

***H*-SUPPLEMENTED MODULES WITH SMALL RADICAL**

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Abstract

We say that a module M is H -supplemented if for every submodule A there is a direct summand B such that $A + X = M$ holds if and only if $B + X = M$. This paper investigates the structure of H -supplemented modules over commutative noetherian rings. After reducing this question to the case of local rings and describing H -supplemented modules with small radical, it is shown that if every direct summand of M is H -supplemented, then M is a direct sum of hollow modules.

In the second part of this paper it is studied some rings whose modules are H -supplemented.

1 Introduction

In this paper all rings are associative with identity elements and all modules are unital right modules. A submodule L of a module M is said *small* in M , written $L \ll M$, provided $M \neq L + X$ for any proper submodule X of M . If every proper submodule of M is small in M , we call M a *hollow* module. The module M will be called a *local* module if $Rad(M)$ is a small maximal submodule of M . Let N be a submodule of a module M . A submodule K of M is called a *supplement* of N in M provided $M = N + K$ and $M \neq N + L$ for any proper submodule L of K . It is well known that K is a supplement of N in M if and only if $M = N + K$ and $N \cap K$ is small in K . M is called *supplemented* if every submodule of M has a supplement. We say that

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a module M is \oplus -supplemented if every submodule has a supplement that is a direct summand of M . The module M is called *H-supplemented* if for every submodule A there exists a direct summand B such that $A + X = M$ holds if and only if $B + X = M$. It is clear that if $M = B \oplus C$, then C is a supplement of A in M . So every *H-supplemented* module is \oplus -supplemented. The module M is called completely *H-supplemented* (\oplus -supplemented) if every direct summand of M is *H-supplemented* (\oplus -supplemented). The structure of finitely generated *H-supplemented* modules over commutative local rings is given in [20, Satz 3.2]. In Section 2, we will be concerned with the structure of *H-supplemented* modules over commutative noetherian rings. It is shown that in studying of *H-supplemented* or completely *H-supplemented*, one may restrict to the case of modules over local rings. Our main result (Theorem 2.9) describes the structure of *H-supplemented* and completely *H-supplemented* modules with small radical over commutative local noetherian rings: Let R be a commutative noetherian local ring with maximal ideal m . Let M be an R -module with $\text{Rad}M \ll M$. The following are equivalent:

- (i) M is *H-supplemented*;
- (ii) M is completely *H-supplemented*;
- (iii) $M \cong \bigoplus_{k \in K} \frac{R}{I_k}$ where I_k ($k \in K$) are ideals of R such that:
 - (a) there exists $e \geq 1$ such that the set $\{k \in K \mid m^e \not\subseteq I_k\}$ is finite, and
 - (b) the ideals I_k ($k \in K$) are linearly ordered by inclusion.

It is proved also that every completely *H-supplemented* module over commutative noetherian rings is a direct sum of hollow modules.

We conclude this paper by studying some rings whose modules are *H-supplemented*. Among other characterizations, it is proved that for a commutative ring R , every R -module is *H-supplemented* if and only if the ring R is artinian principal.

2 *H-supplemented* modules over commutative noetherian rings

Definition 2.1. A family of modules $\{M_\alpha : \alpha \in \Lambda\}$ is called *locally-semi-transfinitely-nilpotent* (*lsTn*) if for any subfamily M_{α_i} ($i \in \mathbb{N}$) with distinct α_i and any family of non-isomorphisms $f_i : M_{\alpha_i} \rightarrow M_{\alpha_{i+1}}$, and for every $x \in M_{\alpha_1}$, there exists $n \in \mathbb{N}$ (depending on x) such that $f_n \cdots f_2 f_1(x) = 0$.

Lemma 2.2. Let R be a commutative noetherian local ring with maximal ideal m . Let $M = \bigoplus_{k \in K} Rx_k$ such that $\text{Rad}(M) \ll M$. Then the family $(Rx_k)_{k \in K}$ is *lsTn*.

