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H-SUPPLEMENTED MODULES WITH SMALL RADICAL

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Abstract

We say that a module M is H-supplemented if for every submodule A there is a direct summand B such that $A + X = M$ holds if and only if $B + X = M$. This paper investigates the structure of H-supplemented modules over commutative noetherian rings. After reducing this question to the case of local rings and describing H-supplemented modules with small radical, it is shown that if every direct summand of M is H-supplemented, then M is a direct sum of hollow modules.

In the second part of this paper it is studied some rings whose modules are H-supplemented.

1 Introduction

In this paper all rings are associative with identity elements and all modules are unital right modules. A submodule L of a module M is said *small* in M, written $L \ll M$, provided $M \neq L + X$ for any proper submodule X of M. If every proper submodule of M is small in M, we call M a *hollow* module. The module M will be called a *local* module if $Rad(M)$ is a small maximal submodule of M. Let N be a submodule of a module M. A submodule K of M is called a *supplement* of N in M provided $M = N + K$ and $M \neq N + L$ for any proper submodule L of K. It is well known that K is a supplement of N in M if and only if $M = N + K$ and $N \cap K$ is small in K. M is called *supplemented* if every submodule of M has a supplement. We say that

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a module M is $\oplus$-*supplemented* if every submodule has a supplement that is a direct summand of M. The module M is called H-*supplemented* if for every submodule A there exists a direct summand B such that $A + X = M$ holds if and only if $B + X = M$. It is clear that if $M = B \oplus C$, then C is a supplement of A in M. So every H-supplemented module is $\oplus$-supplemented. The module M is called completely H-supplemented ($\oplus$-supplemented) if every direct summand of M is H-supplemented ($\oplus$-supplemented). The structure of finitely generated H-supplemented modules over commutative local rings is given in [20, Satz 3.2]. In Section 2, we will be concerned with the structure of H-supplemented modules over commutative noetherian rings. It is shown that in studying of H-supplemented or completely H-supplemented, one may restrict to the case of modules over local rings. Our main result (Theorem 2.9) describes the structure of H-supplemented and completely H-supplemented modules with small radical over commutative local noetherian rings: Let R be a commutative noetherian local ring with maximal ideal m. Let M be an R-module with $RadM \ll M$. The following are equivalent:

(i) M is H-supplemented;

(ii) M is completely H-supplemented;

(iii) $M \cong \oplus_{k\in K} \frac{R}{I_k}$ where I_k $(k \in K)$ are ideals of R such that:

(a) there exists $e \geq 1$ such that the set $\{k \in K \mid m^e \nsubseteq I_k\}$ is finite, and

(b) the ideals I_k $(k \in K)$ are linearly ordered by inclusion.

It is proved also that every completely H-supplemented module over commutative noetherian rings is a direct sum of hollow modules.

We conclude this paper by studying some rings whose modules are H-supplemented. Among other characterizations, it is proved that for a commutative ring R, every R-module is H-supplemented if and only if the ring R is artinian principal.

2 H-supplemented modules over commutative noetherian rings

Definition 2.1. *A family of modules $\{M_\alpha : \alpha \in \Lambda\}$ is called locally-semi-transfinitely-nilpotent ($lsTn$) if for any subfamily $M_{\alpha_i} (i \in \mathbb{N})$ with distinct α_i and any family of non-isomorphisms $f_i : M_{\alpha_i} \to M_{\alpha_{i+1}}$, and for every $x \in M_{\alpha_1}$, there exists $n \in \mathbb{N}$ (depending on x) such that $f_n \cdots f_2 f_1(x) = 0$.*

Lemma 2.2. *Let R be a commutative noetherian local ring with maximal ideal m. Let $M = \oplus_{k\in K} Rx_k$ such that $Rad(M) \ll M$. Then the family $(Rx_k)_{k\in K}$ is $lsTn$.*

