East-West J. of Mathematics: Vol. 10, No 1 (2008) pp. 15-25

SOME CONSTRUCTIONS IN THE CATEGORY OF POINTED G-SETS

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Abstract

This paper deals with the study of products, co-products, equalizers, co-equalizers, intersections, pullbacks and pushouts in the category of pointed G-sets. Further, it is shown that the category of pointed G-sets is complete and finitely complete.

1. Introduction

Motivated by the idea of pointed sets, pointed mappings, G-sets and G-morphisms, the notions of pointed G-sets and pointed G-morphisms have been defined, henceforth the category of pointed G-sets, denoted by $\mathcal{G}$-Sets*, has been constructed by taking into account pointed G-sets as the objects of the category and pointed G-morphisms as the morphisms of the category. Results regarding special morphisms like monomorphisms, epimorphisms, coretractions and retractions in the category $\mathcal{G}$-Sets* have been proved in [4]. In the present analysis, we study some more properties of the category $\mathcal{G}$-Sets* and show that the category $\mathcal{G}$-Sets* has products, co-products, equalizers, co-equalizers, intersections, pullbacks and pushouts. After showing the existence of these notions, we obtain that the category $\mathcal{G}$-Sets* is complete and finitely complete.

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Key words: pointed G-sets, pointed G-morphisms and category of pointed G-sets.
2000 AMS Mathematics Subject Classification: 18A05, 20C05

2. Preliminaries

We begin with the following definitions and results that will be needed in the sequel [8,9,10]:

Definition 2.1. Let G be a group and X be a set. Then X is said to be a G-set if there exists a mapping $\phi : G \times X \to X$ such that for all $a, b \in G$ and $x \in X$ the following conditions are satisfied:

(i) $\phi(ab, x) = \phi(a, \phi(b, x))$,

(ii) $\phi(e, x) = x$,

where e is the identity of G. The G-set X defined above will be denoted by the pair (X, ϕ).
For the sake of convenience, one can denote $\phi(a, x)$ as ax. Under this notation, above conditions become

(i) $(ab)x = a(bx)$,

(ii) $ex = x$.

Definition 2.2. Let (X, ϕ) be a G-set. Then a subset A of X is called a G-subset of X if (A, ϕ) is also a G-set.

Definition 2.3. Let X and Y be two G-sets. Then a mapping $f : X \to Y$ is called a G-morphism from X to Y if $f(ax) = af(x)$ for all $a \in G, x \in X$.

Definition 2.4. Let $\{X_i\}_{i\in I}$ be a family of G-sets. Then the product of $\{X_i\}_{i\in I}$, denoted by $\prod\limits_{i\in I} X_i$, is defined to be the set $\{f : I \to \cup X_i \mid f(i) \in X_i \text{ for all } i \in I\}$.

Proposition 2.1 [10, Theorem 3.2]. Let $\{X_i\}_{i\in I}$ be a family of G-sets. Then, the product $\prod\limits_{i\in I} X_i$ of the family $\{X_i\}_{i\in I}$ is a G-set.

Definition 2.5. A pointed set (X, x') is said to be a pointed G-set if there exists a mapping $\phi : G \times X \to X$ such that

(i) (X, ϕ) is a G-set,

(ii) $\phi(g, x') = x'$ for all $g \in G$.

Definition 2.6. Let (X, x') be a pointed G-set. Then a pointed set (A, x') is called a pointed G-subset of (X, x') if A is a G-subset of X.

Definition 2.7. Let (X, x') be a pointed G-set and (A, x'), (B, x') be two pointed G-subsets of (X, x') such that $A \cap B = \emptyset$. Then disjoint union of

(A, x') and (B, x') is defined to be the set $(A \cup B, x')$.

Definition 2.8. Let (X, x') be a pointed G-set and (A, x'), (B, x') be two pointed G-subsets of (X, x'). Then intersection of (A, x') and (B, x') is defined to be the set $(A \cap B, x')$.

Definition 2.9. Let (X, x') and (Y, y') be two pointed G-sets. Then their product is defined to be the ordered pair $(X \times Y, (x', y'))$.

