

# INVERTIBLE MATRICES OVER SEMIFIELDS

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## Abstract

A *semifield* is a commutative semiring  $(S, +, \cdot)$  with zero 0 and identity 1 such that  $(S \setminus \{0\}, \cdot)$  is a group. Then every field is a semifield. It is known that a square matrix  $A$  over a field  $F$  is an invertible matrix over  $F$  if and only if  $\det A \neq 0$ . In this paper, invertible matrices over a semifield which is not a field are characterized. It is shown that if  $S$  is a semifield which is not a field, then a square matrix  $A$  over  $S$  is an invertible matrix over  $S$  if and only if every row and every column of  $A$  contains exactly one nonzero element.

## 1 Introduction

A *semiring* is a triple  $(S, +, \cdot)$  such that  $(S, +)$  and  $(S, \cdot)$  are semigroups and for all  $x, y, z \in S$ ,  $x \cdot (y + z) = x \cdot y + x \cdot z$  and  $(y + z) \cdot x = y \cdot x + z \cdot x$ . A semiring  $(S, +, \cdot)$  is called *additively* [*multiplicatively*] *commutative* if  $x + y = y + x$  [ $x \cdot y = y \cdot x$ ] for all  $x, y \in S$ . We call  $(S, +, \cdot)$  *commutative* if it is both additively and multiplicatively commutative. An element  $0 \in S$  is called a *zero* of a semiring  $(S, +, \cdot)$  if  $x + 0 = 0 + x = x$  and  $x \cdot 0 = 0 \cdot x = 0$  for all  $x \in S$  and by an *identity* of  $(S, +, \cdot)$  we mean an element  $1 \in S$  such that  $x \cdot 1 = 1 \cdot x = x$  for all  $x \in S$ . Note that a zero and an identity of a semiring are unique.

If a semiring  $(S, +, \cdot)$  has a zero 0 [an identity 1], we say that an element  $x \in S$  is *additively* [*multiplicatively*] *invertible* over  $S$  if there exists an element

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$y \in S$  such that  $x + y = y + x = 0$  [ $x \cdot y = y \cdot x = 1$ ]. Note that such a  $y$  is unique and may be written as  $-x$  [ $x^{-1}$ ]. Observe that if  $x$  is additively invertible, then for all  $a \in S$ ,  $ax + a(-x) = a(x - x) = a0 = 0$ ,  $a(-x) + ax = a(-x + x) = a0 = 0$ ,  $xa + (-x)a = (x - x)a = 0a = 0$  and  $(-x)a + xa = (-x + x)a = 0a = 0$ . Thus  $-ax = a(-x)$  and  $-xa = (-x)a$ . Since  $\cdot$  is distributive over  $+$  in a semiring  $(S, +, \cdot)$ , the following fact holds.

**Proposition 1.1.** *Let  $S$  be a additively commutative semiring with zero  $0$ .*

*If  $x_1, \dots, x_k$  are additively invertible over  $S$ , then  $\sum_{i=1}^k a_i x_i$  and  $\sum_{i=1}^k x_i a_i$  are*

*additively invertible over  $S$  for all  $a_1, \dots, a_k \in S$ . Moreover,  $-\sum_{i=1}^k a_i x_i =$*

$$\sum_{i=1}^k a_i(-x_i) \text{ and } -\sum_{i=1}^k x_i a_i = \sum_{i=1}^k (-x_i) a_i.$$

**Proof** Let  $x_1, \dots, x_k$  be additively invertible in  $S$  and  $a_1, \dots, a_k \in S$ . Then

$$\begin{aligned} \sum_{i=1}^k a_i(-x_i), \sum_{i=1}^k (-x_i) a_i \in S. \text{ Since } S \text{ is additively commutative, } \sum_{i=1}^k a_i x_i + \\ \sum_{i=1}^k a_i(-x_i) = \sum_{i=1}^k (a_i x_i + a_i(-x_i)) = \sum_{i=1}^k a_i(x_i - x_i) = \sum_{i=1}^k a_i 0 = 0 \\ \text{and } \sum_{i=1}^k x_i a_i + \sum_{i=1}^k (-x_i) a_i = \sum_{i=1}^k (x_i a_i + (-x_i) a_i) = \sum_{i=1}^k (x_i - x_i) a_i = \\ \sum_{i=1}^k 0 a_i = 0, \text{ proving our Lemma. } \square \end{aligned}$$

A commutative semiring  $(S, +, \cdot)$  with zero  $0$  and identity  $1$  is called a *semifield* if  $(S \setminus \{0\}, \cdot)$  is a group. Then every field is a semifield. It is clearly seen that the following fact holds in any semifield.

