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Endomorphism Ring of Quasi-rp-injective and Quasi-lp-injective modules

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Abstract

Let R be a ring. A right R-module N is called an M-p-injective module if any homomorphism from an M-cyclic submodule of M to N can be extended to an endomorphism of M. Generalizing this notion, we investigated the class of M-rp-injective modules and M-lp-injective modules, and proved that for a finitely generated Kasch module M, if M is quasi-rp-injective, then there is a bijection between the class of maximal submodules of M and the class of minimal left right ideals of its endomorphism ring S. In this paper, we give some characterizations and properties of the structure of endomorphisms ring of M-rp-injective modules and M-lp-injective modules and the relationships between them.

Lemma 1. *Let M be a right R- module and $S := End(M_R)$. Then M is a*

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quasi-rp-injective module if and only if for any non-zero element $s \in S$, there exists $t \in S$ with $ts \neq 0$ satisfying one of the following equivalent conditions:
(1) $l_S Ker(ts) = Sts$;
(2) For any $v \in S$, *if* $Ker(ts) \subset Ker(v)$, *then* $Sv \subseteq Sts$;
(3) For any $u \in S$, $l_S(Im(u) \cap Ker(ts)) = l_S(Im(u)) + Sts$.

Proof. Note that the condition (1) is a characterization of quasi-rp-injective modules (see [23]), and the proof of (1) $\Leftrightarrow$ (2) is routine.

We now prove (2) $\Rightarrow$ (3). It is easy to see that $l_S(Im(u)) \subset l_S(Im(u) \cap Ker(ts))$. Take any $fts \in Sts$ and $m \in Im(u) \cap Ker(ts)$. Then $fts(m) = 0$. This shows that $fts \in l_S(Im(u) \cap Ker(ts))$, and hence $Sts \subset l_S(Im(u) \cap Ker(ts))$. It follows that $l_S(Im(u)) + Sts \subset l_S(Im(u) \cap Ker(ts))$. Conversely, let $v \in l_S(Im(u) \cap Ker(ts))$. Then $v(Im(u) \cap Ker(ts)) = 0$. It follows that $Ker(tsu) \subset Ker(vu)$. This implies that $Svu \subset Stsu$. Then there is an $f \in S$ such that $vu = ftsu$. Hence $(v - fts)u = 0$, and therefore $v - fts \in l_S(Im(u))$ or equivalently, $v \in l_S(Im(u)) + Sts$. This shows that $l_S(Im(u) \cap Ker(ts)) \subset l_S(ImU)) + Sts$, proving our assertion.

(3) $\Rightarrow$ (1) Taking $u = 1_M$, we get the result immediately. □

By a similar argument, we can prove the following lemma for quasi-*lp*-modules:

Lemma 2. *Let M be a right R- module and $S := End(M_R)$. Then M is a quasi-lp-injective module if and only if for any non-zero element $s \in S$, there exists $t \in S$ with $st \neq 0$ satisfying one of the following equivalent conditions:*
(1) $l_S Ker(st) = Sst$;
(2) For any $v \in S$, *if* $Ker(st) \subset Ker(v)$, *then* $Sv \subseteq Sst$;
(3) For any $u \in S$, $l_S(Im(u) \cap Ker(st)) = l_S(Im(u)) + Sst$.

Recall that a right R- module is finitely cogenerated if and only if for any family $\{A_i | i \in I\}$ of submodules with $\bigcap_{i \in I} A_i = 0$, there is a finite subset I_0 of I such that $\bigcap_{i \in I_0} A_i = 0$.

Lemma 3. *Let M be a right R-module, $S = End(M_R)$ and $\triangle$ the set of all $s \in S$ such that $Ker(s)$ is essential in M. If $Soc(M)$ is essential in M, then $\triangle = l_S(Soc(M))$.*
Proof. Take any $s \in \triangle$, we have $0 \neq s$ and $Ker(s)$ is essential in M. It follows that $Ker(s) \supset Soc(M)$, and hence $l_S(Ker(s)) \subset l_S(Soc(M))$. Since $Ss \subset l_S(Ker(s))$, we have $s \in l_S(Soc(M))$. Conversely, if $s \in l_S(Soc(M))$, then $Soc(M) \subset Ker(s)$. Because $Soc(M)$ is essential in M, and then $Ker(s)$

is essential in M, showing that $s \in \triangle$. $\square$

Recall that a right R-module M is called a semiartinian module if and only if for any proper submodule U of M, we have $Soc(M/U) \neq 0$. If M is finitely cogenerated or semiartinian, then $Soc(M)$ is an essential submodule of M.