Definition 2.10. Let (X, x') and (Y, y') be two pointed G-sets. Then a mapping $f : (X, x') \to (Y, y')$ is called a pointed G-morphism if

(i) f is a G-morphism i.e., $f(ax) = af(x)$ for all $a \in G$, $x \in X$,

(ii) $f(x') = y'$.

We shall use the following lemmas in our main results:

Lemma 2.1. Let X and Y be two pointed G-sets. Then

(i) The cartesian product of any two pointed G-sets is a pointed G-set,

(ii) Disjoint union of pointed G-subsets is a pointed G-subset,

(iii) Intersection of a finite family of pointed G-subsets is a pointed G-subset.

Proof. The proof of Lemma 2.1 is trivial.

Lemma 2.2. Let (X, ϕ) be a G-set. Then

(i) for any $x, y \in X$, a relation $\sim_G$ on X defined by $x \sim_G y \Leftrightarrow y = \phi(g, x)$ for some $g \in G$, is an equivalence relation,

(ii) the set $X/\sim_G$ of all G-equivalence classes is a G-set.

Proof(i). Let e be the identity element of G. Then for every $x \in X$, we have $x = \phi(e, x)$ implying thereby $x \sim_G x$. Now, suppose $x \sim_G y$. Then $y = \phi(g, x)$ for some $g \in G$. So, we have $\phi(g^{-1}, y) = \phi(g^{-1}, \phi(g, x)) = \phi(g^{-1}g, x) = \phi(e, x) = x$ implies that $y \sim_G x$. Further, suppose $x \sim_G y$ and $y \sim z$. Then there exist $g_1, g_2 \in G$ such that $y = \phi(g_1, x)$ and $z = \phi(g_2, y)$. Thus we have $\phi(g_2 g_1, x) = \phi(g_2, \phi(g_1, x)) = \phi(g_2, y) = z$ implying thereby $x \sim_G z$. Consequently, $\sim_G$ is an equivalence relation.
We call this equivalence relation as G-equivalence relation and corresponding equivalence class as G-equivalence class.

Proof(ii). Proof is trivial.

Lemma 2.3. Let (X, x') be a pointed G-set and $\sim_G$ be a G-equivalence relation on X. Then the pointed set $(X/{\sim_G}, [x'])$ is a pointed G-set.

Proof. Define a mapping $\phi : G \times X/{\sim_G} \to X/{\sim_G}$ by $\phi(g, [x]) = [gx]$ for all $g \in G, x \in X$. It can be easily seen that $(X/{\sim_G}, \phi)$ is a G-set and for any $a \in G$, we get $\phi(a, [x']) = [ax'] = [x']$ which shows that $(X/{\sim_G}, [x'])$ is a pointed G-set.

3. Main Results

Theorem 3.1. The category $\mathcal{G}$-Sets* has finite products.

Proof. If (X, x') and (Y, y') are two pointed G-sets, then in view of Lemma 2.1 (i), $(X \times Y, (x', y'))$ is also a pointed G-set. Define natural projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ by $p_1(x, y) = x$ and $p_2(x, y) = y$ for all $x \in X$, $y \in Y$. Trivially, p_1 and p_2 are pointed G-morphisms. We claim that $(X \times Y, (x', y'))$ together with morphisms p_1 and p_2 is the categorical product of (X, x') and (Y, y').

Let $\alpha_1 : (Z, z') \to (X, x')$ and $\alpha_2 : (Z, z') \to (Y, y')$ be two morphisms in $\mathcal{G}$-Sets*, then we can define a mapping $\eta : (Z, z') \to (X \times Y, (x', y'))$ by

$$\eta(z) = (\alpha_1(z), \alpha_2(z)) \quad \text{for all } z \in Z.$$

For any $z \in Z$ and $a \in G$, one gets $\eta(az) = (\alpha_1(az), \alpha_2(az)) = a(\alpha_1(z), \alpha_2(z)) = a(\eta(z))$ which shows that η is a G-morphism and also $\eta(z') = (\alpha_1(z'), \alpha_2(z')) = (x', y')$. Therefore η is a pointed G-morphism. Also, we have $(p_1 \circ \eta)(z) = p_1(\eta(z)) = p_1(\alpha_1(z), \alpha_2(z)) = \alpha_1(z)$ which implies $p_1 \circ \eta = \alpha_1$. Similarly, $p_2 \circ \eta = \alpha_2$.