**Proposition 1.2.** *If  $S$  is a semifield, then for all  $x, y \in S$ ,  $xy = 0$  implies  $x = 0$  or  $y = 0$ .*

Let  $\mathbb{R}$  be the set of real numbers,  $\mathbb{Q}$  the set of rational numbers,  $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ ,  $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$ ,  $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$  and  $\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$ . Then  $(\mathbb{R}_0^+, +, \cdot)$  and  $(\mathbb{Q}_0^+, +, \cdot)$  are semifields which are not fields.

For an  $n \times n$  matrix  $A$  over a semiring  $S$  and  $i, j \in \{1, \dots, n\}$ , let  $A_{ij}$  be the entry of  $A$  in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column. Let  $A^t$  denote the transpose of  $A$ , that is,  $A_{ij}^t = A_{ji}$  for all  $i, j \in \{1, \dots, n\}$ . Then  $(A^t)^t = A$  and  $(A + B)^t = A^t + B^t$  for all  $n \times n$  matrices  $A, B$  over  $S$ . We have that for all  $n \times n$  matrices  $A, B$  over a commutative semiring  $S$ ,  $(AB)^t = B^t A^t$ .

Let  $S = (S, +, \cdot)$  be a commutative semiring with zero  $0$  and identity  $1$ . An  $n \times n$  matrix  $A$  over  $S$  is called *invertible* over  $S$  if there is an  $n \times n$  matrix  $B$  over  $S$  such that  $AB = BA = I_n$  where  $I_n$  is the identity  $n \times n$  matrix over  $S$ . Note that such a  $B$  is unique.

It is well-known that a square matrix  $A$  over a field  $F$  is invertible if and only if  $\det A \neq 0$ . A generalization of this result can be found in [1, page 160] as follows: A square matrix  $A$  over a commutative ring  $R$  with identity 1 is invertible over  $R$  if and only if  $\det A$  is a multiplicatively invertible in  $R$ , that is, there exists an element  $r \in R$  such that  $(\det A)r = r(\det A) = 1$ . Characterizations of invertible matrices over some kinds of semirings can be found in [2] and [4].

The above examples of semifields which are not fields have the property that 0 is the only additively invertible element, that is, for  $x, y \in S, x + y = 0$  implies  $x = y = 0$ . In fact, this property is generally true.

**Proposition 1.3.** ([5]) *If  $S$  is a semifield which is not a field, then 0 is the only additively invertible element of  $S$ .*

The purpose of this paper is to show that a square matrix  $A$  over a semifield  $S$  which is not a field is invertible over  $S$  if and only if every row and every column of  $A$  contains exactly one nonzero element.

## 2 Main Result

First, we give some necessary conditions for a square matrix over a commutative semiring  $S$  with zero and identity to be invertible over  $S$ .

**Proposition 2.1.** *Let  $S$  be a commutative semiring with zero 0 and identity 1 and  $A$  an  $n \times n$  matrix over  $S$ . If  $A$  is invertible over  $S$ , then for all  $i, j, k \in \{1, \dots, n\}, j \neq k, A_{ij}A_{ik}$  and  $A_{ji}A_{ki}$  are additively invertible.*

**Proof** Let  $B$  be an  $n \times n$  matrix over  $S$  such that  $AB = BA = I_n$ . Then for all distinct  $p, q \in \{1, \dots, n\}, (AB)_{pq} = 0 = (BA)_{pq}$ , so

$$\sum_{l=1}^n A_{pl}B_{lq} = \sum_{l=1}^n B_{pl}A_{lq} = 0.$$

This shows that for all  $l, p, q \in \{1, \dots, n\}$  with  $p \neq q, A_{pl}B_{lq}$  and  $B_{pl}A_{lq}$  are additively invertible in  $S$ .

