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On the direct sums of uniform modules and QF rings

Le Van An and **Ngo Sy Tung**

* *Department of Mathematics Vinh University*
Vinh city, Vietnam
e-mail: levanan_na@yahoo.com

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e-mail:

Abstract

In this paper, we give some results on direct sums of uniform modules. We also characterized of QF$-$rings by class of modules having finite composition length and characterized of QF$-$rings by class of semiartinian, CS$-$semisimple, SC rings.

1 Introduction

Throughout this paper, all rings are associative with identity and all modules are unital right modules. The socle, the injective hull and the endomorphism ring of M is denoted by $Soc(M)$, $E(M)$, and $End(M)$. If the composition length of a module M is finite, then we denote its length by $l(M)$.

Given two $R-$modules M and N, N is called $M-$projective if for every submodule X of M, any homomorphism $\varphi : N \longrightarrow M/X$ can be lifted to a homomorphism $\psi : N \longrightarrow M$. A module N is called projective if it is $M-$projective for every $R-$module M. On the other hand, N is called quasi - projective if N is $N-$projective. N is called $M-$injective if for every submodule A of M, any homomorphism $\alpha : A \longrightarrow N$ can be extended to a homomorphism $\beta : M \longrightarrow N$. A module N is called injective if it is $M-$ injective for every $R-$module M. On the other hand, N is called quasi$-$injective if N is

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$N-$injective. In particular, a ring R is called right (left) self$-$injective if R_R $({}_RR)$ is quasi$-$injective. For basic properties of injective (projective) modules we refer to [1], [2], [5], [7], [8], [9], [18], [21] and [24].

For a module M consider the following conditions:

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand.

(C_3) If A, B are direct summand of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

A module M is called CS (or extending module) if it satisfies the condition (C_1); a continuous module if it satisfies (C_1) and (C_2), and a quasi$-$continuous if it satisfies (C_1) and (C_3). A module M is called uniform extending if every uniform submodule of M is essential in a direct summand of M. We have the following implications:

Injective $\Rightarrow$ quasi$-$injective $\Rightarrow$ continuous $\Rightarrow$ quasi$-$continuous $\Rightarrow$ CS $\Rightarrow$ uniform extending.

We refer to [5] and [18] for background on CS and (quasi$-$)continuous modules.

A family of submodules of a module M, whose sum in M is direct, is called a local direct summand if every finite subsum is a direct summand of M. A decomposition $M = \oplus_{i\in I} M_i$ is said to be complement (uniform) direct summand if for every (uniform) direct summand A of M there exists a subset J of I such that $M = A \oplus (\oplus_{j\in J} M_j)$.

A module M is called uniserial if the set of all its submodules is linearly ordered by inclusion. If R_R $({}_RR)$ is uniserial, then we call R right (left) uniserial. We call a module M serial if it is a direct sum of uniserial modules. The ring R is called right (left) serial if R_R $({}_RR)$ is a serial module. For basic properties of uniserial (serial) modules and rings we refer to [1], [5], [7], [20], [21] and [24].

For any module M, the submodule $Z(M) =^{def} \{x \in M \mid xI = 0$ for some essential right ideal I of $R\}$, called singular submodule of M. If $Z(M) = M$ then M is called singular, while if $Z(A) = 0$ then A is called nonsingular. For basic properties of singular (nonsingular) modules we refer to [8], [9].

A module M is called (countably) $\Sigma-$uniform extending (CS, quasi$-$injective, injective) if $M^{(A)}$ (respectively, $M^{(\mathbb{N})}$) is uniform extending (CS, quasi$-$injective, injective) for any set A. Note that $\mathbb{N}$ denotes the set of all natural numbers. A ring R is right (left) (countably) $\Sigma-$uniform extending (CS, injective) if R_R $({}_RR)$ is (countably) $\Sigma-$uniform extending (CS, injective). If every right $R-$module is CS, then R is defined to be CS$-$semisimple. A ring R is called right (left) SC if every singular right (left) $R-$module is continuous.

In [10], Harada introduced and investigated the following condition for a given ring R.

$(*)^*$ Every non$-$cosmall right $R-$module contains a nonzero projective direct summand.

In [19], [20], a ring R is called a right co−H−ring if it satisfies $(*)^*$ and the ACC on right annihilator ideals. A ring R is right co−H if and only if R is a right Σ−CS ring.

