I)$. This implies that $M = (\oplus_{i \in I} f(M_i)) \oplus X = f(M) \oplus X$ for some $X \leq M$. Hence M is semi co–Hopfian. □

Note that U is not invariant under the monomorphism f in Example 2.8.

Proposition 2.10 *A direct product $R = \prod_{i \in I} R_i$ of rings R_i is semi co–Hopfian left R–module if and only if each R_i is semi co–Hopfian left R_i–module.*

Proof Clear by Proposition 2.3(3). □

But any direct product of semi co–Hopfian modules need not be semi co–Hopfian.

Example 2.11 Let p be prime and the $\mathbb{Z}$–module $M = \prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$. M is not semi co–Hopfian. For define $f : M \to M$ by $f(a_1 + p\mathbb{Z}, a_2 + p^2\mathbb{Z}, a_3 + p^3\mathbb{Z}, \ldots) = (0, pa_1 + p^2\mathbb{Z}, pa_2 + p^3\mathbb{Z}, \ldots)$. Then f is a $\mathbb{Z}$–monomorphism and $Imf = pM$. If pM is a direct summand of M, then $M = pM \oplus L$ for some submodule L. Since $pL = 0$, $L \subseteq \mathbb{Z}_p \times p\mathbb{Z}_{p^2} \times p^2\mathbb{Z}_{p^3} \times \cdots$. But $(0, 1 + p^2\mathbb{Z}, 0, 0, \ldots) \notin pM \oplus L$. This gives that pM is not a direct summand of M.

Proposition 2.12 *Let a $\mathbb{Z}$–module $M = M' \oplus M''$ be a direct sum of a semisimple module M' and an injective module M'' such that M'' has finite uniform dimension. Then M is semi co–Hopfian.*

Proof Let $f : M \to M$ be any monomorphism. $f(M'')$ is injective and hence $f(M'')$ is contained in M'' (since M is a $\mathbb{Z}$–module). Then $M'' = f(M'') \oplus N$ for some submodule N. But M'' and $f(M'')$ have the same uniform dimension. Therefore $N = 0$. Thus $f(M'') = M''$ and $f(M) = M'' \oplus (f(M) \cap M')$ which is a direct summand of M because M' is semisimple. □

A module M is called *torsion free* if $rm = 0$ then $r = 0$ or $m = 0$ for any $r \in R$ and $m \in M$. Torsion free modules need not be semi co–Hopfian, for example $\mathbb{Z}_{\mathbb{Z}}$.

Proposition 2.13 *Let R be a commutative domain and let M be a torsion free semi co–Hopfian R–module. Then M is injective.*

Proof Let c be any non–zero element of R. Define $f : M \to M$ by $f(m) = cm$, $m \in M$. Then f is a monomorphism. By assumption there exists a submodule

$N \leq M$ such that $M = f(M) \oplus N = cM \oplus N$. Then $cN = 0$ so that $N = 0$. Thus $M = cM$ for all non–zero $c \in R$. Since M is torsion–free it follows that M is injective. $\square$

Now we will consider the descending chain condition (DCC) on non–summands.

Proposition 2.14 *If M has DCC on non–summand submodules, then M is semi co–Hopfian.*

Proof If M is not semi co–Hopfian, then there exists a non–summand submodule M_1 of M such that $M_1 \cong M$. Since M_1 is not semi co–Hopfian there exists a non–summand submodule M_2 of M_1 such that $M_2 \cong M_1$. Repeating this argument we have a strictly descending chain of non–summand submodules of M. $\square$

Moreover,

Proposition 2.15 *Let $\mathcal{P}$ be a property of modules preserved under isomorphism. If a module M has the property $\mathcal{P}$ and satisfies DCC on non–summand submodules with property $\mathcal{P}$, then M is semi co–Hopfian.*

Proof By a proof similar to Proposition 2.14. $\square$

Corollary 2.16 *If M has DCC on its non–semi co–Hopfian submodules, then M is semi co–Hopfian.*

As for the endomorphism ring of a semi co–Hopfian module, note that by [18, Lemma 1.1] if M has a finite uniform dimension and is semi co–Hopfian, then the endomorphism ring $End_R(M)$ is semilocal.