Proof. It is clear that Rx_k ($k \in K$) are local modules since R is a local ring. By [8, Theorem 8], every proper submodule of M is contained in some maximal submodule. Therefore there is $e \geq 1$ such that $m^e M$ is finitely generated by

[18, Satz 2.4]. Let $(I_k = \text{Ann}(x_k))_{k \in K}$. Hence the set $\{k \in K \mid m^e \not\subseteq I_k\}$ is finite. Let $f : Rx_i \rightarrow Rx_j$ be a non-isomorphism and $a \in R$ such that $f(x_i) = ax_j$. Then $aI_i \subseteq I_j$. If $a \notin m$, then a is invertible. Thus $I_i \subseteq I_j$. Hence $I_i \subset I_j$ or $I_i = I_j$. But if $I_i = I_j$, then f will be an isomorphism. Therefore $I_i \subset I_j$.

Let $Rx_{\alpha_i} (i \in \mathbb{N})$ be a subfamily of $Rx_k (k \in K)$ with distinct α_i and $f_i : Rx_{\alpha_i} \rightarrow Rx_{\alpha_{i+1}}$ a family of non-isomorphisms. Let $a_i \in R$ such that $f_i(x_{\alpha_i}) = a_i x_{\alpha_{i+1}}$. Thus for every $n \in \mathbb{N}$, we have $f_n \cdots f_2 f_1(x_{\alpha_1}) = a_n \cdots a_2 a_1 x_{\alpha_{n+1}}$. Since R is noetherian and $\{k \in K \mid m^e \not\subseteq I_k\}$ is finite, there exists $l \in \mathbb{N}$ such that $f_l \cdots f_2 f_1(x_{\alpha_1}) = 0$. Therefore $\{Rx_k : k \in K\}$ is *lsTn*. \square

Lemma 2.3. *Let R be a commutative local ring with maximal ideal m . Let $H_1 = Rx_1$ and $H_2 = Rx_2$ be two cyclic R -modules with $I_i = \text{Ann}(H_i)$ ($i = 1, 2$). Suppose that $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. Then for every isomorphism $f : H_1/mH_1 \rightarrow H_2/mH_2$, there exists an epimorphism g of either H_1 onto H_2 or H_2 onto H_1 such that $\bar{g} = f$ or $\bar{g} = f^{-1}$.*

Proof. Let $f : H_1/mH_1 \rightarrow H_2/mH_2$ be an isomorphism. So there exists $a \in R - m$ such that $f(\bar{x}_1) = a\bar{x}_2$. If $I_1 \subseteq I_2$, then the homomorphism $g : H_1 \rightarrow H_2$ defined by $g(x_1) = ax_2$ is well defined. It is clear that g is an epimorphism and $\bar{g} = f$. If $I_2 \subseteq I_1$, then the homomorphism $h : H_2 \rightarrow H_1$ defined by $h(x_2) = a^{-1}x_1$, where a^{-1} is the inverse of a , is well defined. It is clear that h is an epimorphism and $\bar{h} = f^{-1}$. \square

Note that from [2, Theorem 4.1] it follows that every local module over a commutative ring has local endomorphism ring.

If for each simple direct summand A of $M/\text{Rad}(M)$, there exists a local direct summand K of M such that $(K + \text{Rad}(M))/\text{Rad}(M) = A$, then we say that M has the *lifting property of simple modules*. More generally, if for any direct summand B of $M/\text{Rad}(M)$, there exists a direct summand N of M such that $(N + \text{Rad}(M))/\text{Rad}(M) = B$, we say that M has the *lifting property of direct summands*.