In special case, Let M be a quasi-rp-injective module. If M is finitely cogenerated or semiartinian then we also have $\triangle = l_S(Soc(M))$.

Let K be a subset of S. We denote $Ker(K) = \bigcap_{\varphi \in K} Ker\varphi$.

Theorem 4. *Let M be a quasi-rp-injective module and $S = End(M_R)$. If M is a finitely generated Kasch module, then the following properties hold.*
(1) $Soc({}_SS)$ is essential in ${}_SS$;
(2) $Rad(M) = Ker(Soc({}_SS))$;
(3) If ${}_SS$ is finitely cogenerated as a left S-module, then $l_S(Rad(M))$ is essential in ${}_SS$.

Proof. (1) Since M is a quasi-rp-injective module, for any nonzero element $s \in S$, there is an element $t \in S$ satisfying $ts \neq 0$ and $l_S(Ker(ts)) = Sts$. Since M is a finitely generated module, there exists a maximal submodule T of M such that $Ker(ts) \subset T$. It follows that $l_S(T) \subset l_S(Ker(ts)) = Sts \subset Ss$. By [[23],Theorem 2.4(2b)], $l_S(T)$ is a minimal left ideal of S and hence $Soc({}_SS) \cap Ss \neq 0$. This shows that $Soc({}_SS)$ is essential in S.

(2) Take any maximal submodule T of M. By [[23], Theorem 2.4], $l_S(T)$ is a minimal left ideal of S contained in $Soc({}_SS)$. It follows that $Ker(Soc({}_SS)) \subset T$ for any maximal submodule T of M, showing that $Ker(Soc({}_SS)) \subset Rad(M)$. Conversely, take any minimal left ideal I of S. By [[23], Theorem 2.4], $Ker(I)$ is a maximal submodule of M which implies that $Rad(M) \subset Ker(Soc({}_SS))$, and hence $Rad(M) = Ker(Soc({}_SS))$.

(3) We have from (2) that $l_S(Rad(M)) = l_SKer(Soc({}_SS))$. In other hand, we always have $Soc({}_SS) \subset l_SKer({}_SS))$. Since ${}_SS$ is finitely cogenerated, by [[13],Theorem 9.4.3], $Soc({}_SS)$ is essential in ${}_SS$. It follows that $l_S(Rad(M))$ is essential in ${}_SS$. $\square$

The following corollary is routine:

Corollary 5. *Let M be a quasi-rp-injective module, $S = End(M_R)$. If M is a finitely generated Kasch module and ${}_SS$ is semiartinian as a left S-module, then $l_S(Rad(M))$ is essential in ${}_SS$.*

An element u of S is called a *uniform element* if $u(M)$ is a uniform submodule of M.

Let R be a ring. An element u of R is called a *right uniform element*, if $u \neq 0$ and uR is a uniform right ideal of R.

Lemma 6. *Let M be a right R-module. If u is a uniform element of S, then the set:*

$$A_u = \{s \in S | Ker(s) \cap Im(u) \neq 0\}$$

is a left ideal containing $l_S(Im(u))$.

Proof. Clearly $A_u \neq \emptyset$.

Taking any $s_1, s_2 \in A_u$, we have $Ker(s_1) \cap Im(u) \neq 0, Ker(s_2) \cap Im(u) \neq 0$. Because u is a uniform element of S, $Ker(s_1) \cap Ker(s_2) \cap Im(u) \neq 0$. Hence there exists $m \in M$ such that $s_1(u(m)) = s_2(u(m)) = 0$ with $u(m) \neq 0$. Therefore $Ker(s_1 - s_2) \cap Im(u) \neq 0$, and hence $s_1 - s_2 \in A_u$. Since $Ker(s) \subset Ker(\alpha s)$ for any $\alpha, s \in S$, we have $\alpha s \in A_u$ for all $\alpha \in S, s \in A_u$. Clearly, $l_S(Im(u)) \subset A_u$. This shows that A_u is a left ideal of S containing $l_S(Im(u))$. □

The following lemma is helpful in proving the next theorem.