Proof. It is clear that Rx_k $(k \in K)$ are local modules since R is a local ring. By [8, Theorem 8], every proper submodule of M is contained in some maximal submodule. Therefore there is $e \geq 1$ such that $m^e M$ is finitely generated by

[18, Satz 2.4]. Let $(I_k = Ann(x_k))_{k\in K}$. Hence the set $\{k \in K \mid m^e \nsubseteq I_k\}$ is finite. Let $f : Rx_i \longrightarrow Rx_j$ be a non-isomorphism and $a \in R$ such that $f(x_i) = ax_j$. Then $aI_i \subseteq I_j$. If $a \notin m$, then a is invertible. Thus $I_i \subseteq I_j$. Hence $I_i \subset I_j$ or $I_i = I_j$. But if $I_i = I_j$, then f will be an isomorphism. Therefore $I_i \subset I_j$.

Let $Rx_{\alpha_i}(i \in \mathbb{N})$ be a subfamily of $Rx_k(k \in K)$ with distinct α_i and $f_i : Rx_{\alpha_i} \longrightarrow Rx_{\alpha_{i+1}}$ a family of non-isomorphisms. Let $a_i \in R$ such that $f_i(x_{\alpha_i}) = a_i x_{\alpha_{i+1}}$. Thus for every $n \in \mathbb{N}$, we have $f_n \cdots f_2 f_1(x_{\alpha_1}) = a_n \cdots a_2 a_1 x_{\alpha_{n+1}}$. Since R is noetherian and $\{k \in K \mid m^e \nsubseteq I_k\}$ is finite, there exists $l \in \mathbb{N}$ such that $f_l \cdots f_2 f_1(x_{\alpha_1}) = 0$. Therefore $\{Rx_k : k \in K\}$ is $lsTn$. □

Lemma 2.3. *Let R be a commutative local ring with maximal ideal m. Let $H_1 = Rx_1$ and $H_2 = Rx_2$ be two cyclic R-modules with $I_i = Ann(H_i)$ $(i = 1, 2)$. Suppose that $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. Then for every isomorphism $f : H_1/mH_1 \to H_2/mH_2$, there exists an epimorphism g of either H_1 onto H_2 or H_2 onto H_1 such that $\overline{g} = f$ or $\overline{g} = f^{-1}$.*

Proof. Let $f : H_1/mH_1 \to H_2/mH_2$ be an isomorphism. So there exists $a \in R-m$ such that $f(\overline{x_1}) = a\overline{x_2}$. If $I_1 \subseteq I_2$, then the homomorphism $g : H_1 \to H_2$ defined by $g(x_1) = ax_2$ is well defined. It is clear that g is an epimorphism and $\overline{g} = f$. If $I_2 \subseteq I_1$, then the homomorphism $h : H_2 \to H_1$ defined by $h(x_2) = a^{-1}x_1$, where a^{-1} is the inverse of a, is well defined. It is clear that h is an epimorphism and $\overline{h} = f^{-1}$. □

Note that from [2, Theorem 4.1] it follows that every local module over a commutative ring has local endomorphism ring.

If for each simple direct summand A of $M/Rad(M)$, there exists a local direct summand K of M such that $(K + Rad(M))/Rad(M) = A$, then we say that M has *the lifting property of simple modules*. More generally, if for any direct summand B of $M/Rad(M)$, there exists a direct summand N of M such that $(N + Rad(M))/Rad(M) = B$, we say that M has *the lifting property of direct summands*.

Proposition 2.4. *Let R be a commutative noetherian local ring with maximal ideal m. Let $M = \oplus_{k\in K} H_k$ such that $(H_k)_{k\in K}$ are local submodules of M and $RadM \ll M$. Let $I_k = Ann(H_k)$ $(k \in K)$. Suppose that the set of ideals $(I_k)_{k\in K}$ is totally ordered with respect to set inclusion. Then M is completely H-supplemented.*