A ring R is called right (left) semi−artinian if every nonzero right (left) R−module has a nonzero socle. A ring R is right (left) perfect if every right (left) modules has a projective cover. The ring R is right semiperfect if every finitely generated right (or left) modules has a projective cover. Every right (left) perfect rings is semiperfect rings and right (left) perfect rings is also left (right) semi - artinian rings. For basic properties of semi−artinian, perfect, semiperfect rings we refer to [1], [5], [18], [21] and [24].

A ring R is called QF (quasi−Frobenius) if R is right and left self−injective and Artinian. In [5, 18.1], R is a QF−ring iff R is right or left self−injective, right or left Artinian. QF−rings have been studied in [2], [3], [4], [5], [6], [7], [10], [12], [13], [15], [16], [19], [20] and [24].

In this paper, we give some results on direct sums of uniform modules. We also characterized of QF−rings by class of modules having finite composition length and characterized of QF−rings by class of semiartinian, CS−semisimple, SC rings.

2 Direct sums of uniform modules

Proposition 2.1. *(a) Let M be a module with $l(M) = 2$. Then M is CS.*

(b) Let $M = \oplus_{i=1}^{n} M_i$ be a direct sum of submodules of length 2 such that M_i is M_j−injective for any $i, j = 1, .., n$ and $i \neq j$. Then M is CS.

Proof. (a) $Case$ 1. If M is an indecomposable module of length 2, we will show that M is a uniform module. Suppose that M is not uniform. Consider two non−zero submodules A, B of M such that $A \cap B = 0$. Then, we have $0 \subset A \subset A \oplus B \subset M$ and $0 \neq A \neq A \oplus B \neq M$. Hence $l(M) > 2$, a contradiction. Since M is a uniform module, so that M is CS.

Case 2. If $M = A \oplus B$, with A, B are non−zero submodules of M, then A, B are simple modules (because $l(M) = 2$). Hence M is CS.

(b) By (a), M_i is CS for any $i = 1, 2, ..., n$. By [11, Theorem 8] we have (b). □

Theorem 2.2. Let U_1, U_2 be uniform modules such that $l(U_1) = l(U_2) < \infty$. Set $U = U_1 \oplus U_2$. Then U satisfies (C_3).

Proof. By [1], $End(U_1)$ and $End(U_2)$ are local rings. We show that U satisfies (C_3), i.e., for two direct summands S_1, S_2 of U with $S_1 \cap S_2 = 0$, $S_1 \oplus S_2$ is also a direct summand of U. Note that, since u−dim$(U) = 2$, the following case is trivial:

If one of the $S_i's$ has uniform dimension 2, the other is zero.

Hence we consider the case that both S_1, S_2 are uniform. Write $U = S_2 \oplus$

K. By Azumaya's Lemma (cf. [1, 12.6, 12.7]), either $S_2 \oplus K = S_2 \oplus U_1$ or $S_2 \oplus K = S_2 \oplus U_2$. Since U_1 and U_2 can interchange with each other, we need only to consider one of the two possibilities. Let us consider the case $U = S_2 \oplus K = S_2 \oplus U_1 = U_1 \oplus U_2$. Then, it follows $S_2 \cong U_2$. Write $U = S_1 \oplus H$. Then either $U = S_1 \oplus H = S_1 \oplus U_1$ or $S_1 \oplus H = S_1 \oplus U_2$.

If $U = S_1 \oplus H = S_1 \oplus U_1$, then by modularity, we get $S_1 \oplus S_2 = S_1 \oplus X$ where $X = (S_1 \oplus S_2) \cap U_1$. From here, we get $X \cong S_2 \cong U_2$. Since $l(U_1) = l(U_2) = l(X)$, we have $U_1 = X$, and hence $S_1 \oplus S_2 = S_1 \oplus U_1 = U$.

If $U = S_1 \oplus H = S_1 \oplus U_2$, then by modularity, we get $S_1 \oplus S_2 = S_1 \oplus V$ where $V = (S_1 \oplus S_2) \cap U_2$. From here, we get $V \cong S_2 \cong U_2$. Since $l(U_2) = l(V)$, we have $U_2 = V$, and hence $S_1 \oplus S_2 = S_1 \oplus U_2 = U$.