Finally, we show that η is unique. Suppose, there exists another morphism $\xi : (Z, z') \to (X \times Y, (x', y'))$ in $\mathcal{G}$-Sets* such that $p_1 \circ \xi = \alpha_1$ and $p_2 \circ \xi = \alpha_2$. Then we have $\xi(z) = (p_1(\xi(z)), p_2(\xi(z))) = (\alpha_1(z), \alpha_2(z)) = \eta(z)$ for all $z \in Z$ which implies $\xi = \eta$. This completes the proof.

Next, let (X, x') and (Y, y') be two pointed G-sets. Consider the pointed G-subset of $X \times Y$ with the base point (x', y'), consisting of the elements of the type (x, y') and (x', y) for all $x \in X$, $y \in Y$ and denote it by $(X+Y, (x', y'))$. Trivially, the natural inclusions $u_1 : (X, x') \to (X + Y, (x', y'))$ and $u_2 : (Y, y') \to (X + Y, (x', y'))$ are pointed G-morphisms.

Theorem 3.2. The category $\mathcal{G}$-Sets* has finite co-products.

Proof. If (X, x') and (Y, y') are two pointed G-sets, then obviously $(X + Y, (x', y'))$ is a pointed G-set. Define natural inclusions $u_1 : (X, x') \to (X + Y, (x', y'))$ and $u_2 : (Y, y') \to (X + Y, (x', y'))$ by $u_1(x) = (x, y')$ and

$u_2(y) = (x', y)$ for all $x \in X$, $y \in Y$ which are trivially pointed G-morphisms. We claim that $(X + Y, (x', y'))$ together with morphisms u_1 and u_2 is the categorical co-product of (X, x') and (Y, y').

Let $\alpha_1 : (X, x') \to (Z, z')$ and $\alpha_2 : (Y, y') \to (Z, z')$ be two morphisms in $\mathcal{G}$-Sets*. Define a mapping $\eta : (X + Y, (x', y')) \to (Z, z')$ as follows:

$$\eta(x', y') = z',$$

$$\eta(x, y') = \alpha_1(x) \quad \text{for all } x \in X,$$

$$\eta(x', y) = \alpha_2(y) \quad \text{for all } y \in Y.$$

For any $z \in Z$ and $a \in G$, one gets $\eta(a(x, y')) = \eta(ax, ay') = \eta(ax, y') = \alpha_1(ax) = a(\alpha_1(x)) = a\eta(x, y')$. Similarly, $\eta(a(x', y)) = a\eta(x', y)$. Thus η is a G-morphism and as $\eta(x', y') = z'$, it follows that η is a pointed G-morphism. Obviously, $\eta \circ u_1 = \alpha_1$ and $\eta \circ u_2 = \alpha_2$.

Finally, we show that η is unique. Suppose, there exists another morphism $\xi : (X + Y, (x', y')) \to (Z, z')$ in $\mathcal{G}$-Sets* such that $\xi \circ u_1 = \alpha_1$ and $\xi \circ u_2 = \alpha_2$. Then for any $x \in X$, we have $(\xi \circ u_1)(x) = \alpha_1(x)$ implying thereby $\xi(u_1(x)) = \alpha_1(x)$ which yields $\xi(x, y') = \eta(x, y')$ for all $(x, y') \in X + Y$. Therefore $\xi = \eta$. Similarly, for any $y \in Y$, we have $(\xi \circ u_2)(y) = \alpha_2(y)$ which yields $\xi(u_2(y)) = \alpha_2(y)$ implying thereby $\xi(x', y) = \eta(x', y)$ for all $(x', y) \in X + Y$. Thus $\xi = \eta$ and consequently η is unique. This completes the proof.