Proof. By [9, Proposition A.3], it suffices to prove that for every direct summand N of M, N has the lifting property of direct summands. By [13, Theorem 1], we need only to show that each direct summand of $M/RadM$ lifts to a direct summand of M. By Lemma 2.3 and [3, Theorem 2], M has the lifting property of simple modules. Since the family $(H_k)_{k\in K}$ is $lsTn$ by Lemma 2.2, M has the lifting property of direct summands by [4, Theorem 1]. This proves the result. □

Proposition 2.5. *Let a module $M = \oplus_{i \in I} M_i$ be a direct sum of submodules $M_i (i \in I)$. If for every submodule N of M, we have $N = \oplus_{i \in I}(N \cap M_i)$, then M is (completely) H-supplemented if and only if all $M_i (i \in I)$ are (completely) H-supplemented.*

Proof. Clear. □

Let R denote a commutative ring. Let Ω be the set of all maximal ideal of R. If $m \in \Omega$, M an R-module, we denote as in [19, p. 53] by $K_m(M) = \{x \in M \mid x = 0$ or the only maximal ideal over $Ann_R(x)$ is $m\}$ as the *m-local component* of M. We call M *m-local* if $K_m(M) = M$. In this case M is an R_m-module by the following operation: $(\frac{r}{s})x = rx'$ with $x = sx'$ $(r \in R, s \in R - m)$. The submodules of M over R and over R_m are identical.

For $K(M) = \{x \in M \mid Rx$ is supplemented$\}$ it is easily seen that $K(M) = \{x \in M \mid \frac{R}{Ann_R(x)}$ is semiperfect$\}$, and we always have the decomposition $K(M) = \oplus_{m \in \Omega} K_m(M)$ (see [19, Satz 2.3]). Moreover, if the ring R is noetherian, then for every supplemented R-module M, we have $M = K(M)$ by [19, Satz 2.3 and Satz 2.5].

The next result shows that we can reduce our investigations about H-supplemented and completely H-supplemented modules with $M = K(M)$ over commutative rings to the case of local rings.

Corollary 2.6. *Let M be an R-module over the commutative ring R. The following are equivalent:*

(i) $K(M)$ is H-supplemented (completely H-supplemented);

(ii) $K_m(M)$ is H-supplemented (completely H-supplemented) for all $m \in \Omega$.

Proof. It is an immediate consequence of Proposition 2.5 since for every submodule N of $K(M)$ we have $N = \oplus_{m \in \Omega} N \cap K_m(M)$. □

Proposition 2.7. *Let R be a commutative noetherian ring. Let M be a module with $RadM \ll M$. If M is H-supplemented, then $M = \oplus_{k \in K} H_k$ with H_k $(k \in K)$ are local submodules of M.*

Proof. By [9, Proposition A.3], $M/Rad(M)$ is semisimple and M has the lifting property of direct summands. By [4, Theorem 4], $M = [\oplus_{k \in K} H_k] + Rad(M)$ where $H_k (k \in K)$ are local submodules of M. But $RadM \ll M$. Then $M = \oplus_{k \in K} H_k$. □

Corollary 2.8. *Let R be a commutative noetherian ring. The following are equivalent for an R-module M with $Rad(M) \ll M$:*

(i) M is H-supplemented;

(ii) $M = \oplus_{k \in K} H_k$ is a direct sum of local submodules $H_k (k \in K)$ and M has the lifting property of simple modules.

Proof. (i) ⇒ (ii) By [9, Proposition A.3] and Proposition 2.7.

(ii) $\Rightarrow$ (i) It is clear that $M/Rad(M)$ is semisimple. By Lemma 2.2, the family $\{H_k \mid k \in K\}$ is $lsTn$. The result follows from [4, Theorem 1] and [9, Proposition A.3]. $\square$

Theorem 2.9. *Let R be a commutative noetherian local ring with maximal ideal m. Let M be an R-module with $RadM \ll M$. The following are equivalent:*

(i) M is H-supplemented;

(ii) M is completely H-supplemented;

(iii) $M \cong \oplus_{k\in K} \frac{R}{I_k}$ where I_k $(k \in K)$ are ideals of R such that:

(a) there exists $e \geq 1$ such that the set $\{k \in K \mid m^e \nsubseteq I_k\}$ is finite, and

(b) the ideals I_k $(k \in K)$ are linearly ordered by inclusion.

