

Cohomological classification of Ann-functors

Nguyen Tien Quang and Dang Dinh Hanh

*Department of Mathematics
Hanoi National University of Education
Hanoi, Viet Nam
nguyenquang272002@gmail.com, ddhanhdhsphn@gmail.com*

Abstract

Regular Ann-functor classification problem has been solved with Shukla cohomology. In this paper, we would like to present a solution to the above problem in the general case and in the case of strong Ann-functors with, respectively, Mac Lane cohomology and Hochschild cohomology.

1 Introduction

The definition of *Ann-categories* was presented by N.T.Quang [7] in 1988, which is regarded as a categorization of the ring structure. Each Ann-category \mathcal{A} is Ann-equivalent to its reduced Ann-category. This Ann-category is of the type (R, M, h) , where R is a ring of congruence classes of objects of \mathcal{A} , $M = \text{Aut}(0)$ is the R -bimodule and h is a 3-cocycle in $Z_{MaL}^3(R, M)$ (due to Mac Lane [6]). Then, there exists a bijection between the set of congruence classes of Ann-categories of the type (R, M) and the cohomology group $H_{MaL}^3(R, M)$ (see [11]). For *regular Ann-categories* (whose the commutativity constraint satisfies $c_{X,X} = id$), in the above bijection, the group $H_{MaL}^3(R, M)$ is replaced with the Shukla cohomology group $H_{Sh}^3(R, M)$ [12].

In [4], M.Jibladze and T. Pirashvili presented the definition of *categorical rings* as a slightly modified version of the definition of Ann-categories and classified them by Mac Lane ring cohomology. However, in [9] authors have showed that, the fact that has not been proved the R -bimodule structure on M can be deduced from the axiomatics of categorical rings.

Key words: Ann-category, Ann-functor, classification, Mac Lane ring cohomology, Hochschild cohomology.

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The Ann-functor classification problem has been solved for regular Ann-categories with Shukla cohomology [1,8]. In this paper, we present a solution for this problem in the general case via low-dimensioned cohomology groups of Mac Lane ring cohomology. In particular, Hochschild algebra cohomology has been used to classify strong Ann-functors.

In this paper, for convenience, sometimes we denote by XY the tensor product of the two objects X and Y , instead of $X \otimes Y$.

2 Preliminaries

2.1 The basic concepts

The definition of Ann-categories was presented in [7, 9, 11]. We always suppose that \mathcal{A} is an Ann-category with a collection of constraints:

$$(a^+, c, (O, g, d), a, (I, l, r), \mathcal{L}, \mathcal{R}).$$

Definition 1. Let \mathcal{A} and \mathcal{A}' be Ann-categories. An Ann-functor from \mathcal{A} to \mathcal{A}' is a triple $(F, \check{F}, \tilde{F})$, where (F, \check{F}) is a symmetric monoidal functor respect to the operation \oplus , (F, \tilde{F}) is a monoidal functor respect to the operation \otimes , satisfies the two following commutative diagrams:

$$\begin{array}{ccccc}
 F(X(Y \oplus Z)) & \xleftarrow{\check{F}} & FX.F(Y \oplus Z) & \xleftarrow{id \otimes \check{F}} & FX(FY \oplus FZ) \\
 \downarrow F(L) & & & & \downarrow L' \\
 F(XY \oplus XZ) & \xleftarrow{\check{F}} & F(XY) \oplus F(XZ) & \xleftarrow{\check{F} \oplus \check{F}} & FX.FY \oplus FX.FZ \\
 & & & & \\
 F((X \oplus Y)Z) & \xleftarrow{\check{F}} & F(X \oplus Y).FZ & \xleftarrow{\check{F} \otimes id} & (FX \oplus FY).FZ \\
 \downarrow F(R) & & & & \downarrow R' \\
 F(XZ \oplus YZ) & \xleftarrow{\check{F}} & F(XZ) \oplus F(YZ) & \xleftarrow{\check{F} \oplus \check{F}} & FX.FZ \oplus FY.FZ
 \end{array}$$

The commutation of the above diagrams is called the compatibility of the functor F with the distributivity constraints of the two Ann-categories $\mathcal{A}, \mathcal{A}'$.

We call $\varphi : F \rightarrow G$ an Ann-morphism between two Ann-functors $(F, \check{F}, \tilde{F})$ and $(G, \check{G}, \tilde{G})$ if it is an \oplus -morphism as well as an \otimes -morphism.

An Ann-functor $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ is called an Ann-equivalence if there exists an Ann-functor $(F', \check{F}', \tilde{F}') : \mathcal{A}' \rightarrow \mathcal{A}$ and natural isomorphisms $\alpha : F \circ F' \cong id_{\mathcal{A}'}, \quad \alpha' : F' \circ F \cong id_{\mathcal{A}}$.

An Ann-functor $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ is an Ann-equivalence iff F is a categorical equivalence.