Theorem 3.3. The category $\mathcal{G}$-Sets* has finite intersections.

Proof. Let $\{u_i : (X_i, x') \to (X, x') \mid i = 1, 2, ..., n\}$ be a family of sub-objects of an object (X, x') in $\mathcal{G}$-Sets* where u_i's are inclusion mappings which are trivially pointed G-morphisms. Consider the set $X' = \bigcap_{i=1}^{n} X_i$. In view of Lemma 2.1 (*iii*), X' is a pointed G-set with the basepoint x'. Hence, $(X', x') \in \mathcal{G}$-Sets*. Let $u : (X', x') \to (X, x')$ defined by $u(x) = x$ for all $x \in X'$, be a morphism in $\mathcal{G}$-Sets*. We claim that X' together with $u : (X', x') \to (X, x')$ is the intersection of the family of sub-objects $\{u_i : (X_i, x') \to (X, x') \mid i = 1, 2, ..., n\}$ in $\mathcal{G}$-Sets*.

Consider morphisms $v_i : (X', x') \to (X_i, x')$ defined by $v_i(x) = x$ for all $x \in X'$, $i = 1, 2, ..., n$. Trivially v_i's are G-morphisms and also $v_i(x') = x'$. Therefore v_i's are pointed G-morphisms.

For any $x \in X$, we have $(u_i \circ v_i)(x) = u_i(v_i(x)) = u_i(x) = x = u(x)$ which implies $u_i \circ v_i = u$. Now, for any object $Y \in \mathcal{G}$-Sets, let $f : (Y, y') \to (X, x')$

be a morphism in $\mathcal{G}$-Sets* which factors through each u_i i.e., $f = u_i \circ f_i$.

For any $y \in Y$, one gets $f(y) = (u_i \circ f_i)(y) = u_i(f_i(y)) = f_i(y) \in X_i$ which yields $f(Y) \subseteq X_i$ for each i, $(i = 1, 2, ..., n)$ implying thereby $f(Y) \subseteq \cap X_i = X'$. Therefore, we can define a mapping $\eta : (Y, y') \to (X', x')$ by $\eta(y) = f(y)$ for all $y \in Y$. It can be easily shown that η is a pointed G-morphism. Now, for any $y \in Y$, we get $(u \circ \eta)(y) = u(\eta(y)) = \eta(y) = f(y)$ which implies $u \circ \eta = f$.

It remains to show that η is unique. Suppose there exists another morphism $\xi : (Y, y') \to (X', x')$ in $\mathcal{G}$-Sets* such that $u \circ \xi = f$. Then we have $(u \circ \xi)(y) = f(y)$ yielding thereby $u(\xi(y)) = \eta(y)$ which gives $\xi(y) = \eta(y)$ for all $y \in Y$ implying thereby $\xi = \eta$. This completes the proof.

To prove our next Theorem we require the following propositions:

Proposition 3.1. If $\delta : I \to \bigcup X_i$ is the mapping defined by $\delta(i) = {x_i}'$ for all $i \in I$. Then the set $\prod_{i \in I} X_i$ together with the base point δ is a pointed G-set.

Proof. By Proposition 2.1, $\prod_{i \in I} X_i$ is a G-set under the mapping $\phi : G \times \prod_{i \in I} X_i \to \prod_{i \in I} X_i$ defined by $\phi(a, f) = af$ for all $a \in G$ and $f \in \prod_{i \in I} X_i$, where $af : I \to \bigcup_{i \in I} X_i$ is defined by $(af)(i) = a(f(i))$ for each $i \in I$.

For any $a \in G$, we have $(\phi(a, \delta))(i) = (a\delta)(i) = a(\delta(i)) = a({x_i}') = {x_i}' = \delta(i)$ for all $i \in I$ which implies $\phi(a, \delta) = \delta$ and henceforth $(\prod_{i \in I} X_i, \delta)$ is a pointed G-set.

Proposition 3.2. Projections are pointed G-morphisms.