Proposition 2.4. *Let R be a commutative noetherian local ring with maximal ideal m . Let $M = \bigoplus_{k \in K} H_k$ such that $(H_k)_{k \in K}$ are local submodules of M and $\text{Rad}M \ll M$. Let $I_k = \text{Ann}(H_k)$ ($k \in K$). Suppose that the set of ideals $(I_k)_{k \in K}$ is totally ordered with respect to set inclusion. Then M is completely H -supplemented.*

Proof. By [9, Proposition A.3], it suffices to prove that for every direct summand N of M , N has the lifting property of direct summands. By [13, Theorem 1], we need only to show that each direct summand of $M/\text{Rad}M$ lifts to a direct summand of M . By Lemma 2.3 and [3, Theorem 2], M has the lifting property of simple modules. Since the family $(H_k)_{k \in K}$ is *lsTn* by Lemma 2.2, M has the lifting property of direct summands by [4, Theorem 1]. This proves the result. \square

Proposition 2.5. *Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules $M_i (i \in I)$. If for every submodule N of M , we have $N = \bigoplus_{i \in I} (N \cap M_i)$, then M is (completely) H -supplemented if and only if all $M_i (i \in I)$ are (completely) H -supplemented.*

Proof. Clear. □

Let R denote a commutative ring. Let Ω be the set of all maximal ideal of R . If $m \in \Omega$, M an R -module, we denote as in [19, p. 53] by $K_m(M) = \{x \in M \mid x = 0 \text{ or the only maximal ideal over } \text{Ann}_R(x) \text{ is } m\}$ as the m -local component of M . We call M m -local if $K_m(M) = M$. In this case M is an R_m -module by the following operation: $(\frac{r}{s})x = rx'$ with $x = sx'$ ($r \in R, s \in R - m$). The submodules of M over R and over R_m are identical.

For $K(M) = \{x \in M \mid Rx \text{ is supplemented}\}$ it is easily seen that $K(M) = \{x \in M \mid \frac{R}{\text{Ann}_R(x)} \text{ is semiperfect}\}$, and we always have the decomposition $K(M) = \bigoplus_{m \in \Omega} K_m(M)$ (see [19, Satz 2.3]). Moreover, if the ring R is noetherian, then for every supplemented R -module M , we have $M = K(M)$ by [19, Satz 2.3 and Satz 2.5].

The next result shows that we can reduce our investigations about H -supplemented and completely H -supplemented modules with $M = K(M)$ over commutative rings to the case of local rings.

Corollary 2.6. *Let M be an R -module over the commutative ring R . The following are equivalent:*

- (i) $K(M)$ is H -supplemented (completely H -supplemented);
- (ii) $K_m(M)$ is H -supplemented (completely H -supplemented) for all $m \in \Omega$.

Proof. It is an immediate consequence of Proposition 2.5 since for every submodule N of $K(M)$ we have $N = \bigoplus_{m \in \Omega} N \cap K_m(M)$. □

Proposition 2.7. *Let R be a commutative noetherian ring. Let M be a module with $\text{Rad}M \ll M$. If M is H -supplemented, then $M = \bigoplus_{k \in K} H_k$ with $H_k (k \in K)$ are local submodules of M .*

Proof. By [9, Proposition A.3], $M/\text{Rad}(M)$ is semisimple and M has the lifting property of direct summands. By [4, Theorem 4], $M = [\bigoplus_{k \in K} H_k] + \text{Rad}(M)$ where $H_k (k \in K)$ are local submodules of M . But $\text{Rad}M \ll M$. Then $M = \bigoplus_{k \in K} H_k$. □

Corollary 2.8. *Let R be a commutative noetherian ring. The following are equivalent for an R -module M with $\text{Rad}(M) \ll M$:*

- (i) M is H -supplemented;
- (ii) $M = \bigoplus_{k \in K} H_k$ is a direct sum of local submodules $H_k (k \in K)$ and M has the lifting property of simple modules.

Proof. (i) \Rightarrow (ii) By [9, Proposition A.3] and Proposition 2.7.

(ii) \Rightarrow (i) It is clear that $M/\text{Rad}(M)$ is semisimple. By Lemma 2.2, the family $\{H_k \mid k \in K\}$ is *lsTn*. The result follows from [4, Theorem 1] and [9, Proposition A.3]. \square

Theorem 2.9. *Let R be a commutative noetherian local ring with maximal ideal m . Let M be an R -module with $\text{Rad}M \ll M$. The following are equivalent:*

- (i) M is H -supplemented;
- (ii) M is completely H -supplemented;
- (iii) $M \cong \bigoplus_{k \in K} \frac{R}{I_k}$ where I_k ($k \in K$) are ideals of R such that:
 - (a) there exists $e \geq 1$ such that the set $\{k \in K \mid m^e \not\subseteq I_k\}$ is finite, and
 - (b) the ideals I_k ($k \in K$) are linearly ordered by inclusion.