Thus U satisfies (C_3), as desired. □

Remark . Theorem 2.2 is not true, in general, if U_1, U_2 are uniform modules such that $l(U_1) \neq l(U_2) < \infty$, then $U = U_1 \oplus U_2$ does not satisfy (C_3). For example, consider the special case of $\mathbb{Z}-$modules. Take $U_1 = \mathbb{Z}/2\mathbb{Z}$, $U_2 = \mathbb{Z}/4\mathbb{Z}$. Then $U = U_1 \oplus U_2$ does not satisfy (C_3).

Proof. We have $U = \{(x, y) \mid x \in U_1, y \in U_2\}$. Set $S_1 =< (\bar{1}, \bar{2}) >= \{(\bar{0}, \bar{0}); (\bar{1}, \bar{2})\}$, $S_2 = U_1$. Then $U = S_1 \oplus U_2 = S_2 \oplus U_2$, $S_1 \cap S_2 = 0$. Since $S_1 \oplus S_2 = \{(\bar{0}, \bar{0}); (\bar{1}, \bar{2}); (\bar{1}, \bar{0}); (\bar{0}, \bar{2})\}$, thus $S_1 \oplus S_2$ is not direct summand of U. Hence U does not satisfy (C_3). □

Corollary 2.3. *([6, Proposition 2.2.1]) Let U be a uniform module with finite composition length. Then U is quasi−injective if and only if $U \oplus U$ is CS.*

Proof. If $M = U \oplus U$ is CS, then combining with Theorem 2.2 we can see that M is quasi−continuous. Hence U is a quasi−injective module. □

Corollary 2.4. *Let U_1, U_2 be uniform modules such that $l(U_1) = l(U_2) < \infty$. Set $U = U_1 \oplus U_2$. If U is CS, then U_1 is U_2-injective and U_2 is U_1-injective.*

Proof. If U is CS, then combining with Theorem 2.2 we can see that U is quasi−continuous. The mutual injectivity of the U_1, U_2 follows from [18, 2.10]. □

Corollary 2.5. *Let R be a ring with $R = e_1R \oplus ... \oplus e_nR$ where each e_iR is a uniform right ideal and $\{e_i\}_1^n$ is a system of idempotents. Moreover assume that $l(e_1R) = l(e_2R) = ... = l(e_nR) < \infty$. Then R is right self−injective if and only if $(R \oplus R)_R$ is CS.*

Proof. Combining Corollary 2.3 and Corollary 2.4. □

Corollary 2.6. *Let U_1, U_2 be uniform modules such that $l(U_1) = l(U_2) < \infty$. Set $U = U_1 \oplus U_2$. The following assertions are equivalent:*

(i) U satisfies (C_2);

(ii) If $X \subseteq U$ and $X \cong U_k$ (with $k = 1$ or $k = 2$), then $X \subseteq^{\oplus} M$.

Proof. The implication $(i) \Longrightarrow (ii)$ is clear.

$(ii) \Longrightarrow (i)$. We show that U satisfies (C_2), i.e., for two submodules X, Y of U, with $X \cong Y$ and $Y \subseteq^{\oplus} U$, X is also a direct summand of U. Note that, since $u-\dim(U) = 2$, we have $u-\dim(Y) = 0, 1, 2$, the following case, $u-\dim(Y) = 0$ is trival.

Case 1. If $u-\dim(Y) = 1$, then $U = Y \oplus U_1$ or $U = Y \oplus U_2$. Since U_1 and U_2 can interchange with each other, we need only consider one of the two possibilities. Let us consider the case $U = Y \oplus U_1 = U_1 \oplus U_2$, then it follows $X \cong Y \cong U/U_1 \cong U_2$, by hypothesis (ii), we have $X \subseteq^{\oplus} U$, as required.