Proof. (i) $\Rightarrow$ (iii) By Proposition 2.7, $M \cong \oplus_{k\in K} \frac{R}{I_k}$ where I_k $(k \in K)$ are ideals of R. By [8, Theorem 8], every proper submodule of M is contained in some maximal submodule. Thus there exists $e \geq 1$ such that $m^e M$ is finitely generated by [18, Satz 2.4]. Hence the set $\{k \in K \mid m^e \nsubseteq I_k\}$ is finite. By Lemma 2.2, the family $\frac{R}{I_k}$ $(k \in K)$ is $lsTn$. Since M has the lifting property of direct summands by [9, Proposition A.3], for every pair $(k_1, k_2) \in K \times K$, $R/I_{k_1} \oplus R/I_{k_2}$ has the lifting property of direct summands by [4, Theorem 1]. So the module $R/I_{k_1} \oplus R/I_{k_2}$ is H-supplemented by Corollary 2.8. From [20, Satz 3.2], it follows that $I_{k_1} \subseteq I_{k_2}$ or $I_{k_2} \subseteq I_{k_1}$.

(iii) $\Rightarrow$ (ii) It is clear that $m^e M$ is finitely generated. From [18, Satz 2.4] we deduce that every proper submodule of M is contained in some maximal submodule. So $Rad(M) \ll M$. The result follows from Proposition 2.4.

(ii) $\Rightarrow$ (i) It is clear. $\square$

The following result may be proved in much the same way as [20, Lemma 1.1 (a)].

Proposition 2.10. *Let M_0 be a direct summand of a module M such that for every decomposition $M = N \oplus K$ of M, there exist submodules $N' \leq N$ and $K' \leq K$ such that $M = M_0 \oplus N' \oplus K'$. If M is H-supplemented, then M/M_0 is H-supplemented.*

Let M be an R-module. By $P(M)$ we denote the sum of all radical submodules of M. It is easily seen that if $M = N \oplus K$, then $P(M) = P(N) \oplus P(K)$.

Corollary 2.11. *Let M be an H-supplemented module. If $P(M)$ is a direct summand of M, then $P(M)$ and $M/P(M)$ are H-supplemented.*

Proof. Let L be a submodule of M such that $M = P(M) \oplus L$. Let $M = N \oplus K$. Since $P(M)$ is a direct summand of M and $P(M) = P(N) \oplus P(K)$, there exist submodules $N' \leq N$ and $K' \leq K$ such that $N = P(N) \oplus N'$ and $K = P(K) \oplus K'$. Thus $M = P(M) \oplus N' \oplus K'$. On the other hand, we have $M = P(N) \oplus P(K) \oplus L$. Therefore $M/P(M)$ and M/L are both H-supplemented by Proposition 2.10. $\square$

A module M is called *coatomic* if every proper submodule of M is contained in some maximal submodule.

Proposition 2.12. *Let M be an H-supplemented module over a commutative noetherian ring R. Then $M = P(M) \oplus K$ such that K is coatomic and $P(M)$ and K are both H-supplemented.*

Proof. Since M is H-supplemented, it is $\oplus$-supplemented. Then $M = P(M) \oplus K$ where K is a coatomic submodule of M by [5, Theorem 2.1]. By Corollary 2.11, $P(M)$ and K are H-supplemented. □

Corollary 2.13. *Let M be a completely H-supplemented module over a commutative noetherian local ring R. Then M is a direct sum of hollow submodules.*

Proof. By Proposition 2.12, $M = P(M) \oplus K$ such that K is coatomic and $P(M)$ and K are H-supplemented. By Theorem 2.9, K is a direct sum of local submodules. Moreover, since $P(M)$ is completely $\oplus$-supplemented, $P(M)$ is a direct sum of hollow submodules by [5, Proposition 2.2]. This proves the result. □