Proposition 2.17 *Let M be a module. If the ring ${}_SS = End_R(M)$ is semi co–Hopfian, then ${}_RM$ is semi co–Hopfian. The converse is true if $Ker(\alpha)$ is generated by M whenever $\alpha \in S$ is such that $l_S(\alpha) = 0$.*

Proof (Since M is a right S–module, we will consider Imf as $(M)f$.) Let $f : M \to M$ be a monomorphism. Then $S \cong Sf$. Since ${}_SS$ is semi co–Hopfian there exists an idempotent $e \in S$ such that $Sf = Se$ by Lemma 2.1. Then $Imf = Ime$ is a direct summand of M.

For the converse; let $\alpha \in S$ be such that $l_S(\alpha) = 0$. If we prove that $\alpha S = S$, then S will be semi co–Hopfian by Proposition 2.3. By hypothesis, $Ker(\alpha) = \sum\{Imh \,|\, h \in S,\, Imh \subseteq Ker(\alpha)\}$. If $Imh \subseteq Ker(\alpha)$, we have that

$h = 0$. Then $Ker(\alpha) = 0$ and hence $\varphi : Im\alpha \to M$, defined by $((m)\alpha)\varphi = m$, is an isomorphism. Since M is semi co–Hopfian, $Im\alpha$ is a direct summand of M. Let $\pi : M \to Im\alpha$ be the projection. Then $\alpha(\pi\varphi) = 1 \in \alpha S$. So S is semi co–Hopfian. □

The converse of Proposition 2.17 is not true in general. For example, let ${}_{\mathbb{Z}}M = \mathbb{Z}_{p^\infty}$ for a prime p. Then the endomorphism ring S of M is isomorphic to the ring of p–adic integers. ${}_{\mathbb{Z}}M$ is co–Hopfian but ${}_SS$ is not (see [12]). Since only idempotents in S are 0 and 1, ${}_SS$ is not semi co–Hopfian by Proposition 2.4.

Since a free module generates all its submodules, we have;

Corollary 2.18 *If M is free, then M is semi co–Hopfian if and only if $End_R(M)$ is semi co–Hopfian. In particular, R^n is semi co–Hopfian left R–module if and only if $M_n(R)$ is semi co–Hopfian left $M_n(R)$–module.*

Let M be a module. The elements of $M[X]$ are formal sums of the form $a_0 + a_1X + \cdots + a_kX^k = \sum_{i=1}^k a_iX^i$ with k an integer greater than or equal to 0 and $a_i \in M$. Addition is defined by adding the corresponding coefficients. The $R[X]$–module structure is given by

$$(\sum_{i=0}^k \lambda_iX^i) \cdot (\sum_{j=0}^z a_jX^j) = \sum_{\mu=0}^{k+z} c_\mu X^\mu$$

where $c_\mu = \sum_{i+j=\mu} \lambda_i a_j$, for any $\lambda_i \in R$, $a_j \in M$.

Theorem 2.19 *Let M be an R– module. If $M[X]$ is semi co–Hopfian $R[X]$–module, then M is semi co–Hopfian R–module.*

Proof Let $f : M \to M$ be an injective endomorphism of M. Then $f[X] : M[X] \to M[X]$ with $f[X](\sum m_iX^i) = \sum f(m_i)X^i$ is an injective endomorphism of $M[X]$. Since $M[X]$ is semi co–Hopfian, $Im(f[X]) = (Imf)[X] \leq^\oplus M[X]$. Now we claim that $Imf \leq^\oplus M$. Let $M[X] = (Imf)[X] \oplus K$ for some submodule K of $M[X]$ and K' denote the submodule of M which is generated by the constant polynomials of K. Note that $K' \neq 0$ if $M \neq Imf$. We will show that $M = Imf \oplus K'$. Let $m \in M$. Then $m \in M[X]$ and so $m = g(X) + k(X)$ where $g(X) \in (Imf)[X]$, $k(X) \in K$. Since m is a constant polynomial in $M[X]$, we have $m = g(0) + k(0)$ where $g(0) \in Imf$ and $k(0) \in K'$. Next, take $k' \in Imf \cap K'$. But $k' \in (Imf)[X] \cap K = 0$. □

There exists a left semi co–Hopfian ring that is not right semi co–Hopfian.