Next, let  $i, j, k \in \{1, \dots, n\}$  be such that  $j \neq k$ . Then

$$A_{ij}A_{ik} = (A_{ij}A_{ik})(AB)_{ii} = A_{ij}A_{ik} \left( \sum_{l=1}^n A_{il}B_{li} \right) = A_{ij}A_{ik}A_{ik}B_{ki} + \sum_{\substack{l=1 \\ l \neq k}}^n A_{ij}A_{ik}A_{il}B_{li} =$$

$$A_{ik}^2(B_{ki}A_{ij}) + \sum_{\substack{l=1 \\ l \neq k}}^n A_{ij}A_{il}(B_{li}A_{ik})$$

$$\begin{aligned}
A_{ji}A_{ki} &= (BA)_{ii}A_{ji}A_{ki} \\
&= \left(\sum_{l=1}^n B_{il}A_{li}\right)A_{ji}A_{ki} \\
&= \sum_{\substack{l=1 \\ l \neq j}}^n B_{il}A_{li}A_{ji}A_{ki} + B_{ij}A_{ji}A_{ji}A_{ki} \\
&= \sum_{\substack{l=1 \\ l \neq j}}^n A_{ki}A_{li}(A_{ji}B_{il}) + A_{ji}^2(A_{ki}B_{ij}). \tag{3}
\end{aligned}$$

From (1), (2), (3) and Proposition 1.1, we deduce that  $A_{ij}A_{ik}$  and  $A_{ji}A_{ki}$  are both additively invertible in  $S$ .  $\square$

**Example 1.** Define  $\oplus$  on  $[0, 1]$  by

$$x \oplus y = \max\{x, y\} \text{ for all } x, y \in [0, 1].$$

Then  $([0, 1], \oplus, \cdot)$  is clearly a commutative semiring with zero 0 and identity 1. Moreover, 0 is the only additively invertible element of  $([0, 1], \oplus, \cdot)$ . Let  $A$  be an  $n \times n$  matrix whose entries are in  $[0, 1]$ . Assume that  $A$  is invertible over  $([0, 1], \oplus, \cdot)$ . Then  $AB = BA = I_n$  for some  $n \times n$  matrix  $B$  over  $[0, 1]$ . Thus  $A$  and  $B$  contain neither a zero row nor a zero column. Since 0 is the only additively invertible in  $([0, 1], \oplus, \cdot)$ , by Proposition 2.1, every row and every column of  $A$  and  $B$  contain exactly one nonzero element. Since for  $x, y \in [0, 1]$ ,  $xy = 1$  implies  $x = y = 1$ , we deduce that a nonzero element of  $A$  and  $B$  in each row and each column must be 1.

If  $A$  is an  $n \times n$  matrix over  $[0, 1]$  of this form, then  $A$  is invertible over  $([0, 1], \oplus, \cdot)$ . In fact, this is true for such an  $A$  in any commutative semiring with zero 0 and identity 1 that  $AA^t = A^tA = I_n$ . Since for  $i, j \in \{1, \dots, n\}$ ,

$$\begin{aligned}
(AA^t)_{ij} &= \sum_{l=1}^n A_{il}A_{lj}^t = \sum_{l=1}^n A_{il}A_{jl} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases} \\
(A^tA)_{ij} &= \sum_{l=1}^n A_{il}^tA_{lj} = \sum_{l=1}^n A_{li}A_{lj} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}
\end{aligned}$$

it follows that  $AA^t = A^tA = I_n$ .

**Theorem 2.2.** *Let  $S$  be a semifield which is not a field and  $A$  an  $n \times n$  matrix over  $S$ . Then  $A$  is invertible over  $S$  if and only if every row and every column of  $A$  contains exactly one nonzero element.*

**Proof [Proof.]** It is evident if  $n = 1$ . Assume that  $n > 1$  and  $A$  is invertible over  $S$ . Let  $B$  be an  $n \times n$  matrix over  $S$  such that  $AB = BA = I_n$ . Note that every row and every column must contain at least one nonzero element. To show that every row of  $A$  has exactly one nonzero element, suppose on the contrary that there are  $p, q, q' \in \{1, \dots, n\}$  such that  $q \neq q'$ ,  $A_{pq} \neq 0$  and  $A_{pq'} \neq 0$ . Let  $j \in \{1, \dots, n\}$  be such that  $j \neq p$ . Then

$$0 = (I_n)_{pj} = (AB)_{pj} = \sum_{l=1}^n A_{pl}B_{lj}.$$

By Proposition 1.3,  $A_{pl}B_{lj} = 0$  for all  $l \in \{1, \dots, n\}$ . In particular,  $A_{pq}B_{qj} = 0$ . Since  $A_{pq} \neq 0$ , by Proposition 1.2,  $B_{qj} = 0$ . This shows that

$$B_{qj} = 0 \text{ for all } j \in \{1, \dots, n\} \text{ with } j \neq p. \tag{1}$$