Corollary 2.14. *Let M be an injective H-supplemented module over a commutative noetherian local ring R. Then M is a direct sum of hollow submodules.*

Proof. By Proposition 2.12, $M = P(M) \oplus K$ such that K is coatomic and $P(M)$ and K are H-supplemented. Since M is injective $\oplus$-supplemented, M is completely $\oplus$-supplemented by [6, Proposition 13]. Thus $P(M)$ is a direct sum of hollow submodules by [5, Proposition 2.2]. Furthermore, K is a direct sum of local submodules by Theorem 2.9. □

A module M is called *discrete* if it satisfies the following conditions (D_1) and (D_2):

(D_1) For every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \ll M$;

(D_2) If $A \leq M$ such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M.

Proposition 2.15. *Let M be a socle-free module over a commutative noetherian local ring R. If M is H-supplemented, then M is a finite direct sum of hollow submodules.*

Proof. From Proposition 2.12, we obtain $M = P(M) \oplus K$ such that K is coatomic and $P(M)$ and K are H-supplemented. By [11, Theorem 2.4 and Corollary 2.5], $P(M)$ is a sum of finitely many hollow submodules. By [6, Lemma 3], $P(M)$ has a hollow direct summand L. It follows that L is a hollow discrete non-local module by [11, Theorem 1.3] and [10, Proposition 3]. Applying [9, Corollary 5.5], L has local endomorphism ring. By [13, Proposition 1] and Proposition 2.10, $P(M)/L$ is H-supplemented. By repeating the same reasoning, we conclude that $P(M)$ is a finite direct sum of hollow submodules (see [5, Remark 2.1]). On the other hand, K is completely H-supplemented by Theorem 2.9. Thus K is completely $\oplus$-supplemented. Applying [5, Proposition 2.4], K is a finite direct sum of local submodules. This completes the proof. □

Notation Let m be a maximal ideal of R and $n_1, \ldots, n_k$ non-negative integers. We will denote by $B_m(n_1, \ldots, n_k)$ the direct sum of arbitrarily many copies of $\frac{R}{m^{n_1}}, \ldots, \frac{R}{m^{n_k}}$.

Proposition 2.16. *Let R be a local principal ideal ring (not necessarily a domain) with maximal ideal m. If M is an R-module with $Rad(M) \ll M$, then the following are equivalent:*

(i) M is H-supplemented;

(ii) M is completely H-supplemented;

(iii) $M \cong R^{(a)} \oplus B_m(n_1, \ldots, n_k)$ for some non-negative integers $n_1, \ldots, n_k$ and a.

Proof. By Theorem 2.9 and [12, Lemma 6.3]. □

Remark 2.17. *Let R be a commutative ring and M an R-module. Let m be a maximal ideal of R, $x \in K_m(M)$ and k a positive integer.*

(i) If $Ann_{R_m}(x) = (mR_m)^k$, then $Ann_R(x) = m^k$;

(ii) If R_m is a domain and $Ann_{R_m}(x) = 0$, then $p = Ann_R(x)$ is a prime ideal of R such that $p \in Ass_R(K_m(M))$ and m is the only maximal ideal over p.

By combining the last remark with Proposition 2.16, Corollary 2.6 and [16, Ch. IV, §15, Theorem 33], we get the following result which describes the structure of H-supplemented and completely H-supplemented modules over principal ideal rings.

Proposition 2.18. *Let R be a principal ideal ring (not necessarily a domain) and M an R-module with $Rad(M) \ll M$. The following are equivalent:*

(i) *M is H-supplemented;*

(ii) *M is completely H-supplemented;*

(iii) *$M \cong [\oplus_{i \in I} B_{m_i}(n_{i_1}, \ldots, n_{i_{k_{m_i}}})] \oplus [\oplus_{j \in J} (\frac{R}{p_j})^{(a_j)}]$ with:*

(1) the $m_i (i \in I)$ are maximal ideals of R, the $p_j (j \in J)$ are non-maximal prime ideals of R and $\{n_{i_1}, \ldots, n_{i_{k_{m_i}}}, a_j\}_{(i,j) \in I \times J}$ is a family of positive integers, and

(2) the ring $\frac{R}{p_j}$ is local for all $j \in J$.