Proof. Let $\{(X_i, {x_i}')\}_{i \in I}$ be a family of pointed G-sets. Then in view of the above Proposition 3.1, $\prod_{i \in I} X_i$ with the base point δ is a pointed G-set. For each index $i \in I$, we define projections $p_i : \prod_{i \in I} X_i \to X_i$ by $p_i(f) = f(i)$.

Now, for any $a \in G$, we have $p_i(af) = (af)(i) = a(f(i)) = a(p_i(f))$ which shows that p_i's are G-morphisms. Also we have $p_i(\delta) = \delta(i) = {x_i}'$. Therefore $p_i : (\prod_{i \in I} X_i, \delta) \to (X_i, {x_i}')$ are pointed G-morphisms.

Theorem 3.4. The category $\mathcal{G}$-Sets* has arbitrary products.

Proof. Let $\{(X_i, {x_i}')\}_{i \in I}$ be a family of objects in $\mathcal{G}$-Sets*. Consider the set $\prod_{i \in I} X_i = \{f : I \to \cup X_i\}$ such that $f(i) \in X_i$ for all $i \in I$. In view of the above

Proposition 3.1, the set $(\prod_{i\in I} X_i, \delta)$ is a pointed G-set.

For each $i \in I$, there are natural projections $p_i : (\prod_{i\in I} X_i, \delta) \to (X_i, {x_i}')$ defined by $p_i(f) = f(i)$ for all $f \in \prod_{i\in I} X_i$. In view of the above Proposition 3.2, p_i's are pointed G-morphisms. We claim that $(\prod_{i\in I} X_i, \delta)$ together with the projection mappings $\{p_i\}_{i\in I}$ is the categorical product of the family $\{X_i\}_{i\in I}$ in $\mathcal{G}$-Sets*.

Let $\{q_i : (X, x') \to (X_i, {x_i}')\}$ be a family of morphisms in $\mathcal{G}$-Sets*. Define a mapping $\eta : (X, x') \to (\prod_{i\in I} X_i, \delta)$ such that $x \mapsto \eta(x)$, where $\eta(x) : I \to \cup X_i$ is defined by $(\eta(x))(i) = q_i(x)$ for all $x \in X$, $i \in I$.

For any $a \in G$, one gets $(\eta(ax))(i) = q_i(ax) = a(q_i(x)) = a(\eta(x))(i)$ for all $i \in I$ which implies $\eta(ax) = a(\eta(x))$. Therefore η is a G-morphism. Also, we have $(\eta(x'))(i) = q_i(x') = {x_i}' = \delta(i)$ for all $i \in I$ which implies $\eta(x') = \delta$. Consequently, η is a pointed G-morphism.

Moreover, for any $x \in X$ and $i \in I$, we have $(p_i \circ \eta)(x) = p_i(\eta(x)) = (\eta(x))(i) = q_i(x)$ implying thereby $p_i \circ \eta = q_i$.

For the uniqueness of η, suppose there exists another morphism $\xi : (X, x') \to (\prod_{i\in I} X_i, \delta)$ in $\mathcal{G}$-Sets* such that $p_i \circ \xi = q_i$ for all $i \in I$. Then we have $(p_i \circ \xi)(x) = q_i(x)$ yielding thereby $p_i(\xi(x)) = q_i(x)$ implying $(\xi(x))(i) = \eta(x)(i)$ which gives $\xi(x) = \eta(x)$ for all $x \in X$. Therefore $\xi = \eta$. This completes the proof.

Theorem 3.5. The category $\mathcal{G}$-Sets* has arbitrary co-products.

Proof. Let $\{(X_i, {x_i}')\}_{i\in I}$ be a family of objects in $\mathcal{G}$-Sets*. Consider the set $\bigcup X_i$ together with base point x' such that we identify each ${x_i}'$ with x', then obviously $(\bigcup X_i, x')$ is a pointed G-set.