Case 2. If $u-\dim(Y) = 2$, then $Y = U$, hence $X \cong U$. Let φ be an isomorphism $U \longrightarrow X$. Setting $X_k = \varphi(U_k)$, $k \in \{1, 2\}$, we have $X_k \cong U_k$. Note that $X = \varphi(U) = \varphi(U_1 \oplus U_2) = \varphi(U_1) \oplus \varphi(U_j) = X_1 \oplus X_2$. By the hypothesis (ii), $X_i, X_j \subseteq^{\oplus} U$. By Theorem 2.2, we can see that $X = U$, proving (ii). □

Proposition 2.7. *Let $M = \oplus_{i \in I} M_i$, with all M_i uniform such that $l(M_i) \leqslant 2, \forall i \in I$. The following assertions are equivalent:*

(i) M is a CS module;

(ii) $M_i \oplus M_j$ is CS for every $i, j \in I, i \neq j$ and $l(M_i) = l(M_j) = 2$.

Proof. The implication $(i) \Longrightarrow (ii)$ is clear.

$(ii) \Longrightarrow (i)$. We will show that M is CS. Suppose that $A = M_i \oplus M_j$ is CS, where M_i, M_j are uniform modules such that $l(M_i) = l(M_j) = 2$. By Theorem 2.2, A is the quasi$-$continuous modules. Hence M_i, M_j are relatively injective modules for every $i, j \in I, i \neq j$ and $l(M_i) = l(M_j) = 2$. Since $M = X \oplus Y$, where X is semisimple and Y is a direct sum of relatively injective submodules of length 2. By [5, Lemma 8.14], M is extending. □

Theorem 2.8. *Let $M = \oplus_{i=1}^{n} M_i$ be a decomposition such that: (a) all M_i are uniform; (b) this decomposition of M is complement uniform direct summands; and (c) all $i, j \in \{1, 2, ..., n\}, i \neq j$, M_i can not be properly embedded in M_j. Then the following statements are equivalent:*

(i) M is CS module;

(ii) $M_i \oplus M_j$ is CS, $\forall 1 \leqslant i < j \leqslant n$.

Proof. The implication $(i) \Longrightarrow (ii)$ is clear.

$(ii) \Longrightarrow (i)$. We will show that M is CS. Suppose that $A = M_i \oplus M_j$ is CS, where M_i, M_j are uniform modules such that M_i can not be properly embedded in M_j. Let K be a non$-$zero submodule of M_i and let $f : K \longrightarrow M_j$ be a homomorphism. Then K is a uniform module. Set $U = \{x - f(x) \mid x \in K\} \subseteq M_i \oplus M_j$. Then $U \cong K$. Take $y \in U \cap M_j$, there exists $x \in K$ such that $y = x - f(x)$. Since $x = y + f(x) \in M_j \cap K = 0$, we have $y = 0$. Hence $U \cap M_j = 0$. Since A is a CS module, there exists a direct summand U' of A such that $U \subseteq^{e} U'$. By [22, Lemma 1], A has a decomposition that uniform exchange, we have $A = U' \oplus M_i$ or $A = U' \oplus M_j$.

Case 1. $A = U' \oplus M_i$. Let $\pi_i : U' \oplus M_i \longrightarrow M_i$ be the canonical projection and let $\varphi = \pi_i \mid_{M_j}$. Since $U \subseteq^e U'$, so U' is a uniform closed submodule of A and $U' \cap M_j = 0$. Hence φ is the monomorphism and it implies that φ is isomorphism, i.e., $A = U' \oplus M_j$. Let $\alpha = \pi_j \mid_{M_i}$, where $\pi_j : U' \oplus M_i \longrightarrow M_j$ is the canonical projection. Therefore, for every $x \in K$, $x = f(x) + (x - f(x))$, for some $f(x) \in M_j$ and $x - f(x) \in U'$. It follows that $\alpha(x) = \pi_j \mid_{M_i} (x) = \pi_j \mid_{M_i} (f(x) + (x - f(x))) = \pi_j \mid_{M_i} (f(x)) + \pi_j \mid_{M_i} (x - f(x)) = f(x)$, i.e., f can be extended to a homorphism $\alpha : M_i \longrightarrow M_j$. Hence M_j is M_i-injective.

Case 2. $A = U' \oplus M_j$. Let $\pi_j : U' \oplus M_j \longrightarrow M_j$ be the canonical projection and let $\beta = \pi_j \mid_{M_i}$. Therefore, for every $x \in K$, $x = f(x) + (x - f(x))$, for some $f(x) \in M_j$ and $x - f(x) \in U'$. It would imply that $\beta(x) = \beta[(x - f(x)) + f(x)] = f(x)$, i.e., f can be extended to a homorphism $\beta : M_i \longrightarrow M_j$. Hence M_j is M_i-injective.