Example 2.19. *Let M be a $\mathbb{Z}$-module with $Rad(M) \ll M$. By Proposition 2.18, M is H-supplemented if and only if $M \cong \oplus_{i \in I} B_{p_i \mathbb{Z}}(n_{i_1}, \ldots, n_{i_{k_{m_i}}})$, where the $n_{i_1}, \ldots, n_{i_{k_{m_i}}} (i \in I)$ are positive integers and the $p_i (i \in I)$ are prime integers.*

3 Rings whose modules are H-supplemented

Throughout this section, R is a commutative ring.

Proposition 3.1. *Let R be a commutative ring. The following are equivalent:*
(i) R is artinian principal;
(ii) Every R-module is $\oplus$-supplemented;
(iii) Every R-module is H-supplemented.

Proof. (i) $\Leftrightarrow$ (ii) By [5, Theorem 1.1].

(i) $\Rightarrow$ (iii) Since R is artinian, $M = K(M)$. Let m be a maximal ideal of R. By [12, Theorem 6.9], we have $K_m(M) \cong \oplus_{i\in I} R/m^{n_i}$ where $n_i (i \in I)$ are positive integers. Since R is artinian, there is a non-negative integer k for which $m^k = m^{k+1}$. Therefore $K_m(M)$ is H-supplemented by Theorem 2.9. Consequently, M is H-supplemented by Corollary 2.6.

(iii) $\Rightarrow$ (ii) Clear. $\square$

A family of sets is said to have the finite intersection property, abbreviated f.i.p., if the intersection of every finite subfamily is non-empty. Let M be an R-module. M is linearly compact if whenever $\{x_\alpha + M_\alpha\}_{\alpha\in X}$ is a family of cosets of submodules of M ($x_\alpha \in M$ and M_α is a submodule of M) with the f.i.p., then $\cap_{\alpha\in X} x_\alpha + M_\alpha \neq \emptyset$. One can translate this into a condition about solving congruences. With the above notation $x \in x_\alpha + M_\alpha$ if and only if $x \equiv x_\alpha \; mod M_\alpha$. Thus an R -module M is linearly compact if given any family of congruences $\{x \equiv x_\alpha mod M_\alpha\}_{\alpha\in X}$ of M, being able to find a solution for any finite subset of these congruences implies one can find a solution for all the congruences. R is said to be a *maximal* ring if R is linearly compact as R-module. R is called almost maximal if $\frac{R}{I}$ is a linearly compact R-module for all non-zero ideals I of R.

A commutative ring R is a valuation ring if it satisfies one of the following three equivalent conditions:
(i) For any two elements a and b, either a divides b or b divides a;
(ii) The ideals of R are linearly ordered by inclusion;
(iii) R is a local ring and every finitely generated ideal is principal.

Proposition 3.2. *The following conditions on a commutative ring R are equivalent:*
(i) Every finitely generated R-module is H-supplemented;
(ii) Every finitely generated R-module is $\oplus$-supplemented;
(iii) R is a finite product of almost maximal valuation rings.

Proof. (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) By [5, Proposition 1.4].