We have the following note: in the Definition 1, it is only required that (F, \tilde{F}) is an AC -functor (i.e. an \oplus -functor which is compatible with the associativity and commutativity constraints). Indeed, since $(\mathcal{A}, \oplus), (\mathcal{A}', \oplus)$ are Gr -categories, each A -functor is compatible with the unitivity constraints.

2.2 The third Mac Lane ring cohomology group $H_{MaL}^3(R, M)$

Let R be a ring and M be an R -bimodule. From the definition of Mac Lane ring cohomology [6], we may obtain the description of the elements of the cohomology group $H_{MaL}^3(R, M)$.

The group $Z_{MaL}^3(R, M)$ of 3-cochains of the ring R , with coefficients in the R -bimodules M , consisting of quadruples $(\sigma, \alpha, \lambda, \rho)$, the functions:

$$\alpha, \lambda, \rho : R^3 \rightarrow M$$

and $\sigma : R^4 \rightarrow M$ satisfy the following relations:

- M1. $x\alpha(y, z, t) - \alpha(xy, z, t) + \alpha(x, yz, t) - \alpha(x, y, zt) + \alpha(x, y, z)t = 0$
- M2. $\alpha(x, z, t) + \alpha(y, z, t) - \alpha(x+y, z, t) + \rho(xz, yz, t) - \rho(x, y, zt) + \rho(x, y, z)t = 0$
- M3. $-\alpha(x, y, t) - \alpha(x, z, t) + \alpha(x, y+z, t) + x\rho(y, z, t) - \rho(xy, xz, t) - \lambda(x, yt, zt) + \lambda(x, y, z)t = 0$
- M4. $\alpha(x, y, z) + \alpha(x, y, t) - \alpha(x, y, z+t) + x\lambda(y, z, t) - \lambda(xy, z, t) + \lambda(x, yz, yt) = 0$
- M5. $\lambda(x, z, t) + \lambda(y, z, t) - \lambda(x+y, z, t) + \rho(x, y, z) + \rho(x, y, t) - \rho(x, y, z+t) + \sigma(xz, xt, yz, yt) = 0$
- M6. $\lambda(x, a, b) + \lambda(x, c, d) - \lambda(x, a+c, b+d) - \lambda(x, a, c) - \lambda(x, b, d) + \lambda(x, a+b, c+d) - x\sigma(a, b, c, d) + \sigma(xa, xb, xc, xd) = 0$
- M7. $\rho(a, b, x) + \rho(c, d, x) - \rho(a+c, b+d, x) - \rho(a, c, x) - \rho(b, d, x) + \rho(a+b, c+d, x) - \sigma(ax, bx, cx, dx) + \sigma(a, b, c, d)x = 0$
- M8. $\sigma(a, b, c, d) + \sigma(x, y, z, t) - \sigma(a+x, b+y, c+z, d+t) + \sigma(a, b, x, y) + \sigma(c, d, z, t) - \sigma(a+c, b+d, x+z, y+t) + \sigma(a, c, x, z) + \sigma(b, d, y, t) - \sigma(a+b, c+d, x+y, z+t) = 0$
- M9. $\alpha(0, y, z) = \alpha(x, 0, z) = \alpha(x, y, 0) = 0$
- M10. $\sigma(0, 0, z, t) = \sigma(x, y, 0, 0) = \sigma(0, y, 0, t) = \sigma(x, 0, z, 0) = \sigma(x, 0, 0, t) = 0$.

The subgroup $B_{MaL}^3(R, M) \subset Z_{MaL}^3(R, M)$ of 3-coboundaries consists of the quadruples $(\sigma, \alpha, \lambda, \rho)$ such that there exist the maps $\mu, \nu : R^2 \rightarrow M$ satisfying:

- M11. $\sigma(x, y, z, t) = -\mu(x, y) - \mu(z, t) + \mu(x+z, y+t) + \mu(x, z) + \mu(y, t) - \mu(x+y, z+t)$
- M12. $\alpha(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z$
- M13. $\lambda(x, y, z) = \nu(x, y) + \nu(x, z) - \nu(x, y+z) + x\mu(y, z) - \mu(xy, xz)$

$$\text{M14. } \rho(x, y, z) = -\nu(x, z) - \nu(y, z) + \nu(x + y, z) + \mu(xz, yz) - \mu(x, y)z.$$

$$\text{Finally, } H_{MaL}^3(R, M) = Z_{MaL}^3(R, M)/B_{MaL}^3(R, M).$$

Each Ann-category \mathcal{I} of the type (R, M) has the *structure* f is a collection $f = (\xi, \eta, \alpha, \lambda, \rho)$, where $\xi, \alpha, \lambda, \rho : R^3 \rightarrow M$ and $\eta : R^2 \rightarrow M$ are functions satisfying 17 the equations (see Proposition 5.8 [11]). Now, we define a function $\sigma : R^4 \rightarrow M$, given by:

$$\sigma(x, y, z, t) = \xi(x + y, z, t) - \xi(x, y, z) + \eta(y, z) + \xi(x, z, y) - \xi(x + z, y, t).$$