Lemma 7. *Let M be a right R-module and u a uniform element of S. If $s_0 \in S$ such that $Ker(s_0) \cap Im(u) = 0$, then the set:*

$$B_u = \{t \in S | Ker(ts_0) \cap Im(u) \neq 0\}$$

is a left ideal of S.

Proof. Clearly, $B_u \neq \emptyset$.

Take any $t_1, t_2 \in B_u$. Then $Ker(t_1 s_0) \cap Im(u) \neq 0, Ker(t_2 s_0) \cap Im(u) \neq 0$. Since u is a uniform element of S, $Ker(t_1 s_0) \cap Ker(t_2 s_0) \cap Im(u) \neq 0$. It follows that there is non-zero element $x \in M$ such that $t_1 s_0 u(x) = t_2 s_0 u(x) = 0$. Therefore $Ker((t_1 - t_2)s_0) \cap Im(u) \neq 0$, and hence $t_1 - t_2 \in B_u$.

We now take any $\alpha \in S$ and $t \in B_u$. Then $Ker(ts_0) \cap Im(u) \neq 0$. Since $Ker(ts_0) \subset Ker(\alpha t s_0)$ and $Ker(\alpha t s_0) \cap Im(u) \neq 0$, we can see that $\alpha t \in B_u$. This shows that B_u is a left ideal of S. □

Lemma 8. *Let M be a quasi-lp-injective right R-module. If u is a uniform element of S, then the set:*

$$A_u = \{s \in S | Ker(s) \cap Im(u) \neq 0\}$$

is a maximal left ideal containing $l_S(Im(u))$.

Proof. Applying Lemma 3, we see that A_u is a left ideal of S containing $l_S(Im(u))$. The remainder of the proof is to show the maximality of A_u.

Take any $s \notin A_u$. Then $Ker(s) \cap Im(u) = 0$, and hence $su \neq 0$. Since M is quasi-lp-injective, there exists $t \in S$ such that $sut \neq 0$ and $l_S Ker(sut) = Ssut$. If $m \in Ker(sut)$, then $sut(m) = 0$. It follows from $s \notin A_u$ that $m \in Ker(ut)$. This shows that $Ker(sut) = Ker(ut)$, and hence $ut \in l_S(Ker(sut)) = Ssut$. Thus, there exists $f \in S$ such that $ut = fsut$ which implies that $(1 - fs)ut = 0$. Then $1 - fs \in l_S(ut)$, and hence the element 1 can be written in the form $1 = fs + h$ for some $h \in l_S(ut)$. It follows that $S = Sfs + Sh$. We will prove that $Sh \subset A_u$. Let $gh \in Sh$. Then we have $ghut = 0$. This shows that $0 \neq Im(ut) \subset Ker(gh)$. Since $Im(ut) \subset Im(u)$, we get $Ker(gh) \cap Im(u) \neq 0$, showing that $Sh \subset A_u$. Since $Sfs \subset Ss$, we have $S = A_u + Ss$, proving the maximality of A_u. □

Corollary 9 [[25], Lemma 3.10]. *Let R be a right self-rp-injective ring. If $u \in R$ is a right uniform element, then the set:*

$$M_u = \{x \in R | uR \cap r_R(x) \neq 0\}$$

is a maximal left ideal containing $l_R(u)$. □

Corollary 10 follows directly from Theorem 8.

Corollary 10. *Let M be a quasi-lp-injective right R-module. If S is uniform, then S is local.*

Lemma 11. *Let M be a quasi-lp-injective module and S its endomorphisms ring. We assume that $0 \neq \varphi \in S$ such that $\varphi(M)$ is a simple submodule of M. For any $0 \neq \psi \in S$, if $\varphi(M) \cong \psi(M)$, then $S\varphi \cong S\psi$.*

Proof. Since M is a quasi-lp-injective module and $\psi \neq 0$, there exists $\psi' \in S$ such that $\psi\psi' \neq 0$ and any homomorphism from $\psi\psi'(M) \longrightarrow M$ can be extended to an endomorphism of M. Let $\sigma : \psi(M) = \psi\psi'(M) \longrightarrow \varphi(M)$ be an isomorphism. Then σ can be extended to an endomorphism $\overline{\sigma}$ of M. Let $\iota_1 : \psi(M) \longrightarrow M$ and $\iota_2 : \varphi(M) \longrightarrow M$ be inclusions. We have $\overline{\sigma}\iota_1 = \iota_2\sigma$.