Example 2.20 (Faith-Menal) The left $C2$ ring R which is not right $C2$ is the example: Let D be any countable, existentially closed division ring over a field F, and let $R = D \bigotimes_F F(x)$. Then the trivial extension of D by R,

$T(R, D)$ is a non–Artinian left P–injective, right finite dimensional ring (see [14, Example 7.11 and 8.16]). Since R is left P–injective, it is left $C2$ and so left semi co–Hopfian. If R is right semi co–Hopfian, then R is right co–Hopfian by Proposition 2.5. By Camps–Dicks Theorem (see [14, Theorem C.2]), R is semilocal. Since J(R) is nilpotent, R is right Artinian by Hopkins–Levitzki Theorem. But this is a contradiction.

It is well–known that if ${}_RR$ has $C2$, then $Z_l \subseteq J(R)$. By a proof similar to Lemma 2.3 in [15] we have the following generalization. But we give the proof for completeness.

Proposition 2.21 *If ${}_RR$ is semi co–Hopfian, then $Z_l \subseteq J(R)$.*

Proof Let $a \in Z_l$. Then $l_R(1-a) = 0$ and so $R(1-a) \cong R$. This isomorphism gives an injective endomorphism of R such that $Imf = R(1-a)$. By hypothesis, $R(1-a)$ is a direct summand of R. Then $(1-a)R$ is a direct summand. Let $e^2 = e \in R$ be such that $(1-a)R = eR$. Since $(1-e)(1-a) = 0$, we have $1 - e = 0$. Hence $1 - a$ is right invertible. Since this holds for all $a \in Z_l$, $Z_l \subseteq J(R)$. □

The converse is not true in general. The localization $\mathbb{Z}_{(p)}$ of the ring of integers at the prime p is commutative domain with $Z_l = 0$ but not a division ring. By Proposition 2.4, ${}_RR$ is not semi co–Hopfian.

3 Semi Hopfian Modules

In this section we consider the dual version of semi co–Hopfian modules. A module M is called *semi Hopfian* if any surjective endomorphism of M has a direct summand kernel, i.e. any surjective endomorphism of M splits. Then any Hopfian module is semi Hopfian.

Example 3.1 If R is semisimple Artinian, then a module ${}_RM$ is Hopfian if and only if ${}_RM$ has finite length (see [9]). Also a vector space over a field is Hopfian if and only if it is finite dimensional. Hence an infinite dimensional vector space over a field is semi Hopfian (it is semisimple) but not Hopfian.

Example 3.2 Let p be prime and M be any direct sum of copies of $\mathbb{Z}_{p^2}$. Then we claim that M is a semi Hopfian $\mathbb{Z}$–module. Let $f : M \to M$ be an epimorphism. Since $p^2M = 0$, f is an $\mathbb{Z}_{p^2}$–epimorphism. Since M is a free $\mathbb{Z}_{p^2}$–module, f splits. This implies that M is a semi Hopfian $\mathbb{Z}$–module. But it is well known that M is not a Hopfian $\mathbb{Z}$–module.

Note that for any ring R, R_R is Hopfian if and only if ${}_RR$ is Hopfian if and only if it is Dedekind finite [16, Proposition 1.2]. Hence any ring which is not Dedekind finite is an example of a module which is semi Hopfian but not Hopfian. For example, the ring of linear endomorphisms of an infinite dimensional left vector space over a division ring is not Dedekind finite.

The following characterization can be seen easily.

Lemma 3.3 *The following are equivalent for a module M.*

1) M is semi Hopfian.

2) Any submodule N of M which satisfies $M/N \cong M$ is a direct summand of M.

Hence any module with $D2$ is semi Hopfian. In particular, any quasi–projective module is semi Hopfian by [13, Proposition 4.38]. Therefore, the concept of semi Hopfian modules is a generalization of Hopfian modules and modules with $D2$.

Example 3.4 *There exists a semi Hopfian module which has not $D2$.*

Proof Let ${}_{\mathbb{Z}}M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$, p is a prime number. Then M is not relatively projective but has $D1$ by [10, Example 4]. Hence M has not $D2$ by [13, Lemma 4.23]. Since ${}_{\mathbb{Z}}M$ is Noetherian, it is Hopfian and hence semi Hopfian. $\square$

Note that any semi Hopfian quasi–discrete module has $D2$ (see [13, Lemma 5.1]).

Dual of the summand sum property is the summand intersection property. A module M has the *summand intersection property (SIP)* if the intersection of any two direct summands of M is a direct summand. Note that M has SIP if and only if for every decomposition $M = A \oplus B$ and every R–homomorphism f from A to B, the kernel of f is a direct summand [8].