This function is respect to the associativity-commutativity constraint v in the Ann-category \mathcal{A} , where

$$v = v_{X,Y,Z,T} : (X \oplus Y) \oplus (Z \oplus T) \longrightarrow (X \oplus Z) \oplus (Y \oplus T)$$

is given by the following commutative diagram:

$$\begin{array}{ccccc} (X \oplus Y) \oplus (Z \oplus T) & \xrightarrow{a_+} & ((X \oplus Y) \oplus Z) \oplus T & \xleftarrow{a_+ \oplus T} & (X \oplus (Y \oplus Z)) \oplus T \\ \downarrow v & & & & \downarrow (X \oplus c) \oplus T \\ (X \oplus Z) \oplus (Y \oplus T) & \xrightarrow{a_+} & ((X \oplus Z) \oplus Y) \oplus T & \xleftarrow{a_+ \oplus T} & (X \oplus (Z \oplus Y)) \oplus T \end{array}$$

The quadruple $h = (\sigma, \alpha, \lambda, \rho)$ is a 3-cocycle of the ring R with coefficients in the R -bimodule M due to Mac Lane (Theorem 7.2 [11]) and therefore each reduced Ann-category is of the form (R, M, h) .

3 An equivalence criterion of an Ann-functor

Firstly, we will show a characterized property of Ann-functors, which is related to the associativity-commutativity constraint v .

Definition 2. Let $\mathcal{A}, \mathcal{A}'$ be symmetric monoidal \oplus -categories. Then, the \oplus -functor $(F, \check{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ is called compatible with the constraints v, v' if the following diagram commutes for all $X, Y, Z, T \in \mathcal{A}$

$$\begin{array}{ccccccc} F((X \oplus Y) \oplus (Z \oplus T)) & \xleftarrow{\check{F}} & F(X \oplus Y) \oplus F(Z \oplus T) & \xleftarrow{\check{F} + \check{F}} & (FX \oplus FY) \oplus (FZ \oplus FT) \\ \downarrow F(v) & & & & \downarrow v' \\ F((X \oplus Z) \oplus (Y \oplus T)) & \xleftarrow{\check{F}} & F(X \oplus Z) \oplus F(Y \oplus T) & \xleftarrow{\check{F} + \check{F}} & (FX \oplus FZ) \oplus (FY \oplus FT) \end{array} \quad (1)$$

Then

Lemma 3.1. *Let \oplus -functor $(F, \check{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ be compatible with the unitivity constraints. Then (F, \check{F}) is an AC-functor iff it is compatible with the constraints v, v' .*

Proof. The necessary condition was presented by D. B. A. Epstein (Lemma 1.5 [2]).

Now, assume that the diagram (1) commutes. To prove that the pair (F, \check{F}) is compatible with the commutativity constraints, we consider the following Diagram 1.

In the Diagram 1, the region (I) commutes thanks to the naturality of the morphism v , the regions (II) and (IV) commute since (F, \check{F}) is compatible with the unitivity constraints, the regions (III) and (VII) commute by the coherence theorem in a symmetric monoidal category, the regions (VI) and (VIII) commute thanks to the naturality of \check{F} , the outside region commutes by the diagram (1). Hence, the region (V) commutes. So (F, \check{F}) is compatible with the commutativity constraints.