We note that $\overline{\sigma}|_{\psi(M)} = \sigma$ and $\overline{\sigma}\psi(M) = \sigma\psi(M) = \varphi(M)$. $(*)$

We now define $\gamma : S\varphi \longrightarrow S\psi$ with $\gamma(s\varphi) = s\overline{\sigma}\psi$.

If $s\varphi = s'\varphi$, then $(s - s')\varphi = 0$. It follows that $Im(\varphi) \subset Ker(s - s')$. By $(*)$, we have $Im(\overline{\sigma}\psi) = Im(\varphi) \subset Ker(s - s')$. Then $(s - s')\overline{\sigma}\psi = 0$, and hence $s\overline{\sigma}\psi = s'\overline{\sigma}\psi$. Thus $\gamma(s\varphi) = \gamma(s'\varphi)$, showing that γ is well-defined.

We have $\gamma(s\varphi + s'\varphi) = \gamma((s+s')\varphi) = ((s+s')\overline{\sigma}\psi) = ((s\overline{\sigma}\psi) + s'\overline{\sigma}\psi) = \gamma(s\varphi) + \gamma(s'\varphi)$. In other hand, we have $\gamma(ts\varphi) = \gamma((ts)\varphi) = ts\overline{\sigma}\psi = t(s\overline{\sigma}\psi) = t(\gamma(s\varphi))$. This indicates that γ is an S-homomorphism.

Suppose that $\gamma(s\varphi) = \gamma(s'\varphi)$. Then $s\overline{\sigma}\psi = s'\overline{\sigma}\psi$, and hence we have $Im(\overline{\sigma}\psi) \subset Ker(s-s')$. By $(*)$, we have $Im(\varphi) \subset Ker(s-s')$. Thus $s\varphi = s'\varphi$, proving that γ is one-to-one.

To prove γ is onto, we take any $s\psi \in S\psi$. Let $\sigma^{-1} : \varphi(M) \longrightarrow \psi(M)$ be the inverse of σ. As before, σ^{-1} can be extended to endomorphism $\overline{\sigma^{-1}}$ of M such that $\overline{\sigma^{-1}}\iota_2 = \iota_1\sigma^{-1}$. Then for any $m \in M$ we have:

$$s\overline{\sigma^{-1}}\overline{\sigma}\psi(m) = s\overline{\sigma^{-1}}(\sigma\psi(m)) = s\overline{\sigma^{-1}}(\iota_2\sigma\psi(m)) = s(\overline{\sigma^{-1}}\iota_2)(\sigma\psi(m))$$
$$= s(\iota_1\sigma^{-1})(\sigma\psi(m)) = s\iota_1(\sigma^{-1}\sigma)(\psi(m)) = s\iota_1\psi(m) = s\psi(m).$$

This means that there exists $s\overline{\sigma^{-1}} \in S$ such that $\gamma(s\overline{\sigma^{-1}}\varphi) = s\psi$, showing that γ is onto, and the proof of our lemma is completed. □

Remark 12. Let M be a right R-module, $S = End(M)$ and $\triangle = \{s \in S | Ker(s) \subset^*_> M\}$. Suppose that M is a self-generator. Then $r_S(s) \subset^*_> S$ if and only if $s \in \triangle$.

Proof. Let $s \in \triangle$. Take any $0 \neq t \in S$. Since $Ker(s) \subset^*_> M$, $t(M) \cap Ker(s) \neq 0$. It follows that $Ker(st) \neq 0$. Because M is a self-generator, there is $0 \neq k \in S$ such that $0 \neq k(M) \subset_> Ker(st)$. Therefore, we have $stk = 0$. It means that $tk \in r_S(s)$. Since $0 \neq k(M) \subset_> Ker(st) \subset_> t(M) \cap Ker(s), tk \neq 0$, proving that $r_S(s) \subset^*_> S$.