Hence if $M \oplus M$ has SIP, then M is semi Hopfian. But the converse is not true: Let the $\mathbb{Z}$–module $M = \mathbb{Z} \oplus \mathbb{Z}_2$. Since M is Noetherian, it is semi Hopfian. Denote an R-homomorphism $f : M \to M$ by $f(a, \overline{b}) = (0, \overline{a})$ where $a, b \in \mathbb{Z}$. Then $Ker f = 2\mathbb{Z} \oplus \mathbb{Z}_2$ is not a direct summand. Hence $M \oplus M$ has not SIP.

Now consider torsion free modules. Torsion free modules need not be semi Hopfian.

Example 3.5 *There exists a torsion free module which is not semi Hopfian.*

Proof Let F be the direct sum of copies of $\mathbb{Z}$ and the $\mathbb{Z}$–module $M = \mathbb{Q} \oplus F \oplus F \oplus \cdots$. Then there exists an epimorphism $\alpha : F \to \mathbb{Q}$. Define $\theta : M \to M$ be such that $\theta(q, f_1, f_2, \ldots) = (\alpha(f_1), f_2, \ldots)$ where $q \in \mathbb{Q}, f_i \in F$ for all i. Then θ is a $\mathbb{Z}$–epimorphism and $Ker\theta = \mathbb{Q} \oplus Ker\alpha \oplus 0 \oplus 0 \cdots$. If $Ker\theta$ was a direct summand of M, then $Ker\alpha$ would have to be a direct summand of F, this is a contradiction. So M is not semi Hopfian. □

Unlike Proposition 2.13, there exists a commutative domain R and a torsion free semi Hopfian R–module M which is not injective or not projective. For example, $M_{\mathbb{Z}} = \mathbb{Z}$ is not injective and $M_{\mathbb{Z}} = \mathbb{Q}$ is not projective.

Recall that M has *hollow dimension* $n \in \mathbb{N}$ if there exists an epimorphism from M to a direct sum of n nonzero modules but no epimorphism from M to a direct sum of more than n nonzero modules (see [3, 5.2]). In [3, 5.4(3)], it is proved that a semi Hopfian module with finite hollow dimension (i.e. dual Goldie dimension) is Hopfian. Any module with finite hollow dimension is Dedekind finite, for let M be a module with finite hollow dimension and $M \oplus K \cong M$. Consider the isomorphism $\varphi : M \longrightarrow M \oplus K$ and the projection $\pi : M \oplus K \longrightarrow M$. Clearly, $\pi\varphi$ is a surjection. Since M has finite hollow dimension, $Ker(\pi\varphi) \ll M$ by [3, 5.4(3)]. Then $Ker(\pi\varphi) = \varphi^{-1}(Ker\pi) = \varphi^{-1}(K) \ll M$ implies that $\varphi\varphi^{-1}(K) = K \ll M \oplus K$. But $K \leq^{\oplus} M \oplus K$. Hence $K = 0$, i.e. M is Dedekind finite.

So the following result which is known in the literature generalizes [3, 5.4(3)]. A module M is called *generalized Hopfian* (in [5]) if any surjective endomorphism of M has a small kernel.

Proposition 3.6 *The following are equivalent for a module M.*

1) M is Hopfian.

2) M is Dedekind finite and semi Hopfian.

3) M is generalized Hopfian and semi Hopfian.

Proof (3) $\Leftrightarrow$ (1) $\Rightarrow$ (2) are obvious.
(2) $\Rightarrow$ (1) (see also [11, Exc. 1.8]) Let $f : M \longrightarrow M$ be a surjection. Since M is semi Hopfian f splits. Then there exists an endomorphism $g : M \longrightarrow M$ such that $fg = 1$. But Dedekind finiteness of $End_R(M)$ implies $gf = 1$. Hence, f is an injection. □

It is also known that a semi Hopfian module with finite hollow dimension has a semilocal endomorphism ring [3, 19.2].

Now we investigate some properties of semi Hopfian modules.

Proposition 3.7 *Any direct summand of a semi Hopfian module is semi Hopfian.*

Proof Let K be a direct summand of a semi Hopfian module M and $f : K \to K$ be a surjection. Then $M = K \oplus K'$ for some K', and $f \oplus 1_{K'} : M \to M$ is also a surjection. Thus $Ker(f \oplus 1_{K'}) = Kerf \leq^{\oplus} M$ and hence $Kerf \leq^{\oplus} K$. $\square$

Proposition 3.8 *If M/N is semi Hopfian for every non–summand submodule N of a module M, then M is semi Hopfian.*

Proof Suppose that M is not semi Hopfian. Then there exists a surjective endomorphism $f : M \to M$ such that $Kerf$ is not a direct summand of M. But by assumption $M/Kerf \cong M$ is semi Hopfian, a contradiction. $\square$

Any direct sum of semi Hopfian modules need not be semi Hopfian.