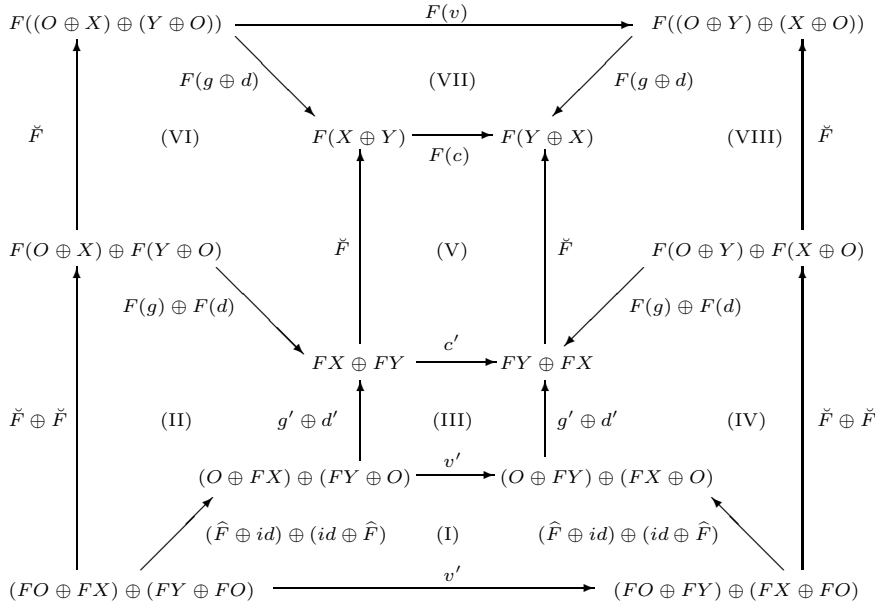
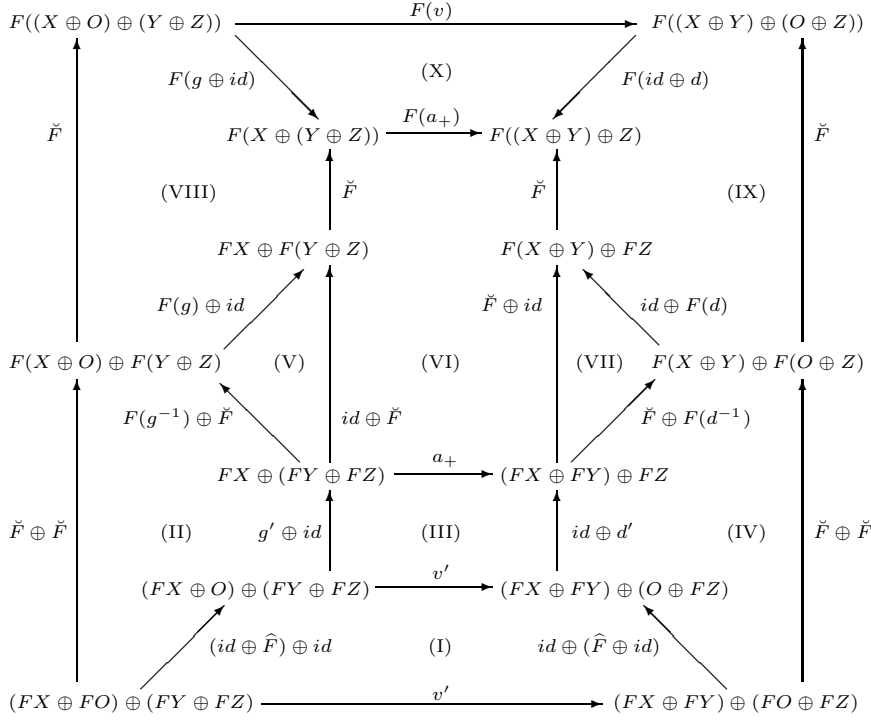


Diagram 1

Next, we consider the following Diagram 2.

In the Diagram 2, the region (I) commutes thanks to the naturality of the morphism v ; the first component of the region (II) commutes since (F, \check{F}) is

compatible with the unitivity constraints, the second one commutes thanks to the composition of morphisms, so the region (II) commutes; the regions (III) and (X) commute thanks to the coherence theorem in a symmetric monoidal category; the first component of the region (IV) commutes thanks to the composition of morphisms, the second one commutes since (F, \check{F}) is compatible with the unitivity constraints, so the region (IV) commutes; the region (V) and (VII) commute thanks to the composition of morphisms; the regions (VIII) and (IX) commute thanks to the naturality of \check{F} ; the outside region commutes thanks to the diagram (1). Therefore, the region (V) commutes, i.e., the pair (F, \check{F}) is compatible with the associativity constraints.



□

Proposition 3.2. *In the definition of Ann-functors, the condition that (F, \check{F}) is a symmetric monoidal \oplus -functor is equivalent to the two following conditions:*

- (a) (F, \check{F}) is compatible with the unitivity constraints respect to the operation \oplus ,
- (b) (F, \check{F}) is compatible with the constraints v, v' .

Proof. Directly deduced from Lemma 3.1. □

4 Ann-functors and the low-dimensional cohomology groups of rings due to Mac Lane

4.1 Ann-functors of the type (p, q)

Now, we will show that each Ann-functor $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ induces an Ann-functor \overline{F} on their reduced Ann-categories, and this correspondence is 1-1. Firstly, we have

Theorem 4.1 (Theorem 4.6 [11]). *Let \mathcal{A} and \mathcal{A}' be Ann-categories. Then, each Ann-functor $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ induces the ring homomorphism:*

$$F_0 : \begin{array}{ccc} \Pi_0(\mathcal{A}) & \rightarrow & \Pi_0(\mathcal{A}') \\ \text{cls}X & \mapsto & \text{cls}FX \end{array}$$

and the group homomorphism

$$F_1 : \begin{array}{ccc} \Pi_1(\mathcal{A}) & \rightarrow & \Pi_1(\mathcal{A}') \\ u & \mapsto & \gamma_{F_0}^{-1}(Fu) \end{array}$$

satisfying

$$F_1(su) = F_0(s)F_1(u); \quad F_1(us) = F_1(u)F_0(s).$$

Furthermore, F is a equivalence iff F_0, F_1 are isomorphisms.

