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INVERTIBLE MATRICES OVER SEMIFIELDS

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Abstract

A *semifield* is a commutative semiring $(S, +, \cdot)$ with zero 0 and identity 1 such that $(S \smallsetminus \{0\}, \cdot)$ is a group. Then every field is a semifield. It is known that a square matrix A over a field F is an invertible matrix over F if and only if $\det A \neq 0$. In this paper, invertible matrices over a semifield which is not a field are characterized. It is shown that if S is a semifield which is not a field, then a square matrix A over S is an invertible matrix over S if and only if every row and every column of A contains exactly one nonzero element.

1 Introduction

A *semiring* is a triple $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups and for all $x, y, z \in S$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$. A semiring $(S, +, \cdot)$ is called *additively* [*multiplicatively*] *commutative* if $x + y = y + x$ [$x \cdot y = y \cdot x$] for all $x, y \in S$. We call $(S, +, \cdot)$ *commutative* if it is both additively and multiplicatively commutative. An element $0 \in S$ is called a *zero* of a semiring $(S, +, \cdot)$ if $x + 0 = 0 + x = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$ and by an *identity* of $(S, +, \cdot)$ we mean an element $1 \in S$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in S$. Note that a zero and an identity of a semiring are unique.

If a semiring $(S, +, \cdot)$ has a zero 0 [an identity 1], we say that an element $x \in S$ is *additively* [*multiplicatively*] *invertible* over S if there exists an element

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$y \in S$ such that $x+y = y+x = 0$ $[x \cdot y = y \cdot x = 1]$. Note that such a y is unique and may be written as $-x$ $[x^{-1}]$. Observe that if x is additively invertible, then for all $a \in S$, $ax+a(-x) = a(x-x) = a0 = 0$, $a(-x)+ax = a(-x+x) = a0 = 0$, $xa + (-x)a = (x-x)a = 0a = 0$ and $(-x)a + xa = (-x+x)a = 0a = 0$. Thus $-ax = a(-x)$ and $-xa = (-x)a$. Since $\cdot$ is distributive over $+$ in a semiring $(S, +, \cdot)$, the following fact holds.

Proposition 1.1. *Let S be a additively commutative semiring with zero 0. If $x_1, \dots, x_k$ are additively invertible over S, then $\sum_{i=1}^{k} a_i x_i$ and $\sum_{i=1}^{k} x_i a_i$ are additively invertible over S for all $a_1, \dots, a_k \in S$. Moreover, $-\sum_{i=1}^{k} a_i x_i = \sum_{i=1}^{k} a_i(-x_i)$ and $-\sum_{i=1}^{k} x_i a_i = \sum_{i=1}^{k} (-x_i) a_i$.*

Proof Let $x_1, \dots, x_k$ be additively invertible in S and $a_1, \dots, a_k \in S$. Then $\sum_{i=1}^{k} a_i(-x_i)$, $\sum_{i=1}^{k} (-x_i) a_i \in S$. Since S is additively commutative, $\sum_{i=1}^{k} a_i x_i + \sum_{i=1}^{k} a_i(-x_i) = \sum_{i=1}^{k} (a_i x_i + a_i(-x_i)) = \sum_{i=1}^{k} a_i(x_i - x_i) = \sum_{i=1}^{k} a_i 0 = 0$ and $\sum_{i=1}^{k} x_i a_i + \sum_{i=1}^{k} (-x_i) a_i = \sum_{i=1}^{k} (x_i a_i + (-x_i) a_i) = \sum_{i=1}^{k} (x_i - x_i) a_i = \sum_{i=1}^{k} 0 a_i = 0$, proving our Lemma. $\square$

A commutative semiring $(S, +, \cdot)$ with zero 0 and identity 1 is called a *semifield* if $(S \smallsetminus \{0\}, \cdot)$ is a group. Then every field is a semifield. It is clearly seen that the following fact holds in any semifield.