Conversely, take any $0 \neq m \in M$. Since M is self-generator, there is a nonzero element $t \in S$ such that $t(M) \subset mR$. Since $r_S(s) \subset^*_> S$, there exists a nonzero element $k \in S$ such that $0 \neq tk \in r_S(s)$. It follows that $stk = 0$, i.e. $0 \neq tk(M) \subset_> Ker(s)$. Thus we have $Ker(s) \cap mR \neq 0$. This shows that $Ker(s) \subset^*_> M$, and hence $s \in \triangle$. □

The following theorem gives a property of the endomorphisms ring of quasi-lp-injective modules.

Theorem 13. *Let M be a quasi-lp-injective module which is self-generator. Denote $S = End(M)$, $\triangle = \{s \in S | Ker(s) \subset^*_> M\}$ and $J(S)$ the Jacobson radical of the ring S. Then $J(S) \subseteq \triangle$. Especially, if S is left Kasch, then $J(S) = \triangle$.*

Proof. Suppose on the contrary that $J(S) \nsubseteq \triangle$. Then there is a nonzero element $\alpha \in J(S)$ with $\alpha \notin \triangle$. Since M is a self generator, there exists $0 \neq \beta \in S$ such that $Ker(\alpha) \cap Im(\beta) = 0$, and hence $\alpha\beta \neq 0$. By the quasi-lp-injectivity of M, there is an element $\gamma \in S$ such that $\alpha\beta\gamma \neq 0$ satisfying

$l_S(Ker(\alpha\beta\gamma)) = S\alpha\beta\gamma$. Put $s = \alpha\beta\gamma$ and $t = \beta\gamma$. Clearly, $Ker(\alpha) \cap Im(t) = 0$, and hence $t \notin J(S)$. If $m \in Ker(s)$, then $\gamma(m) \in Ker(\alpha\beta) = Ker(\beta)$, and therefore $t(m) = \beta\gamma(m) = 0$. This follows that $Ker(s) = Ker(t)$. Since $t \in l_S(Ker(t)) = l_S(Ker(s)) = Ss$, we have $t = us$ for some $u \in S$. Therefore $t \in J(S)$, a contradiction. This shows that $J(S) \subseteq \triangle$.

We now assume in addition that S is left Kasch. Take any $0 \neq s \in \triangle$ and $0 \neq t \in S$. If $S(1-ts) \neq S$, then $S(1-ts) \subset H$ for some maximal left ideal H of S. Since S is left Kasch, there exists $0 \neq k \in S$ such that $H = l_S(k)$. Therefore $k \in r_S l_S(k) = r_S(H)$. Moreover $1-ts \in H$, hence $r_S(H) \subset r_S(1-ts)$. It follows that $0 \neq k \in r_S(1-ts)$. Since $s \in \triangle$, $Ker(ts) \subset^*_> M$. Applying Remark 12, we have $r_S(ts) \subset^*_> S$. From the fact that, $r_S(1-ts) \cap r_S(ts) = 0$, we see that $r_S(1-ts) = 0$, this is a contradiction. Therefore $S(1-ts) = S$, proving that $1-ts$ is invertible. Hence $s \in J(S)$, and we get $J(S) = \triangle$, completing our proof. □

Combining the Theorem 13 and Lemma 3, we easily get the following corollary.

Corollary 14. *Let M be a quasi-lp-injective module which is a self-generator. Let $S = End_R(M)$ be left Kasch. Denote the Jacobson radical of S by $J(S)$. If $Soc(M)$ is essential in M, then $J(S) = l_S(Soc(M))$.*

The following routine remark is helpful to prove the next theorem:

Remark 15. *Let M be a right R-module and $\Omega = \bigoplus_{i=1}^{n} A_i$ a direct sum of uniform submodules of M. If B is a submodule of M such that $B \cap A_i \neq 0, i = 1, 2, ..., n$, then $B \cap \Omega$ is an essential submodule of Ω.*

Lemma 16. *Let M be a quasi-lp-injective module and $\Omega = \bigoplus_{i=1}^{n} Im(u_i)$ a direct sum of homomorphic image of uniform elements of S. If $K \subset S$ is a maximal left ideal not of the form A_u in Lemma 8, then for any right uniform element u, there exists $k \in K$ such that $Ker(1-k) \cap \Omega$ is essential in Ω.*