Proof. (i) \Rightarrow (iii) By Proposition 2.7, $M \cong \bigoplus_{k \in K} \frac{R}{I_k}$ where I_k ($k \in K$) are ideals of R . By [8, Theorem 8], every proper submodule of M is contained in some maximal submodule. Thus there exists $e \geq 1$ such that $m^e M$ is finitely generated by [18, Satz 2.4]. Hence the set $\{k \in K \mid m^e \not\subseteq I_k\}$ is finite. By Lemma 2.2, the family $\frac{R}{I_k}$ ($k \in K$) is *lsTn*. Since M has the lifting property of direct summands by [9, Proposition A.3], for every pair $(k_1, k_2) \in K \times K$, $R/I_{k_1} \oplus R/I_{k_2}$ has the lifting property of direct summands by [4, Theorem 1]. So the module $R/I_{k_1} \oplus R/I_{k_2}$ is H -supplemented by Corollary 2.8. From [20, Satz 3.2], it follows that $I_{k_1} \subseteq I_{k_2}$ or $I_{k_2} \subseteq I_{k_1}$.

(iii) \Rightarrow (ii) It is clear that $m^e M$ is finitely generated. From [18, Satz 2.4] we deduce that every proper submodule of M is contained in some maximal submodule. So $\text{Rad}(M) \ll M$. The result follows from Proposition 2.4.

(ii) \Rightarrow (i) It is clear. \square

The following result may be proved in much the same way as [20, Lemma 1.1 (a)].

Proposition 2.10. *Let M_0 be a direct summand of a module M such that for every decomposition $M = N \oplus K$ of M , there exist submodules $N' \leq N$ and $K' \leq K$ such that $M = M_0 \oplus N' \oplus K'$. If M is H -supplemented, then M/M_0 is H -supplemented.*

Let M be an R -module. By $P(M)$ we denote the sum of all radical submodules of M . It is easily seen that if $M = N \oplus K$, then $P(M) = P(N) \oplus P(K)$.

Corollary 2.11. *Let M be an H -supplemented module. If $P(M)$ is a direct summand of M , then $P(M)$ and $M/P(M)$ are H -supplemented.*

Proof. Let L be a submodule of M such that $M = P(M) \oplus L$. Let $M = N \oplus K$. Since $P(M)$ is a direct summand of M and $P(M) = P(N) \oplus P(K)$, there exist submodules $N' \leq N$ and $K' \leq K$ such that $N = P(N) \oplus N'$ and $K = P(K) \oplus K'$. Thus $M = P(M) \oplus N' \oplus K'$. On the other hand, we have $M = P(N) \oplus P(K) \oplus L$. Therefore $M/P(M)$ and M/L are both H -supplemented by Proposition 2.10. \square

A module M is called *coatomic* if every proper submodule of M is contained in some maximal submodule.

Proposition 2.12. *Let M be an H -supplemented module over a commutative noetherian ring R . Then $M = P(M) \oplus K$ such that K is coatomic and $P(M)$ and K are both H -supplemented.*

Proof. Since M is H -supplemented, it is \oplus -supplemented. Then $M = P(M) \oplus K$ where K is a coatomic submodule of M by [5, Theorem 2.1]. By Corollary 2.11, $P(M)$ and K are H -supplemented. \square

Corollary 2.13. *Let M be a completely H -supplemented module over a commutative noetherian local ring R . Then M is a direct sum of hollow submodules.*

Proof. By Proposition 2.12, $M = P(M) \oplus K$ such that K is coatomic and $P(M)$ and K are H -supplemented. By Theorem 2.9, K is a direct sum of local submodules. Moreover, since $P(M)$ is completely \oplus -supplemented, $P(M)$ is a direct sum of hollow submodules by [5, Proposition 2.2]. This proves the result. \square