For each $i \in I$, define natural inclusions $u_i : (X_i, {x_i}') \to (\bigcup X_i, x')$ by $u_i(x) = x$ for all $x \in X_i$. Trivially u_i's are pointed G-morphisms. We claim that $(\bigcup X_i, x')$ together with the natural inclusions $\{u_i\}_{i\in I}$ is the categorical co-product of the family $(X_i, {x_i}')$ in $\mathcal{G}$-Sets*.

Let $\{q_i : (X_i, {x_i}') \to (Y, y')\}$ be a family of morphisms in $\mathcal{G}$-Sets*. Define a mapping $\eta : (\bigcup X_i, x') \to (Y, y')$ by $\eta(x) = q_i(x)$ for all $x \in X_i$, $i \in I$.

For any $a \in G$, we get $\eta(ax) = q_i(ax) = a(q_i(x)) = a(\eta(x))$ which shows that η is a G-morphism and also $\eta(x') = \eta(x_i') = q_i(x_i{}') = y'$. Therefore, η is a pointed G-morphism.

Moreover, for any $x \in X_i$, we have $(\eta \circ u_i)(x) = \eta(u_i(x)) = \eta(x) = q_i(x)$ which implies $\eta \circ u_i = q_i$.

It remains to show that η is unique. Suppose there exists another morphism $\xi : (\bigcup X_i, x') \to (Y, y')$ in $\mathcal{G}$-Sets* such that $\xi \circ u_i = q_i$ for all $i \in I$. Then we have $(\xi \circ u_i)(x) = q_i(x)$ which implies $\xi(u_i(x)) = \eta(x)$ yielding thereby $\xi(x) = \eta(x)$ for all $x \in X_i$. Therefore $\xi = \eta$ which completes the proof.

Theorem 3.6. The category $\mathcal{G}$-Sets* has equalizers and co-equalizers.

Proof. If $\alpha, \beta : (X, x') \to (Y, y')$ are two morphisms in $\mathcal{G}$-Sets*, then we have two morphisms $\alpha, \beta : X \to Y$ in $\mathcal{G}$-Sets. In view of [11, Theorem 3.3], the equalizer of α, β is (K, i), where $K = \{x \in X \mid \alpha(x) = \beta(x)\} \subseteq X$. Obviously, K is a G-subset of X under the mapping $\phi : G \times K \to K$ define by $\phi(g, k) = gk$ for all $g \in G,\ k \in K$. Also, we get $\phi(g, x') = gx' = x'$ and hence (K, x') is a pointed G-subset of (X, x').

Furthermore, $i : K \to X$ is an inclusion morphism in $\mathcal{G}$-Sets and also $i(x') = x'$ which shows that i is a pointed G-morphism.

The above discussion yields that in the category $\mathcal{G}$-Sets*

$$(K, x') \xrightarrow{i} (X, x') \xrightarrow{\alpha} (Y, y') = (K, x') \xrightarrow{i} (X, x') \xrightarrow{\beta} (Y, y').$$

Let there be a morphism $u : (Z, z') \to (X, x')$ in $\mathcal{G}$-Sets* such that

$$(Z, z') \xrightarrow{u} (X, x') \xrightarrow{\alpha} (Y, y') = (Z, z') \xrightarrow{u} (X, x') \xrightarrow{\beta} (Y, y').$$

Then

$$Z \xrightarrow{u} X \xrightarrow{\alpha} Y = Z \xrightarrow{u} X \xrightarrow{\beta} Y$$

holds in $\mathcal{G}$-Sets.

Since $\mathcal{G}$-Sets has equalizers, for the morphism $u : Z \to X$, we have $Im(u) \subseteq K$ by [11, Theorem 3.3]. Therefore, by the universal property of equalizer there exists a unique morphism $\eta : Z \to K$ defined by $\eta(z) = u(z)$ for all $z \in Z$ in $\mathcal{G}$-Sets such that

$$Z \xrightarrow{\eta} K \xrightarrow{i} X = Z \xrightarrow{u} X.$$

Now, η is a G-morphism and also $\eta(z') = u(z') = x'$ which show that η is a pointed G-morphism. Thus, we have

$$(Z, z') \xrightarrow{\eta} (K, x') \xrightarrow{i} (X, x') = (Z, z') \xrightarrow{u} (X, x').$$

This shows that (K, x') together with the morphism i is the equalizer of the pair of morphisms α and β in $\mathcal{G}$-Sets*.