Proposition 1.2. *If S is a semifield, then for all $x, y \in S$, $xy = 0$ implies $x = 0$ or $y = 0$.*

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ and $\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$. Then $(\mathbb{R}_0^+, +, \cdot)$ and $(\mathbb{Q}_0^+, +, \cdot)$ are semifields which are not fields.

For an $n \times n$ matrix A over a semiring S and $i, j \in \{1, \dots, n\}$, let A_{ij} be the entry of A in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column. Let A^t denote the transpose of A , that is, $A^t_{ij} = A_{ji}$ for all $i, j \in \{1, \dots, n\}$. Then $(A^t)^t = A$ and $(A+B)^t = A^t + B^t$ for all $n \times n$ matrices A, B over S. We have that for all $n \times n$ matrices A, B over a commutative semiring S, $(AB)^t = B^t A^t$.

Let $S = (S, +, \cdot)$ be a commutative semiring with zero 0 and identity 1. An $n \times n$ matrix A over S is called *invertible* over S if there is an $n \times n$ matrix B over S such that $AB = BA = I_n$ where I_n is the identity $n \times n$ matrix over S. Note that such a B is unique.

It is well-known that a square matrix A over a field F is invertible if and only if $\det A \neq 0$. A generalization of this result can be found in [1, page 160] as follows: A square matrix A over a commutative ring R with identity 1 is invertible over R if and only if $\det A$ is a multiplicatively invertible in R, that is, there exists an element $r \in R$ such that $(\det A)r = r(\det A) = 1$. Characterizations of invertible matrices over some kinds of semirings can be found in [2] and [4].

The above examples of semifields which are not fields have the property that 0 is the only additively invertible element, that is, for $x, y \in S, x + y = 0$ implies $x = y = 0$. In fact, this property is generally true.

Proposition 1.3. ([5]) *If S is a semifield which is not a field, then 0 is the only additively invertible element of S.*

The purpose of this paper is to show that a square matrix A over a semifield S which is not a field is invertible over S if and only if every row and every column of A contains exactly one nonzero element.

2 Main Result

First, we give some necessary conditions for a square matrix over a commutative semiring S with zero and identity to be invertible over S.

Proposition 2.1. *Let S be a commutative semiring with zero 0 and identity 1 and A an $n \times n$ matrix over S. If A is invertible over S, then for all $i, j, k \in \{1, \dots, n\}, j \neq k, A_{ij}A_{ik}$ and $A_{ji}A_{ki}$ are additively invertible.*

Proof Let B be an $n \times n$ matrix over S such that $AB = BA = I_n$. Then for all distinct $p, q \in \{1, \dots, n\}$, $(AB)_{pq} = 0 = (BA)_{pq}$, so

$$\sum_{l=1}^{n} A_{pl}B_{lq} = \sum_{l=1}^{n} B_{pl}A_{lq} = 0.$$

This shows that for all $l, p, q \in \{1, \dots n\}$ with $p \neq q, A_{pl}B_{lq}$ and $B_{pl}A_{lq}$ are additively invertible in S.

Next, let $i, j, k \in \{1, \dots, n\}$ be such that $j \neq k$. Then

$$A_{ij}A_{ik} = (A_{ij}A_{ik})(AB)_{ii} = A_{ij}A_{ik}(\sum_{l=1}^{n} A_{il}B_{li}) = A_{ij}A_{ik}A_{ik}B_{ki} + \sum_{\substack{l=1 \\ l\neq k}}^{n} A_{ij}A_{ik}A_{il}B_{li} =$$

$$A_{ik}^2(B_{ki}A_{ij}) + \sum_{\substack{l=1 \\ l\neq k}}^{n} A_{ij}A_{il}(B_{li}A_{ik})$$

$$\begin{aligned}A_{ji}A_{ki} &= (BA)_{ii}A_{ji}A_{ki}\\ &= (\sum_{l=1}^{n} B_{il}A_{li})A_{ji}A_{ki}\\ &= \sum_{\substack{l=1\\ l\neq j}}^{n} B_{il}A_{li}A_{ji}A_{ki} + B_{ij}A_{ji}A_{ji}A_{ki}\\ &= \sum_{\substack{l=1\\ l\neq j}}^{n} A_{ki}A_{li}(A_{ji}B_{il}) + A_{ji}^2(A_{ki}B_{ij}). \qquad (3)\end{aligned}$$