Corollary 2.14. *Let M be an injective H -supplemented module over a commutative noetherian local ring R . Then M is a direct sum of hollow submodules.*

Proof. By Proposition 2.12, $M = P(M) \oplus K$ such that K is coatomic and $P(M)$ and K are H -supplemented. Since M is injective \oplus -supplemented, M is completely \oplus -supplemented by [6, Proposition 13]. Thus $P(M)$ is a direct sum of hollow submodules by [5, Proposition 2.2]. Furthermore, K is a direct sum of local submodules by Theorem 2.9. \square

A module M is called *discrete* if it satisfies the following conditions (D_1) and (D_2) :

(D_1) For every submodule A of M , there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \ll M$;

(D_2) If $A \leq M$ such that M/A is isomorphic to a direct summand of M , then A is a direct summand of M .

Proposition 2.15. *Let M be a socle-free module over a commutative noetherian local ring R . If M is H -supplemented, then M is a finite direct sum of hollow submodules.*

Proof. From Proposition 2.12, we obtain $M = P(M) \oplus K$ such that K is coatomic and $P(M)$ and K are H -supplemented. By [11, Theorem 2.4 and Corollary 2.5], $P(M)$ is a sum of finitely many hollow submodules. By [6, Lemma 3], $P(M)$ has a hollow direct summand L . It follows that L is a hollow discrete non-local module by [11, Theorem 1.3] and [10, Proposition 3]. Applying [9, Corollary 5.5], L has local endomorphism ring. By [13, Proposition 1] and Proposition 2.10, $P(M)/L$ is H -supplemented. By repeating the same

reasoning, we conclude that $P(M)$ is a finite direct sum of hollow submodules (see [5, Remark 2.1]). On the other hand, K is completely H -supplemented by Theorem 2.9. Thus K is completely \oplus -supplemented. Applying [5, Proposition 2.4], K is a finite direct sum of local submodules. This completes the proof. \square

Notation Let m be a maximal ideal of R and n_1, \dots, n_k non-negative integers. We will denote by $B_m(n_1, \dots, n_k)$ the direct sum of arbitrarily many copies of $\frac{R}{m^{n_1}}, \dots, \frac{R}{m^{n_k}}$.

Proposition 2.16. *Let R be a local principal ideal ring (not necessarily a domain) with maximal ideal m . If M is an R -module with $\text{Rad}(M) \ll M$, then the following are equivalent:*

- (i) M is H -supplemented;
- (ii) M is completely H -supplemented;
- (iii) $M \cong R^{(a)} \oplus B_m(n_1, \dots, n_k)$ for some non-negative integers n_1, \dots, n_k and a .

Proof. By Theorem 2.9 and [12, Lemma 6.3]. \square

Remark 2.17. *Let R be a commutative ring and M an R -module. Let m be a maximal ideal of R , $x \in K_m(M)$ and k a positive integer.*

- (i) *If $\text{Ann}_{R_m}(x) = (mR_m)^k$, then $\text{Ann}_R(x) = m^k$;*
- (ii) *If R_m is a domain and $\text{Ann}_{R_m}(x) = 0$, then $p = \text{Ann}_R(x)$ is a prime ideal of R such that $p \in \text{Ass}_R(K_m(M))$ and m is the only maximal ideal over p .*

By combining the last remark with Proposition 2.16, Corollary 2.6 and [16, Ch. IV, §15, Theorem 33], we get the following result which describes the structure of H -supplemented and completely H -supplemented modules over principal ideal rings.