The pair (F_0, F_1) is called *the pair of induced homomorphisms* of the Ann-functor $(F, \check{F}, \tilde{F})$. If $\mathcal{S}, \mathcal{S}'$ are, respectively, the reduced Ann-categories of $\mathcal{A}, \mathcal{A}'$, then the functor $\overline{F} : \mathcal{S} \rightarrow \mathcal{S}'$ given by

$$\overline{F}(s) = F_0(s), \quad \overline{F}(s, u) = (F_0s, F_1u)$$

is called the *reduced functor* of $(F, \check{F}, \tilde{F})$ on reduced Ann-categories.

Proposition 4.2. *Let \overline{F} be the induced functor of the Ann-functor $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$. Then the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ H \uparrow & & \downarrow G' \\ \mathcal{S} & \xrightarrow{\overline{F}} & \mathcal{S}' \end{array}$$

commutes, where H, G' are canonical Ann-equivalences (see Proposition 5.4[11]), and therefore \overline{F} induces an Ann-functor.

Proof. This Proposition is naturally extended from Proposition 2 [10]. □

Definition 3. Let $\mathcal{S} = (R, M, h)$, $\mathcal{S}' = (R', M', h')$ be Ann-categories. A functor $F : \mathcal{S} \rightarrow \mathcal{S}'$ is called a functor of the type (p, q) if

$$F(x) = p(x), \quad F(x, a) = (p(x), q(a)),$$

where $p : R \rightarrow R'$ is a ring homomorphism and $q : M \rightarrow M'$ is a group homomorphism satisfying

$$q(xa) = p(x)q(a), \quad q(ax) = q(a)p(x),$$

for $x \in R, a \in M$.

Proposition 4.3. Let $\mathcal{A} = (R, M, h)$, $\mathcal{A}' = (R', M', h')$ be Ann-categories and $(F, \check{F}, \tilde{F})$ is an Ann-functor from \mathcal{A} to \mathcal{A}' . Then, $(F, \check{F}, \tilde{F})$ is a functor of the type (p, q) .

Proof. For $x, y \in R$,

$$\check{F}_{x,y} : F(x) \oplus F(y) \rightarrow F(x \oplus y), \quad \tilde{F}_{x,y} : F(x) \otimes F(y) \rightarrow F(x \otimes y)$$

are morphisms in the Ann-category \mathcal{A}' . Hence, $F(x) + F(y) = F(x + y)$ and $F(x).F(y) = F(xy)$, so the map $p : R \rightarrow R'$ given by $p(x) = F(x)$ is a ring homomorphism.

Assume that $F(x, a) = (p(x), q_x(a))$. Since (F, \check{F}) is a Gr-functor, according to Theorem 5 [10], $q_x = q$ for all $x \in R$. Moreover, q is a group homomorphism:

$$q(a + b) = q(a) + q(b) \tag{2}$$

for all $a, b \in M$.

Since (F, \tilde{F}) is an \otimes -functor, the following diagram

$$\begin{array}{ccc} Fx \otimes Fy & \xrightarrow{\tilde{F}} & F(x \otimes y) \\ F((x,a) \otimes F((y,b))) \downarrow & & \downarrow F((x,a) \otimes (y,b)) \\ Fx \otimes Fy & \xrightarrow{\tilde{F}} & F(x \otimes y) \end{array}$$

commutes, for all morphisms $(x, a), (y, b)$. So, we have:

$$\begin{aligned} F((x, a) \otimes (y, b)) &= F(x, a) \otimes F(y, b) \\ \Leftrightarrow q_{xy}(ay + xb) &= q_x(a)F(y) + F(x)q_y(b) \end{aligned} \tag{3}$$

Applying $q_x = q_y = q_{xy} = q$ to the relation (3), we have:

$$q(ay + xb) = q(a)F(y) + F(x)q(b) \tag{4}$$

Applying $x = 1$ to (4), we have:

$$q(ay) = q(a)F(y) = q(a)p(y) \tag{5}$$

Applying $y = 1$ to (4), we have:

$$q(xb) = F(x)q(b) = p(x)q(b) \tag{6}$$

If R' -bimodule M' is regarded as an R -bimodule thanks to the actions $xa' = p(x).a', a'x = a'p(x)$, from the equations (2), (5), (6) we may show that $q : M \rightarrow M'$ is a homomorphism between R -bimodules. \square

4.2 Classification of Ann-functors

The existence problem of Ann-functors between Ann-categories has been solved for the regular Ann-categories (Theorem 5.1 [8], Theorem 4.2 [1]) thanks to Shukla cohomology. In this section, we will solve that problem in the general case.