Thus $M = \oplus_{i=1}^{n} M_i$ is a finite direct summand of relatively injective modules M_i. By [11, Theorem 8], M is CS. □

3 QF−rings

Theorem 3.1. *Let R be a right quasi−continuous, right semi−artinian ring such that $(R \oplus R)_R$ is extending and R satisfies the ACC on right annihilator ideals, then R is the QF−ring.*

Proof. We show that R is a right $\Sigma-$extending ring. By [4, Theorem 3.2], R has finite right uniform dimension. Then by [21, 5.1, page 189], R is semiperfect. We have

$$R = e_1 R \oplus ... \oplus e_n R,$$

where $\{e_i\}_{i=1}^{n}$ is a set of mutually orthogonal primitive idempotents of R with all $e_i R$ are uniform by R_R is extending. Let M be a right local module. By [7, 18.23.4], there exists $i \in \{1, .., n\}$ such that $M \cong e_i R / X$. If $X = 0$, then M is a projective module. If $X \neq 0$, then by $e_i R$ is uniform, X is an essential submodule of $e_i R$. Hence M is a singular module.

Let U be a two−generated module, i.e, $U = u_1 R + u_2 R$ for some $u_1, u_2 \in U$, then there exists an epimorphism $\varphi : (R \oplus R)_R \longrightarrow U$. Let $K = ker\varphi$. There exist two submodules P_1, P_2 of $(R \oplus R)_R$ such that $R \oplus R = P_1 \oplus P_2$ and K is an essential submodule of P_1. Now

$$U = \varphi(R \oplus R) = \varphi(P_1) \oplus \varphi(P_2),$$

where $\varphi(P_1) \cong P_1/K$, so that $\varphi(P_1)$ is singular; and $\varphi(P_2) \cong P_2$, so that $\varphi(P_2)$ is projective. If U is uniform module and $U \neq Z(U)$, then $U = U_1 \oplus U_2$ with $U_1 \neq 0$ and projective, U_2 singular. Hence $U = U_1$, i.e., U is a projective module. If $U = Z(U)$, then U is the singular module.

Let V be a uniform module. We prove that every submodule N of V, then or $N \subseteq Z(V)$ or $Z(V) \subseteq N$. If N is not a submodule of $Z(V)$, then there exists $x \in N \backslash Z(V)$ such that xR is not singular. If $Z(V)$ is the submodule of xR, then $Z(V) \subseteq N$, as required. If $Z(V)$ is not a submodule of xR, then there exists $y \in Z(V)$ such that $yR \not\subseteq xR$. Note that $xR + yR$ is a uniform module and $Z(xR + yR) \neq xR + yR$. Therefore $xR + yR$ is projective. We imply $xR + yR$ is a local module. Set $I = xR \cap yR \neq 0$. We consider two modules xR/I and yR/I with two maximal submodules are X/I and Y/I, respectively. We have $(xR+yR)/(xR+Y) = (xR+Y+yR)/(xR+Y) \cong yR/(yR\cap(xR+Y)) = yR/Y$, and $(xR + yR)/(yR + X) \cong xR/X$. Note that yR/Y and xR/X are simple modules, hence $xR+Y$ and $yR+X$ are maximal submodules of $(xR+yR)$. By modularity, $(xR+Y)\cap(yR+X) = X+[(xR+Y)\cap yR] = X+Y+I = X+Y$, thus $xR + Y \neq yR + X$, a contradiction (because $xR + yR$ is a local module). Thus $Z(V) \subseteq N$.

We aim to show next that $V/Z(V)$ is a uniserial module. We consider two submodules A, B of V such that $Z(V) \subseteq A$ and $Z(V) \subseteq B$. Assume that $A \not\subseteq B$ and $B \not\subseteq A$, then there exist $\alpha \in A\backslash B$ and $\beta \in B\backslash A$ with $\alpha, \beta \notin Z(V)$. Therefore $\alpha R + \beta R$ is a local module, a contradiction. Thus $A \subseteq B$ or $B \subseteq A$, i.e., $V/Z(V)$ is the uniserial module.