Proof. Since $K \neq A_{u_1}$, there exists $s_0 \in K$ such that $Ker(s_0) \cap Im(u_1) = 0$ and hence $s_0 u_1 \neq 0$. Since M is quasi-lp-injective, there is $0 \neq t_1 \in S$ such that $s_0 t u_1 t_1 \neq 0$ and $l_S Ker(s_0 u_1 t_1) = S s_0 u_1 t_1$. If $m \in Ker(s_0 u_1 t_1)$, then $s_0 u_1 t_1(m) = 0$. Since $s_0 \notin A_{u_1}$, we see that $m \in Ker(u_1 t_1)$. Hence, $Ker(u_1 t_1) \subset Ker(s_0 u_1 t_1)$, and this implies that $Ker(s_0 u_1 t_1) = Ker(u_1 t_1)$. Thus we have $u_1 t_1 \in l_S(Ker(s_0 u_1 t_1)) = S s_0 u_1 t_1$. Hence, there is $f_1 \in S$ such that $u_1 f_1 = f_1 s_0 u_1 t_1$ and therefore $(1-f_1 s_0) u_1 t_1 = 0$. Then $1-f_1 s_0 \in l_S(u_1 t_1)$. Let $k_1 = f_1 s_0$. Then, we have $k_1 \in K$ and $Ker(1-k_1) \cap Im(u_1) \supset Ker(1-$

$k_1) \cap Im(u_1t_1) \neq 0$. If $Ker(1-k_1) \cap Im(u_i) \neq 0$ for all $i = 1, 2, 3, ..., n$, by Remark 11, we are well done. Suppose that $Ker(1-k_1) \cap Im(u_2) = 0$. Since u_2 is uniform, $(1-k_1)u_2$ is also uniform. As before, there exists $\alpha_1 \in K$ such that $Ker(1-\alpha_1) \cap Im((1-k_1)u_2) \neq 0$. Let $k_2 = \alpha_1 + k_1 - \alpha_1 k_1$. Then $k_2 \in K$ and $Ker(1-k_2) \cap Im(u_i) \neq 0, i = 1, 2$. Continuing this process, we will obtain $k \in K$ such that $Ker(1-k) \cap Im(u_i) \neq 0$ for all $i \in \mathbb{N}$, and the proof is complete. □

Theorem 17. *Let M be a quasi-lp-injective module which is self-generator with n-Goldie-dimension and $S = End_R(M)$.*
(1) If $I \subset S$ is a maximal left ideal, then $I = A_u$ for some right uniform element $u \in S$.
(2) $S/J(S)$ is semisimple.

Proof. (1) Since M is a self-generator, every submodule of M contains an M-cyclic submodule. Combining with assumption being n-Goldie-dimension of M, there is a direct sum $\Omega = \bigoplus_{i=1}^{n} Im(u_i) \subset^{*}_{>} M$, where each u_i is an uniform element of S. In the contrary, suppose that I is not of the form A_u for some right uniform element of $u \in S$. By Lemma 16, there exists an element $s \in I$ such that $Ker(1-s) \cap \Omega$ is essential in Ω. It follows that $1 - s \in J(S) \subset I$, a contradiction. Thus $I = A_u$ for some right uniform element $u \in S$.

(2) Take any $s \in \bigcap_{i=1}^{n} A_{u_i}$. We have $Ker(s) \cap Im(u_i) \neq 0$, for all i=1,...,n. Since u_i is uniform, and $\Omega = \bigoplus_{i=1}^{n} Im(u_i) \subset^{*}_{>} M$, $Ker(s) \subset^{*}_{>} M$. It follows that $s \in J(S)$ and hence $\bigcap_{i=1}^{n} A_{u_i} = J(S)$. This shows that $S/J(S)$ is semisimple. □

Corollary 18.*Let R be a right self-lp-injective which has finite Goldie dimension. Then the following statements hold:*

(1) If $I \subset R$ be a maximal left ideal, then $I = A_u$ for some right uniform element $u \in R$.
(2) $R/J(R)$ is semisimple.

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