Definition 4. *If $F : (R, M, h) \rightarrow (R', M', h')$ is a functor of the type (p, q) , then F induces 3-cocycles $h_* = q_*h = q(h)$, $h'^* = p^*h' = h'p$, for example*

$$\begin{aligned} \sigma'^*(x, y, z, t) &= \sigma'(p(x), p(y), p(z), p(t)) \\ \sigma_*(x, y, z, t) &= q(\sigma(x, y, z, t)). \end{aligned}$$

The function $k = p^*h' - q_*h$ is called an obstruction of the functor of the type (p, q) .

Then we have

Theorem 4.4. *The functor $F : (R, M, h) \rightarrow (R', M', h')$ of the type (p, q) is an Ann-functor iff the obstruction $\bar{k} = 0$ in $H^3_{M\alpha L}(R, M')$.*

Proof. Let $(F, \check{F}, \tilde{F}) : (R, M, h) \rightarrow (R', M', h')$ be an Ann-functor of the type (p, q) . Since $\check{F}_{x,y} = (\bullet, \mu(x, y))$, $\tilde{F}_{x,y} = (\bullet, \nu(x, y))$ where $\mu, \nu : R^2 \rightarrow M'$, we may identify \check{F}, \tilde{F} with μ, ν and call μ, ν the pair of associated functions with F, \tilde{F} . According to Lemma 3.1, $(F, \check{F}, \tilde{F})$ is compatible with the pair of constraints (v, v') , i.e., the diagram (1) commutes, so we have:

$$\begin{aligned} 7. \quad \sigma'^*(x, y, z, t) - \sigma_*(x, y, z, t) &= \mu(x, y) + \mu(z, t) - \mu(x + z, y + t) - \mu(x, z) \\ &\quad - \mu(y, t) + \mu(x + y, z + t) \end{aligned}$$

Since F is compatible with the associativity constraint of multiplication, the distributivity constraints of the Ann-categories \mathcal{A} and \mathcal{A}' , we have:

$$8. \quad \alpha'^*(x, y, z) - \alpha_*(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z$$

9. $\lambda'^*(x, y, z) - \lambda_*(x, y, z) = \nu(x, y+z) - \nu(x, y) - \nu(x, z) + x\mu(y, z) - \mu(xy, xz)$
 10. $\rho'^*(x, y, z) - \rho_*(x, y, z) = \nu(x+y, z) - \nu(x, z) - \nu(y, z) + \mu(x, y)z - \mu(xz, yz)$

From the equations (7) – (10), we have:

$$h'^* - h_* = \delta g \quad (11)$$

where $g = (-\mu, \nu)$. Hence the obstruction of the functor F vanishes in the cohomology group $H_{MaL}^3(R, M)$.

Conversely, assume that the obstruction of the functor F vanishes in the cohomology group $H_{MaL}^3(R, M')$. Then there exists a 2-cochain $g = (\mu, \nu)$ such that $h'^* - h_* = \delta g$. Take \check{F}, \tilde{F} be functor morphisms associated with the functions $-\mu, \nu$, we can verify that $(F, \check{F}, \tilde{F})$ is an Ann-functor. \square

Theorem 4.5. *If there exists an Ann-functor $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$, of the type (p, q) then:*

- (a) *There exists a bijection between the set of congruence classes of Ann-functors of the type (p, q) and the cohomology group $H_{MaL}^2(R, M')$ of the ring R with coefficients in R -bimodule M' .*
 (b) *There exists a bijection*

$$\text{Aut}(F) \rightarrow Z_{MaL}^1(R, M')$$

between the group of automorphisms of the Ann-functor F and the group $Z_{MaL}^1(R, M')$.

Proof. (a) Suppose that there exists $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$, which is an Ann-functor of the type (p, q) . According to Theorem 4.4, we have

$$\overline{h'^* - h_*} = 0.$$

Hence, there exists a 2-cochain k such that

$$h'^* - h_* = \delta k.$$

Let the 2-cochain k be fixed. Now, we assume that

$$(G, \check{G}, \tilde{G}) : (R, M, h) \rightarrow (R', M', h')$$

is an Ann-functor of the type (p, q) . Then, from the proof of the Theorem 4.4, we have

$$h'^* - h_* = \delta g.$$

Hence, $k - g$ is a 2-cocycle. Consider the correspondence:

$$\Phi : class(G) \mapsto class(k - g)$$

from the set of the congruence classes of Ann-functors of the type (p, q) to the group $H_{MaL}^2(R, M')$.

Firstly, we prove that the above correspondence is a map. Indeed, suppose that

$$(G', \check{G}', \tilde{G}') : (R, M, h) \rightarrow (R', M', h')$$

is also an Ann-functor of the type (p, q) and $u : G \rightarrow G'$ is an Ann-morphism. Since u is an \oplus -morphism as well as an \otimes -morphism, we have:

$$g' = g - \delta(u) \tag{12}$$

So

$$k - g' = k - g + \delta(u).$$

Thus $\overline{k - g} = \overline{k - g'} \in H_{MaL}^2(R, M')$.