Example 3.9 Let p be prime and $M_1 = \mathbb{Z}_p$ and M_2 an infinite direct sum of copies of $\mathbb{Z}_{p^2}$. Then M_1 is simple and M_2 is semi Hopfian by Example 3.2. But $M = M_1 \oplus M_2$ is not semi Hopfian $\mathbb{Z}$–module. For, define $f : M \to M$ by $f(a_1 + p\mathbb{Z}, a_2 + p^2\mathbb{Z}, a_3 + p^2\mathbb{Z}, \ldots) = (a_2 + p\mathbb{Z}, a_3 + p^2\mathbb{Z}, \ldots)$. Then f is a $\mathbb{Z}$–epimorphism and $Kerf = \mathbb{Z}_p \oplus p\mathbb{Z}_{p^2} \oplus 0 \oplus 0 \cdots$ is not a direct summand of M since $p\mathbb{Z}_{p^2}$ is not a direct summand of $\mathbb{Z}_{p^2}$.

Any direct product of semi Hopfian modules need not be semi Hopfian: If we let the $\mathbb{Z}$–module $M = \mathbb{Z}_p \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \cdots$, then M is not semi Hopfian by a proof similar to Example 3.9.

Proposition 3.10 *Let $M = \oplus_{i \in I} M_i$, where M_i is invariant under any surjection of M for all $i \in I$. Then M is semi Hopfian if and only if M_i is semi Hopfian for all $i \in I$.*

Proof The necessity is clear from Proposition 3.7. For the sufficiency, let $f : M \to M$ be a surjection. Then $f|_{M_i} : M_i \to M_i$ is a surjection for all $i \in I$. Since $Ker(f|_{M_i}) \leq^{\oplus} M_i$ for all i we have that $Kerf = \oplus_{i \in I} Ker(f|_{M_i}) \leq^{\oplus} M$. $\square$

Note that M_2 is not invariant under the surjection f of M in Example 3.9.

As a dual of DCC on non–summands we consider the ascending chain condition (ACC) on non–summands.

Theorem 3.11 *If M has ACC on non–summand submodules, then M is semi Hopfian.*

Proof Assume that $f : M \to M$ is a surjection and $Kerf$ is a non–summand of M. Then $Kerf \subseteq Kerf^2 \subseteq Kerf^3 \subseteq \cdots$ is an ascending chain of non–summand submodules of M. By hypothesis there exists an integer n such that $Ker(f^n) = Ker(f^{n+1})$. Now we claim that $Kerf = 0$. Let $x \in M$ be such that $f(x) = 0$. Since f is surjective, $f(a_1) = x$ for some $a_1 \in M$. Again, $f(a_2) = a_1$ for some $a_2 \in M$. By repeating this argument, we have $f(a_n) = a_{n-1}$ for some $a_n \in M$. Then $f(a_1) = f^2(a_2) = \cdots = f^n(a_n) = x$. Hence $f(x) = f^{n+1}(a_n) = 0$ implies that $a_n \in Ker(f^{n+1}) = Ker(f^n)$. As a result, $x = 0$. So we have a contradiction. □

Proposition 3.12 *Let $\mathcal{P}$ be a property of modules preserved under isomorphism. If a module M has the property $\mathcal{P}$ and satisfies ACC on non–summand submodules N such that M/N has the property $\mathcal{P}$, then M is semi Hopfian.*

Proof Suppose that M is not semi Hopfian. Then there exists a non–summand submodule N_1 of M such that $M/N_1 \cong M$. Since M/N_1 has the property $\mathcal{P}$ but is not semi Hopfian, there exists a non–summand submodule N_2/N_1 of M/N_1 such that $M/N_2 \cong M/N_1$. N_2 is also a non–summand of M. Continuing in this way we get an ascending chain $0 \subset N_1 \subset N_2 \subset \cdots$ of non–summand submodules of M. But this is a contradiction. □

Corollary 3.13 *If M is semi co–Hopfian and satisfies ACC on non–summand submodules N such that M/N is semi co–Hopfian, then M is semi Hopfian.*