(iii) $\Rightarrow$ (i) Suppose that $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$, where R_i is an almost maximal valuation ring. We can write $1_R = e_1 + e_2 + \cdots + e_n$, where e_i is the identity element of the ring R_i and 1_R is the identity element of the ring R. Let M be a finitely generated R-module. Then $M = e_1M \oplus e_2M \oplus \cdots \oplus e_nM$. Let $1 \leq i \leq n$. Note that e_iM can be regarded as an R_i-module as well as an R-module, and its submodules are the same in both cases, because $(r_1 + r_2 + \cdots + r_n)e_ix = r_ie_ix$, where $r_j \in R_j$ for $1 \leq j \leq n$ and $x \in M$. Since R_i is an almost maximal valuation ring, then ${}_{R_i}(e_iM)$ is a finite direct sum of cyclic submodules by [1, Theorem 4.5]. Thus ${}_{R_i}(e_iM)$ is a finite direct sum of local submodules. Since R_i is a valuation ring, ${}_{R_i}(e_iM)$ is H-supplemented by [20, Satz 3.2]. It follows that ${}_RM$ is H-supplemented by Proposition 2.5. □

According to [20, Satz 3.2] and Corollary 2.6, every finitely generated H-supplemented module over a commutative ring is a finite direct sum of local submodules. From [7, Corollary 6], it follows that every finitely generated H-supplemented module is completely $\oplus$-supplemented. In general a finitely generated completely $\oplus$-supplemented module need not be H-supplemented (see e.g. [9, Lemma A.4] and [7, Corollary 6]).

It was shown in [7, Proposition 6] that a direct sum of two hollow modules is always completely $\oplus$-supplemented.

A module M is called finitely presented if $M \cong \frac{F}{K}$ for some finitely generated free module F and finitely generated submodule K of F.

Proposition 3.3. *The following conditions are equivalent on a commutative local ring R:*

(i) Every finitely generated completely $\oplus$-supplemented module is H-supplemented;

(ii) Every finitely presented module is $\oplus$-supplemented;

(iii) Every finitely presented module is H-supplemented;

(iv) R is a valuation ring.

Proof. (i) $\Rightarrow$ (iv) Let I and J be two ideals of R. By [7, Proposition 6], the module $M = R/I \oplus R/J$ is completely $\oplus$-supplemented. By hypothesis, M is H-supplemented. This gives $I \subseteq J$ or $J \subseteq I$ by [20, Satz 3.2]. Therefore R is a valuation ring.

(iv) $\Rightarrow$ (i) Let M be a finitely generated completely $\oplus$-supplemented module. By [7, Proposition 11], $M = \oplus_{i=1}^k H_i$ is a direct sum of local submodules H_i $(1 \leq i \leq k)$. Since R is a valuation ring, the ideals $Ann_R(H_i)$ $(1 \leq i \leq k)$ are linearly ordered by inclusion. Thus M is H-supplemented by [20, Satz 3.2].

(iv) $\Rightarrow$ (iii) Let M be a finitely presented R-module. By [14, Theorem 1], M is a finite direct sum of cyclic submodules. Since R is a valuation ring, M is H-supplemented by [20, Satz 3.2].

(iii) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iv) By [5, Proposition 1.5]. □

As in [9, p. 93], we call an ideal *m-isolated* if it is contained in at most one maximal ideal, m.

Proposition 3.4. *The following conditions are equivalent on a commutative ring R:*

(i) Every finitely generated completely $\oplus$-supplemented module is H-supplemented;

(ii) For every maximal ideal m of R, the collection of m-isolated ideals of R is linearly ordered by inclusion.

Proof. (i) $\Rightarrow$ (ii) Let m be a maximal ideal of R. Let I_1 and I_2 be two m-isolated ideals. Then the module $M = R/I_1 \oplus R/I_2$ is completely $\oplus$-supplemented by [7, Proposition 6]. By assumption, M is H-supplemented. Therefore $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$ by [20, Satz 3.2].

(ii) $\Rightarrow$ (i) Let M be a finitely generated completely $\oplus$-supplemented module. By [7, Proposition 11], $M = \oplus_{i=1}^{k} R/I_i$ is a direct sum of local modules R/I_i $(1 \leq i \leq k)$. It is clear that for every i $(1 \leq i \leq k)$, there exists a maximal ideal m_i such that the ideal I_i is m_i-isolated. By Corollary 2.6 and [20, Satz 3.2], M is H-supplemented. $\square$

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