Now, we prove that Φ is an injection. Assume that

$$(G, \check{G}, \tilde{G}), (G', \check{G}', \tilde{G}') : (R, M, h) \rightarrow (R', M', h')$$

are Ann-functors of the type (p, q) and satisfying

$$\overline{k - g} = \overline{k - g'} \in H_{MaL}^2(R, M').$$

Then, there exists an 1-cochain u such that

$$k - g = k - g' - \delta(u)$$

That means

$$g' = g - \delta(u).$$

Hence, the following diagrams:

$$\begin{array}{ccc} G(x) \oplus G(y) & \xrightarrow{\check{G}} & G(x \oplus y) & & G(x) \otimes G(y) & \xrightarrow{\check{G}} & G(x \otimes y) \\ \downarrow u_x \oplus u_y & & \downarrow u_{x \oplus y} & & \downarrow u_x \otimes u_y & & \downarrow u_{x \otimes y} \\ G'(x) \oplus G'(y) & \xrightarrow{\check{G}'} & G'(x \oplus y) & & G'(x) \otimes G'(y) & \xrightarrow{\check{G}' } & G'(x \otimes y) \end{array}$$

commute, it means that $u : G \rightarrow G'$ is an Ann-morphism. Therefore,

$$class(G) = class(G').$$

Finally, we must prove that the correspondence Φ is a surjection. Indeed, assume that g is an arbitrary 2-cocycle. We have:

$$\delta(k - g) = \delta k - \delta g = \delta k = h'^* - h_*.$$

Then, according to Theorem 4.4, there exists an Ann-functor

$$(G, \check{G}, \tilde{G}) : (R, M, h) \rightarrow (R', M', h')$$

of the type (p, q) , and the pair of isomorphisms \check{G}, \tilde{G} associated with the 2-cochain $k - g$.

Clearly, $\Phi(G) = \bar{g}$. So Φ is a surjection.

(b) Assume that $F = (F, \check{F}, \tilde{F}) : (R, M, h) \rightarrow (R', M', h')$ is an Ann-functor of the type (p, q) and $u \in \text{Aut}(F)$. Then, from the equation (12) with $g' = g$, we have $\delta(u) = 0$, i.e., $u \in Z_{MaL}^1(R, M')$. \square

5 Ann-functors and Hochschild cohomology

In this section, we will consider special Ann-functors which are related to the low-dimensional Hochschild groups [3].

Following, we will find a condition for the existence of Ann-functors

$$F = (F, id, \tilde{F}) : (R, M, h) \rightarrow (R', M', h')$$

of the type $(p, 0)$, where $p : R \rightarrow R'$ is a ring homomorphism.

Suppose that there exists an Ann-functor

$$F = (F, id, \tilde{F} = \nu) : (R, M, h) \rightarrow (R', M', h')$$

of the type $(p, 0)$. Then, the equations (7) - (10) turn into:

13. $\sigma'^*(x, y, z, t) = 0$
14. $\alpha'^*(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z$
15. $\lambda'^*(x, y, z) = \nu(x, y + z) - \nu(x, y) - \nu(x, z)$
16. $\rho'^*(x, y, z) = \nu(x + y, z) - \nu(x, z) - \nu(y, z)$

and the Theorem 4.4 turns into

Corollary 5.1. *Let $p : R \rightarrow R'$ be a ring homomorphism. There exists an Ann-functor (F, id, \tilde{F}) from (R, M, h) to (R', M', h') of the type $(p, 0)$ iff $\bar{h}^* = 0 \in H_{MaL}^3(R, M')$.*

Each cocycle of \mathbb{Z} -algebras due to Hochschild is a multi-linear function. This suggests us the following definition:

Definition 5. *An Ann-functor*

$$(F, id, \tilde{F}) : (R, M, h) \rightarrow (R', M', h')$$

of the type $(p, 0)$ is called a **strong Ann-functor** if the function $\nu : R^2 \rightarrow M'$ corresponding to \tilde{F} is bi-additive.

If ν is a normal bi-additive function, then ν is a 2-cocycle of the \mathbb{Z} -algebra R with coefficients in R -bimodule M' due to Hochschild. Then, in the equations (13)-(16), α'^* is a normal multi-linear function and other functions are equal to 0. So, we have

$$h'^* \equiv \alpha'^* = \delta(\nu),$$

in which $\delta(\nu)$ is a 3-coboundary of the ring R with coefficients in R -bimodule M' due to Hochschild. Then, we have the following proposition, as a direct corollary of Theorem 4.4.