We now proceed to prove the result for co-equalizers:

Let R be a relation on Y such that for any $y_1, y_2 \in Y$, $y_1 R y_2 \Leftrightarrow y_1 = \alpha(x)$ and $y_2 = \beta(x)$ for some $x \in X$. Consider a smallest equivalence relation $\bar{R}$ on Y containing R. Then in view of Lemma 2.3, $(Y/\bar{R}, [y'])$ forms a pointed G-set.

Consider the projection mapping $p : (Y, y') \to (Y/\bar{R}, [y'])$, then in view of [11, Theorem 3.4], $p : Y \to Y/\bar{R}$ is the co-equalizer of α and β in $\mathcal{G}$-Sets. Obviously, p is a pointed G-morphism such that

$$(X, x') \xrightarrow{\alpha} (Y, y') \xrightarrow{p} (Y/\bar{R}, [y']) = (X, x') \xrightarrow{\beta} (Y, y') \xrightarrow{p} (Y/\bar{R}, [y']).$$

For any $(Z, z') \in \mathcal{G}$-Sets*, let $q : (Y, y') \to (Z, z')$ be another G-morphism such that

$$(X, x') \xrightarrow{\alpha} (Y, y') \xrightarrow{q} (Z, z') = (X, x') \xrightarrow{\beta} (Y, y') \xrightarrow{q} (Z, z')$$

holds in $\mathcal{G}$-Sets*. Then

$$X \xrightarrow{\alpha} Y \xrightarrow{q} Z = X \xrightarrow{\beta} Y \xrightarrow{q} Z$$

holds in $\mathcal{G}$-Sets. Therefore, by the universal property of co-equalizer there exists a unique morphism $\eta : Y/\bar{R} \to Z$ defined by $\eta([y]) = q(y)$ for all $y \in Y$ in $\mathcal{G}$-Sets such that

$$Y \xrightarrow{p} Y/\bar{R} \xrightarrow{\eta} Z = Y \xrightarrow{q} Z$$

implying $\eta \circ p = q$.

Now, η is a G-morphism and also $\eta([z']) = q(y') = z'$ which in turn yields that η is a pointed G-morphism. Thus, we have

$$(Y, y') \xrightarrow{p} (Y/\bar{R}, [y']) \xrightarrow{\eta} (Z, z') = (Y, y') \xrightarrow{q} (Z, z')$$

holds in $\mathcal{G}$-Sets*. This completes the proof.

Theorem 3.7. The category $\mathcal{G}$-Sets* has pullbacks and pushouts.

Proof. In view of Theorem 3.1 and Theorem 3.6, the category $\mathcal{G}$-Sets* has finite products and equalizers. Therefore, the category $\mathcal{G}$-Sets* has pullbacks [1, Theorem 3.7].

Again, in view of Theorem 3.2 and Theorem 3.6, the category $\mathcal{G}$-Sets* has finite co-products and co-equalizers. Taking in to account the dual of [1, Theorem 3.7], the category $\mathcal{G}$-Sets* has pushouts.

Theorem 3.8. The category $\mathcal{G}$-Sets* is complete.

Proof. In view of Theorem 3.4 and Theorem 3.6, the category $\mathcal{G}$-Sets* has arbitrary products and equalizers. Therefore, the category $\mathcal{G}$-Sets* is left complete [3, pp. 26]. Also, in view of Theorem 3.5 and Theorem 3.6, the category $\mathcal{G}$-Sets* has arbitrary co-products and co-equalizers. Therefore, the category $\mathcal{G}$-Sets* is right complete [3, pp. 26]. Henceforth, the category $\mathcal{G}$-Sets* is complete [3, pp. 26].

In view of [1, Theorem 6.3], completeness implies finitely completeness, the following is an immediate corollary to Theorem 3.8.

Corollary 3.1. The category $\mathcal{G}$-Sets* is finitely complete.

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