From (1), (2), (3) and Proposition 1.1, we deduce that $A_{ij}A_{ik}$ and $A_{ji}A_{ki}$ are both additively invertible in S. □

Example 1. Define $\oplus$ on $[0,1]$ by

$$x \oplus y = \max\{x, y\} \text{ for all } x, y \in [0,1].$$

Then $([0,1], \oplus, \cdot)$ is clearly a commutative semiring with zero 0 and identity 1. Moreover, 0 is the only additively invertible element of $([0,1], \oplus, \cdot)$. Let A be an $n \times n$ matrix whose entries are in $[0,1]$. Assume that A is invertible over $([0,1], \oplus, \cdot)$. Then $AB = BA = I_n$ for some $n \times n$ matrix B over $[0,1]$. Thus A and B contain neither a zero row nor a zero column. Since 0 is the only additively invertible in $([0,1], \oplus, \cdot)$, by Proposition 2.1, every row and every column of A and B contain exactly one nonzero element. Since for $x, y \in [0,1], xy = 1$ implies $x = y = 1$, we deduce that a nonzero element of A and B in each row and each column must be 1.

If A is an $n \times n$ matrix over $[0,1]$ of this form, then A is invertible over $([0,1], \oplus, \cdot)$. In fact, this is true for such an A in any commutative semiring with zero 0 and identity 1 that $AA^t = A^tA = I_n$. Since for $i, j \in \{1, \ldots, n\}$,

$$(AA^t)_{ij} = \sum_{l=1}^{n} A_{il}A^t_{lj} = \sum_{l=1}^{n} A_{il}A_{jl} = \begin{cases} 0 & \text{if } i \neq j,\\ 1 & \text{if } i = j,\end{cases}$$

$$(A^tA)_{ij} = \sum_{l=1}^{n} A^t_{il}A_{lj} = \sum_{l=1}^{n} A_{li}A_{lj} = \begin{cases} 0 & \text{if } i \neq j,\\ 1 & \text{if } i = j,\end{cases}$$

it follows that $AA^t = A^tA = I_n$.

Theorem 2.2. *Let S be a semifield which is not a field and A an $n \times n$ matrix over S. Then A is invertible over S if and only if every row and every column of A contains exactly one nonzero element.*

Proof [**Proof.**] It is evident if $n = 1$. Assume that $n > 1$ and A is invertible over S. Let B be an $n \times n$ matrix over S such that $AB = BA = I_n$. Note that every row and every column must contain at least one nonzero element. To show that every row of A has exactly one nonzero element, suppose on the contrary that there are $p, q, q' \in \{1, \ldots, n\}$ such that $q \neq q'$, $A_{pq} \neq 0$ and $A_{pq'} \neq 0$. Let $j \in \{1, \ldots, n\}$ be such that $j \neq p$. Then

$$0 = (I_n)_{pj} = (AB)_{pj} = \sum_{l=1}^{n} A_{pl} B_{lj}.$$

By Proposition 1.3, $A_{pl} B_{lj} = 0$ for all $l \in \{1, \ldots, n\}$. In particular, $A_{pq} B_{qj} = 0$. Since $A_{pq} \neq 0$, by Proposition 1.2, $B_{qj} = 0$. This shows that

$$B_{qj} = 0 \text{ for all } j \in \{1, \ldots, n\} \text{ with } j \neq p. \tag{1}$$

Also, we have

$$1 = (I_n)_{qq} = (BA)_{qq} = \sum_{l=1}^{n} B_{ql} A_{lq} \tag{2}$$

and

$$0 = (I_n)_{qq'} = (BA)_{qq'} = \sum_{l=1}^{n} B_{ql} A_{lq'}. \tag{3}$$

Then (1) and (2) yield $B_{qp} A_{pq} = 1$. Also, from Proposition 1.3 and (3), we have $B_{qp} A_{pq'} = 0$. Hence

$$A_{pq'} = 1 A_{pq'} = (B_{qp} A_{pq}) A_{pq'} = A_{pq} (B_{qp} A_{pq'}) = A_{pq} 0 = 0$$

which is a contradiction. Hence every row contains exactly one nonzero element.