Proof Take the property $\mathcal{P}$ as being semi co–Hopfian and apply Proposition 3.12. □

Corollary 3.14 *If M is semi Hopfian and satisfies DCC on non–summand semi Hopfian submodules, then M is semi co–Hopfian.*

Proof Clear by Proposition 2.15. □

Corollary 3.15 *If M satisfies ACC on non–summand submodules N such that M/N is not semi Hopfian, then M is semi Hopfian.*

Proof It follows from Proposition 3.12 by letting $\mathcal{P}$ be the property of not being semi Hopfian. $\square$

The following result can be seen by a proof similar to Theorem 2.19.

Theorem 3.16 *Let M be an R–module. If $M[X]$ is semi Hopfian $R[X]$–module, then M is semi Hopfian R–module.*

4 Algebra of Continuous functions

Definition 4.1 A topological space X is said to be *semi Hopfian* (resp. *semi co–Hopfian*) in the category of topological spaces **Top** if every surjective (resp. injective) continuous map $\alpha : X \to X$, there exists a continuous map $\beta : X \to X$ such that $\alpha \circ \beta = id_X$ (resp. $\beta \circ \alpha = id_X$).

Definition 4.2 Let K be a commutative ring and K–**alg** denote the category of K–algebras. A K–algebra A is said to be *semi Hopfian* (resp. *semi co–Hopfian*) as an K–algebra if for any surjective (resp. injective) K–algebra homomorphism $\alpha : A \to A$ there exists a K–algebra homomorphism $\beta : A \to A$ such that $\alpha \circ \beta = 1_A$ (resp. $\beta \circ \alpha = 1_A$)

For any compact Hausdorff space X, $C(X)$ denote the $\mathbb{R}$–algebra of continuous functions from X to $\mathbb{R}$. Varadarajan [16, Theorem 5.3] prove that if X is a compact Hausdorff space, then $C(X)$ is Hopfian (resp. co–Hopfian) as an $\mathbb{R}$–algebra if and only if X is co–Hopfian (resp. Hopfian) as a topological space. Here we prove that if X is semi co–Hopfian (resp. semi Hopfian), then $C(X)$ is semi Hopfian (resp. semi co–Hopfian) in the category of $\mathbb{R}$–algebras. The converse of this result is open.

If $\varphi : X \to Y$ is a continuous map of compact Hausdorff spaces, there is an induced homomorphism $\varphi^* : C(Y) \to C(X)$ in the $\mathbb{R}$–algebra given by $\varphi^*(g) = g \circ \varphi$ for every $g \in C(Y)$. Also given any $\mathbb{R}$–algebra homomorphism $\alpha : C(Y) \to C(X)$, there is a unique continuous map $\varphi : X \to Y$ such that $\alpha = \varphi^*$ (see [16]).

Proposition 4.3 [16, Proposition 5.2] *Let $\varphi : X \to Y$ be a continuous map of a compact Hausdorff spaces. Then $\varphi^* : C(Y) \to C(X)$ is injective (resp. surjective) if and only if $\varphi : X \to Y$ is surjective (resp. injective).*

Theorem 4.4 *Let X be a compact Hausdorff space. If X is a semi co–Hopfian (resp. semi Hopfian) topological space, then $C(X)$ is a semi Hopfian (resp. semi co–Hopfian) $\mathbb{R}$–algebra.*

Proof Assume that X is a semi co–Hopfian topological space. Let $\alpha : C(X) \to C(X)$ be a surjective $\mathbb{R}$–algebra homomorphism. Then there exists a unique continuous map $\varphi : X \to X$ such that $\alpha = \varphi^*$. By Proposition 4.3, φ is injective. By assumption, there exists a continuous map $\gamma : X \to X$ such that $\gamma \circ \varphi = id_X$. Then γ is surjective and again $\gamma^* : C(X) \to C(X)$ is an injective $\mathbb{R}$–algebra homomorphism. Let $f \in C(X)$. Then $(\alpha \circ \gamma^*)(f) = \alpha(\gamma^*(f)) = \alpha(f \circ \gamma) = \varphi^*(f \circ \gamma) = f \circ \gamma \circ \varphi = f \circ id_X = f$. So we have that $\alpha \circ \gamma^* = 1_{C(X)}$. Hence α splits.

The result in parenthesis can be seen similarly. $\square$

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