Proposition 2.18. *Let R be a principal ideal ring (not necessarily a domain) and M an R -module with $\text{Rad}(M) \ll M$. The following are equivalent:*

- (i) M is H -supplemented;
- (ii) M is completely H -supplemented;
- (iii) $M \cong [\oplus_{i \in I} B_{m_i}(n_{i_1}, \dots, n_{i_{k_{m_i}}})] \oplus [\oplus_{j \in J} (\frac{R}{p_j})^{(a_j)}]$ with:
 - (1) *the $m_i (i \in I)$ are maximal ideals of R , the $p_j (j \in J)$ are non-maximal prime ideals of R and $\{n_{i_1}, \dots, n_{i_{k_{m_i}}}, a_j\}_{(i,j) \in I \times J}$ is a family of positive integers, and*
 - (2) *the ring $\frac{R}{p_j}$ is local for all $j \in J$.*

Example 2.19. *Let M be a \mathbb{Z} -module with $\text{Rad}(M) \ll M$. By Proposition 2.18, M is H -supplemented if and only if $M \cong \oplus_{i \in I} B_{p_i \mathbb{Z}}(n_{i_1}, \dots, n_{i_{k_{m_i}}})$, where the $n_{i_1}, \dots, n_{i_{k_{m_i}}} (i \in I)$ are positive integers and the $p_i (i \in I)$ are prime integers.*

3 Rings whose modules are *H*-supplemented

Throughout this section, R is a commutative ring.

Proposition 3.1. *Let R be a commutative ring. The following are equivalent:*

- (i) *R is artinian principal;*
- (ii) *Every R -module is \oplus -supplemented;*
- (iii) *Every R -module is H -supplemented.*

Proof. (i) \Leftrightarrow (ii) By [5, Theorem 1.1].

(i) \Rightarrow (iii) Since R is artinian, $M = K(M)$. Let m be a maximal ideal of R . By [12, Theorem 6.9], we have $K_m(M) \cong \bigoplus_{i \in I} R/m^{n_i}$ where $n_i (i \in I)$ are positive integers. Since R is artinian, there is a non-negative integer k for which $m^k = m^{k+1}$. Therefore $K_m(M)$ is H -supplemented by Theorem 2.9. Consequently, M is H -supplemented by Corollary 2.6.

(iii) \Rightarrow (ii) Clear. □

A family of sets is said to have the finite intersection property, abbreviated f.i.p., if the intersection of every finite subfamily is non-empty. Let M be an R -module. M is linearly compact if whenever $\{x_\alpha + M_\alpha\}_{\alpha \in X}$ is a family of cosets of submodules of M ($x_\alpha \in M$ and M_α is a submodule of M) with the f.i.p., then $\bigcap_{\alpha \in X} x_\alpha + M_\alpha \neq \emptyset$. One can translate this into a condition about solving congruences. With the above notation $x \in x_\alpha + M_\alpha$ if and only if $x \equiv x_\alpha \pmod{M_\alpha}$. Thus an R -module M is linearly compact if given any family of congruences $\{x \equiv x_\alpha \pmod{M_\alpha}\}_{\alpha \in X}$ of M , being able to find a solution for any finite subset of these congruences implies one can find a solution for all the congruences. R is said to be a *maximal* ring if R is linearly compact as R -module. R is called almost maximal if $\frac{R}{I}$ is a linearly compact R -module for all non-zero ideals I of R .

A commutative ring R is a valuation ring if it satisfies one of the following three equivalent conditions:

- (i) For any two elements a and b , either a divides b or b divides a ;
- (ii) The ideals of R are linearly ordered by inclusion;
- (iii) R is a local ring and every finitely generated ideal is principal.

Proposition 3.2. *The following conditions on a commutative ring R are equivalent:*

- (i) *Every finitely generated R -module is H -supplemented;*
- (ii) *Every finitely generated R -module is \oplus -supplemented;*
- (iii) *R is a finite product of almost maximal valuation rings.*

Proof. (i) \Rightarrow (ii) Clear.

(ii) \Rightarrow (iii) By [5, Proposition 1.4].