Set $E_i = E(e_iR)$, we show that E_i is a projective module. Set $Z = Z(E_i)$. By E_i is the uniform module, E_i/Z is also a uniserial module. By R is right semi - artinian, there is an infinite strictly ascending chain

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq ... \subseteq S_m \subseteq ...,$$

where $S_1/Z = Soc(E_i/Z)$, $S_2/S_1 = Soc(E_i/S_1)$, ... , $S_{m+1}/S_m = Soc(E_i/S_m)$, ... By E_i/Z is the uniserial module, S_m is the unique maximal submodule of S_{m+1} and Z is also the unique maximal submodule of S_1. Hence S_m is the local module for all m. But S_m is not the singular module, thus S_m is projective. We prove that, there is k such that $S_k = E_i$. Suppose that there exist $p < q$ such that $S_p \cong S_q$. Note that we have $S_p \subseteq S_q$. Let $f : S_q \longrightarrow S_p$ be an isomorphism and set $Z^* = f^{-1}(Z)$, then $Z^* = Z$ (by Z is singular). Now

$$S_q/Z = S_q/Z^* \cong S_p/Z,$$

thus $l(S_q/Z) = l(S_p/Z)$, a contradiction. Hence $S_p \ncong S_q$ for all $p \neq q$. By S_m is the projective, local module, there exists $j \in \{1, ..., n\}$ such that $S_m \cong e_jR$ (see [1, 27.11]). By the set $\{1, ..., n\}$ is finite, there is k such that $S_k = S_{k+1} = ...$ Thus $S_k = E_i$, i.e., E_i is a projective module. Therefore $E(R_R)$ is projective. By [10 , Theorem 3.6], R satisfies $(*)^*$. Thus R is the right co$-$H$-$ring, i.e., R is the $\Sigma-$extending ring. By [4 , Corollary 3.6], R is the QF$-$ring. $\square$

Theorem 3.2. *Let R be a right continuous, right semi$-$artinian, right countably $\Sigma-$uniform extending ring then R is the QF$-$ring.*

Proof. By [4, Theorem 3.2], R has finite right uniform dimension. We have

$$R_R = R_1 \oplus ... \oplus R_n,$$

where each R_i is the uniform module. Since R is right countably $\Sigma-$uniform extending, thus R is right $\Sigma-$injective (see [13, Proposition 2.5]). Hence R is the QF$-$ring. □

Theorem 3.3. *Let R be a left CS, right and left semi$-$artinian ring such that $(R \oplus R)_R$ is extending and R_R satisfies (C_3), then R is the QF$-$ring.*

Proof. By [4, Corollary 3.3], R is right perfect. By $(R \oplus R)_R$ is the extending module, we have

$$R = e_1R \oplus ... \oplus e_nR,$$

where $\{e_i\}_{i=1}^n$ is a set of mutually orthogonal primitive idempotents of R with all e_iR uniform and $End(e_iR)$ local. Let A be an arbitrary set, then $R^{(A)} = \bigoplus_{i \in I} M_i$ with all M_i are uniform and $End(M_i)$ local. In particular, where each direct summand M_i, there exists $k \in \{1, ..., n\}$ such that $M_i \cong e_kR$. Since $(R \oplus R)_R$ is CS, $M_i \oplus M_j$ is CS for all $i, j \in I$ and $i \neq j$. By [17, Lemma 11], $R^{(A)}$ is uniform extending.

We show that $R^{(A)}$ is a CS module. Let A be a closed submodule of $R^{(A)}$, set $\Gamma = \{\oplus_{\alpha \in \wedge} U_\alpha \mid U_\alpha \subset A$, all U_α is uniform and $\oplus_{\alpha \in \bigwedge} U_\alpha$ is locally direct summand of $R^A\}$. Γ is non$-$empty set by [17, Proposition 6]. We can find a maximal member $\oplus_{j \in J} U_j$ in Γ by Zorn's lemma. Since R is right perfect and $R^{(A)}$ is a projective right $R-$module, the decomposition $R^{(A)} = \bigoplus_{i \in I} M_i$ is complement direct summand. Hence by [18, Theorem 2.25], every local direct summand of $R^{(A)}$ is a direct summand. It follows that $\oplus_{j \in J} U_j$ is a direct summand of $R^{(A)}$. Set $R^{(A)} = \oplus_{j \in J} U_j \oplus X$. By modularity, $A = \oplus_{j \in J} U_j \oplus (X \cap A)$. By $X \cap A$ is closed in A, and A is also closed in $R^{(A)}$, so that $X \cap A$ is closed in $R^{(A)}$. Therefore $X \cap A = 0$ by maximality of $\oplus_{j \in J} U_j$. Thus $R^{(A)} = A \oplus X$, i.e., $R^{(A)}$ is a CS module. Note that R is a right quasi$-$continuous ring. Hence R is right $\Sigma-$CS. By [4, Corollary 3.6], R is the QF$-$ring. □