Also, we have

$$1 = (I_n)_{qq} = (BA)_{qq} = \sum_{l=1}^n B_{ql}A_{lq} \tag{2}$$

and

$$0 = (I_n)_{qq'} = (BA)_{qq'} = \sum_{l=1}^n B_{ql}A_{lq'}. \tag{3}$$

Then (1) and (2) yield  $B_{qp}A_{pq} = 1$ . Also, from Proposition 1.3 and (3), we have  $B_{qp}A_{pq'} = 0$ . Hence

$$A_{pq'} = 1A_{pq'} = (B_{qp}A_{pq})A_{pq'} = A_{pq}(B_{qp}A_{pq'}) = A_{pq}0 = 0$$

which is a contradiction. Hence every row contains exactly one nonzero element.

Since  $A^tB^t = (BA)^t = (AB)^t = B^tA^t = (I_n)^t = I_n$ , from the above proof, we have that every row of  $A^t$  contains exactly one nonzero element. Hence every column of  $A$  contains exactly one nonzero element.

Conversely, assume that every row and every column contains exactly one nonzero element of  $S$ . Then

$$\begin{aligned} &\text{for each } i \in \{1, \dots, n\}, \text{ there is a unique } k_i \in \{1, \dots, n\} \\ &\text{such that } A_{ik_i} \neq 0 \end{aligned} \tag{4}$$

and

$$\text{for all distinct } i, j \text{ in } \{1, \dots, n\}, k_i \neq k_j. \tag{5}$$

Define an  $n \times n$  matrix  $B$  over  $S$  by

$$B_{ij} = \begin{cases} A_{ji}^{-1} & \text{if } A_{ji} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Let  $i, j \in \{1, \dots, n\}$  be given. Then

$$\begin{aligned} (AB)_{ij} &= \sum_{l=1}^n A_{il}B_{lj} \\ &= A_{ik_i}B_{k_i j} && \text{from (4)} \\ &= \begin{cases} A_{ik_i}A_{jk_i}^{-1} & \text{if } A_{jk_i} \neq 0, \\ 0 & \text{if } A_{jk_i} = 0, \end{cases} && \text{from (6)} \\ &= \begin{cases} A_{ik_i}A_{ik_i}^{-1} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} && \text{from (4)} \\ &= \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \\ &= (I_n)_{ij}. \end{aligned}$$

From (4) and (5), we have  $\{k_1, \dots, k_n\} = \{1, \dots, n\}$ . It follows that  $i = k_s$  and  $j = k_t$  for some  $s, t \in \{1, \dots, n\}$ , so

$$\begin{aligned} (BA)_{ij} &= (BA)_{k_s k_t} \\ &= \sum_{l=1}^n B_{k_s l}A_{lk_t} \\ &= B_{k_s t}A_{tk_t} && \text{from (4)} \\ &= \begin{cases} A_{tk_s}^{-1}A_{tk_t} & \text{if } A_{tk_s} \neq 0, \\ 0 & \text{if } A_{tk_s} = 0, \end{cases} && \text{from (6)} \\ &= \begin{cases} A_{tk_t}^{-1}A_{tk_t} & \text{if } k_s = k_t, \\ 0 & \text{if } k_s \neq k_t, \end{cases} && \text{from (4)} \\ &= \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \\ &= (I_n)_{ij}. \end{aligned}$$

This shows that  $AB = BA = I_n$ . Hence  $A$  is invertible over  $S$ .

Therefore the theorem is proved.  $\square$

We note here that Reutenauer and Straubing [3] have shown that if  $A$  and  $B$  are  $n \times n$  matrices over any commutative semiring with zero and identity, then  $AB = I_n$  implies  $BA = I_n$ . However, its given proof is quite complicated.

**Example 2.** Let  $n > 1$  and

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Since  $\det A = 1$ ,  $A$  is invertible over the field  $\mathbb{R} [\mathbb{Q}]$ . However, by Theorem 2.2,  $A$  is not invertible over the semifield  $(\mathbb{R}_0^+, +, \cdot)$   $[(\mathbb{Q}_0^+, +, \cdot)]$ . If

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & n \\ n-1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n-2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

then  $B$  is invertible over the semifield  $(\mathbb{R}_0^+, +, \cdot)$   $[(\mathbb{Q}_0^+, +, \cdot)]$ , so  $B$  is invertible over the field  $(\mathbb{R}, +, \cdot)$   $[(\mathbb{Q}, +, \cdot)]$ .

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