Since $A^t B^t = (BA)^t = (AB)^t = B^t A^t = (I_n)^t = I_n$, from the above proof, we have that every row of A^t contains exactly one nonzero element. Hence every column of A contains exactly one nonzero element.

Conversely, assume that every row and every column contains exactly one nonzero element of S. Then

$$\text{for each } i \in \{1, \ldots, n\}, \text{ there is a unique } k_i \in \{1, \ldots, n\} \text{ such that } A_{ik_i} \neq 0 \tag{4}$$

and

$$\text{for all distinct } i, j \text{ in } \{1, \ldots, n\}, k_i \neq k_j. \tag{5}$$

Define an $n \times n$ matrix B over S by

$$B_{ij} = \begin{cases} A_{ji}^{-1} & \text{if } A_{ji} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Let $i, j \in \{1, \ldots, n\}$ be given. Then

$$\begin{aligned}
(AB)_{ij} &= \sum_{l=1}^{n} A_{il} B_{lj} \\
&= A_{ik_i} B_{k_i j} && \text{from (4)} \\
&= \begin{cases} A_{ik_i} A_{jk_i}^{-1} & \text{if } A_{jk_i} \neq 0, \\ 0 & \text{if } A_{jk_i} = 0, \end{cases} && \text{from (6)} \\
&= \begin{cases} A_{ik_i} A_{ik_i}^{-1} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} && \text{from (4)} \\
&= \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \\
&= (I_n)_{ij}.
\end{aligned}$$

From (4) and (5), we have $\{k_1, \ldots, k_n\} = \{1, \ldots, n\}$. It follows that $i = k_s$ and $j = k_t$ for some $s, t \in \{1, \ldots, n\}$, so

$$\begin{aligned}
(BA)_{ij} &= (BA)_{k_s k_t} \\
&= \sum_{l=1}^{n} B_{k_s l} A_{l k_t} \\
&= B_{k_s t} A_{t k_t} && \text{from (4)} \\
&= \begin{cases} A_{tk_s}^{-1} A_{tk_t} & \text{if } A_{tk_s} \neq 0, \\ 0 & \text{if } A_{tk_s} = 0, \end{cases} && \text{from (6)} \\
&= \begin{cases} A_{tk_t}^{-1} A_{tk_t} & \text{if } k_s = k_t, \\ 0 & \text{if } k_s \neq k_t, \end{cases} && \text{from (4)} \\
&= \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \\
&= (I_n)_{ij}.
\end{aligned}$$

This shows that $AB = BA = I_n$. Hence A is invertible over S.

Therefore the theorem is proved. □

We note here that Reutenauer and Straubing [3] have shown that if A and B are $n \times n$ matrices over any commutative semiring with zero and identity, then $AB = I_n$ implies $BA = I_n$. However, it given proof is quite complicated.

Example 2. Let $n > 1$ and

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Since $\det A = 1$, A is invertible over the field $\mathbb{R}$ $[\mathbb{Q}]$. However, by Theorem 2.2, A is not invertible over the semifield $(\mathbb{R}_0^+, +, \cdot)$ $[(\mathbb{Q}_0^+, +, \cdot)]$. If

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & n \\ n-1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n-2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

then B is invertible over the semifield $(\mathbb{R}_0^+, +, \cdot)$ $[(\mathbb{Q}_0^+, +, \cdot)]$, so B is invertible over the field $(\mathbb{R}, +, \cdot)$ $[(\mathbb{Q}, +, \cdot)]$.

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