Proposition 3.4. *Let R be a ring with $R = e_1R \oplus ... \oplus e_nR$ where each e_iR is an uniform right ideal and $\{e_i\}_1^n$ is a system of idempotents. Moreover assume that $l(e_1R) = l(e_2R) = ... = l(e_nR) < \infty$. The following assertions are equivalent:*

(a) $(R \oplus R)_R$ is CS;

(b) R is right self$-$injective;

(c) R is left self$-$injective;

(d) R is the QF$-$ring.

Proof. $(a) \Longleftrightarrow (b)$. By Corollary 2.5.

$(b) \Longleftrightarrow (d)$ and $(c) \Longleftrightarrow (d)$. Note that $l(R_R) = l(e_1R) + ... + l(e_nR) < \infty$, thus R is right artinian. Therefore $(b) \Longleftrightarrow (d)$ and $(c) \Longleftrightarrow (d)$ by [5, 18.1]. □

Proposition 3.5. *Let R be a right quasi$-$continuous, CS$-$semisimple ring; then R is a QF$-$ring.*

Proof. By [5, 13.5], R is right and left artinian, $R_R = R_1 \oplus ... \oplus R_n$ where each R_i is uniform module such that $l(R_i) < \infty$. By R_R is quasi$-$continuous, R_i is R_j-injective for any $i \neq j$. In particular, $R_i \oplus R_i$ satisfies (C_3), by Corollary 2.3. Note that $R_i \oplus R_i$ is a CS module. Thus $R_i \oplus R_i$ is quasi$-$continuous, i.e., R_i is quasi$-$injective. Since R_i is an injective module, so R is right self$-$injective. This shows that R is a QF$-$ring. □

Corollary 3.6. *([12, Corollary 3.2]) Let R be a right quasi$-$continuous, right SC ring. If $R_R^{(\mathbb{N})}$ is CS, then R is a QF$-$ring.*

Proof. By [12, Theorem 3.1], R is CS$-$semisimple. By Proposition 3.5, R is a QF$-$ring. □

Proposition 3.7. *Let R be a ring such that R_R does not contain a direct summand semisimple module. If R is CS$-$semisimple ring then R is a QF$-$ring.*

Proof. By [5, 13.5], R is right and left artinian, $R_R = R_1 \oplus ... \oplus R_n$ where each R_i is uniform module such that $l(R_i) \leqslant 2$. By R_R does not contain a direct summand semisimple module, we have $l(R_i) = 2$. By Theorem 2.2, $R_i \oplus R_j$ satisfies (C_3). Note that $R_i \oplus R_j$ is a CS module. Thus $R_i \oplus R_j$ is quasi$-$continuous, i.e., R_i is R_j-injective. Since R_i is injective, thus R is right self$-$injective. This shows that R is a QF$-$ring. □

Theorem 3.8. *Let R be a right quasi$-$continuous ring. If R has finite right uniform dimension and the direct sum of any two uniform right $R-$module is CS, then R is a QF$-$ring.*

Proof. By [6, Theorem 3.1.1], R is a right artinian ring, and uniform right $R-$modules have length at most two. By R_R is a CS module with finite uniform dimension, we have $R_R = R_1 \oplus ... \oplus R_n$ where each R_i is uniform module such that $l(R_i) \leqslant 2$. Since R_R is quasi$-$continuous, R_i is R_j-injective for any $i \neq j$. In particular, $R_i \oplus R_i$ satisfies (C_3) (by Theorem 2.2). Note that $R_i \oplus R_i$ is a CS module. Thus $R_i \oplus R_i$ is quasi$-$continuous, i.e., R_i is quasi$-$injective. Since R_i is an injective module, R is right self$-$injective. This shows that R is a QF$-$ring. □

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