(iii) \Rightarrow (i) Suppose that $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$, where R_i is an almost maximal valuation ring. We can write $1_R = e_1 + e_2 + \cdots + e_n$, where e_i is the

identity element of the ring R_i and 1_R is the identity element of the ring R . Let M be a finitely generated R -module. Then $M = e_1M \oplus e_2M \oplus \cdots \oplus e_nM$. Let $1 \leq i \leq n$. Note that e_iM can be regarded as an R_i -module as well as an R -module, and its submodules are the same in both cases, because $(r_1 + r_2 + \cdots + r_n)e_ix = r_ie_ix$, where $r_j \in R_j$ for $1 \leq j \leq n$ and $x \in M$. Since R_i is an almost maximal valuation ring, then ${}_{R_i}(e_iM)$ is a finite direct sum of cyclic submodules by [1, Theorem 4.5]. Thus ${}_{R_i}(e_iM)$ is a finite direct sum of local submodules. Since R_i is a valuation ring, ${}_{R_i}(e_iM)$ is H -supplemented by [20, Satz 3.2]. It follows that ${}_R M$ is H -supplemented by Proposition 2.5. \square

According to [20, Satz 3.2] and Corollary 2.6, every finitely generated H -supplemented module over a commutative ring is a finite direct sum of local submodules. From [7, Corollary 6], it follows that every finitely generated H -supplemented module is completely \oplus -supplemented. In general a finitely generated completely \oplus -supplemented module need not be H -supplemented (see e.g. [9, Lemma A.4] and [7, Corollary 6]).

It was shown in [7, Proposition 6] that a direct sum of two hollow modules is always completely \oplus -supplemented.

A module M is called finitely presented if $M \cong \frac{F}{K}$ for some finitely generated free module F and finitely generated submodule K of F .

Proposition 3.3. *The following conditions are equivalent on a commutative local ring R :*

- (i) *Every finitely generated completely \oplus -supplemented module is H -supplemented;*
- (ii) *Every finitely presented module is \oplus -supplemented;*
- (iii) *Every finitely presented module is H -supplemented;*
- (iv) *R is a valuation ring.*

Proof. (i) \Rightarrow (iv) Let I and J be two ideals of R . By [7, Proposition 6], the module $M = R/I \oplus R/J$ is completely \oplus -supplemented. By hypothesis, M is H -supplemented. This gives $I \subseteq J$ or $J \subseteq I$ by [20, Satz 3.2]. Therefore R is a valuation ring.

(iv) \Rightarrow (i) Let M be a finitely generated completely \oplus -supplemented module. By [7, Proposition 11], $M = \bigoplus_{i=1}^k H_i$ is a direct sum of local submodules H_i ($1 \leq i \leq k$). Since R is a valuation ring, the ideals $\text{Ann}_R(H_i)$ ($1 \leq i \leq k$) are linearly ordered by inclusion. Thus M is H -supplemented by [20, Satz 3.2].

(iv) \Rightarrow (iii) Let M be a finitely presented R -module. By [14, Theorem 1], M is a finite direct sum of cyclic submodules. Since R is a valuation ring, M is H -supplemented by [20, Satz 3.2].

(iii) \Rightarrow (ii) Clear.

(ii) \Rightarrow (iv) By [5, Proposition 1.5]. \square

As in [9, p. 93], we call an ideal m -isolated if it is contained in at most one maximal ideal, m .

Proposition 3.4. *The following conditions are equivalent on a commutative ring R :*

- (i) Every finitely generated completely \oplus -supplemented module is *H-supplemented*;
(ii) For every maximal ideal m of R , the collection of m -isolated ideals of R is linearly ordered by inclusion.

Proof. (i) \Rightarrow (ii) Let m be a maximal ideal of R . Let I_1 and I_2 be two m -isolated ideals. Then the module $M = R/I_1 \oplus R/I_2$ is completely \oplus -supplemented by [7, Proposition 6]. By assumption, M is *H-supplemented*. Therefore $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$ by [20, Satz 3.2].

(ii) \Rightarrow (i) Let M be a finitely generated completely \oplus -supplemented module. By [7, Proposition 11], $M = \bigoplus_{i=1}^k R/I_i$ is a direct sum of local modules R/I_i ($1 \leq i \leq k$). It is clear that for every i ($1 \leq i \leq k$), there exists a maximal ideal m_i such that the ideal I_i is m_i -isolated. By Corollary 2.6 and [20, Satz 3.2], M is *H-supplemented*. \square

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