Proposition 5.2. *Let $F : (R, M, h) \rightarrow (R', M', h')$ be a functor of the type $(p, 0)$. There exists a strong Ann-functor (F, id, \tilde{F}) iff its cohomology class $\overline{h'^*} = 0$ in the cohomology group $H_{Hochs}^3(R, M')$.*

Theorem 5.3. *If there exists a strong Ann-functor $(F, id, \tilde{F}) : (R, M, h) \rightarrow (R', M', h')$ of the type $(p, 0)$, then:*

- (a) *There exists a bijection between the set of congruence classes of strong Ann-functors of the type $(p, 0)$ and the cohomology group $H_{Hochs}^2(R, M')$ of the ring R with coefficients in the R -bimodule M' .*
- (b) *There exists a bijection*

$$Aut(F) \rightarrow Z_{Hochs}^1(R, M')$$

between the group of automorphisms of the Ann-functor F and the group $Z_{Hochs}^1(R, M')$.

Proof. (a) The restriction Φ_H of the map Φ , referred in Theorem 4.5, on the set of congruence classes of strong Ann-functors gives us an injection to the group $H_{Hochs}^2(R, M')$. Moreover, it is easy to see that Φ_H is also a surjection.

(b) Assume that $F = (F, id, \tilde{F}) : (R, M, h) \rightarrow (R', M', h')$ is a strong Ann-functor of the type $(p, 0)$ and $u \in Aut(F)$. Then u is bi-linear respect to the \oplus . So $u \in Z_{Hochs}^1(R, M')$. The converse also holds. \square

6 Application

In this section, as an application, we show the relation between the ring extension problem and the obstruction theory of Ann-functors.

We have known that, for a ring A (not necessary to have the unit) determined a *ring of bimultiplication* M_A . A bimultiplication σ of the ring A is a pair of maps $a \rightarrow \sigma a$, $a \rightarrow a\sigma$ from A to A , which satisfy:

$$\begin{aligned} \sigma(a+b) &= \sigma a + \sigma b & ; & & (a+b)\sigma &= a\sigma + b\sigma \\ \sigma(ab) &= (\sigma a)b & ; & & (ab)\sigma &= a(b\sigma) & ; & & a(\sigma b) = (a\sigma)b \end{aligned}$$

for all $a, b \in A$. The sum and the product of two bimultiplication are, respectively, the addition and the composition of maps .

Each element $c \in A$ induces a bimultiplication μ_c determined by

$$\mu_c a = ca; \quad a\mu_c = ac, \quad \forall a \in A$$

and it is called an *inner bimultiplication of A*. The map: $\mu : A \rightarrow M_A$ is a ring homomorphism and if A has the unit 1 then $\mu(1) = 1$. Since:

$$\sigma\mu_c = \mu_{\sigma c}; \quad \mu_c\sigma = \mu_{c\sigma}$$

the image μA of this homomorphism is a two-sided ideal of M_A . Let us denote $P_A = M_A/\mu(A)$.

Let R be a ring with the unit $1 \neq 0$. Each ring extension of A by R induces a regular ring homomorphism

$$\theta : R \rightarrow P_A$$

i.e., $\theta(1) = 1$ and any two elements of θR are permutable [The bimultiplications σ and τ are called *permutable* if $\varphi(a\psi) = (\varphi a)\psi$ and $\psi(a\varphi) = (\psi a)\varphi$ for all $a \in A$].

Inversely, according to [5], each regular ring homomorphism $\theta : R \rightarrow P_A$ induces a R -bimodule structure on *the bicenter* $C_A = \{c \in A | ca = ac = 0, \forall a \in A\}$, with the operators

$$xc = (\varphi x)c, cx = c(\varphi x); c \in C_A, x \in A, \varphi x \in \theta x.$$

The *obstruction* $\bar{h} \in H_{MacL}^3(R, C_A)$ of the homomorphism θ is defined as follows. For each $x \in R$, let us choose $\varphi(x) \in \theta(x)$. Then there exists the factor set $f, g : R^2 \rightarrow A$ satisfying:

$$\begin{aligned} \mu_{f(x,y)} &= \sigma(x+y) - \sigma(x) - \sigma(y) \\ \mu_{g(x,y)} &= \sigma(xy) - \sigma(x)\sigma(y), \end{aligned}$$

for $x, y \in R$.

From the association, commutation, distribution on properties of the ring A the collection of functions whose values belong to C_A has the formal type $h = (\xi, \eta, \alpha, \lambda, \rho) = \delta(f, g)$. The collection h is an element in $Z^3_{MacL}(R, C_A)$, and its cohomology class is called *the obstruction of the homomorphism θ* .

Now, we present the ring extension problem by the language of Ann-functors. Consider the category \mathcal{M}_A whose objects are elements of the ring M_A , and if φ, λ are bimultiplications of A , then let us denote:

$$Hom(\varphi, \lambda) = \{c \in A \mid \lambda = \mu_c + \varphi\}.$$

The composition of two arrows is the operation $+$ in A . The operations \oplus and \otimes are given by:

$$\begin{aligned} \varphi \oplus \lambda &= \varphi + \lambda, & \varphi, \lambda \in \mathcal{M}_A \\ c \oplus d &= c + d, & c, d \in A \\ \varphi \otimes \lambda &= \varphi \circ \lambda, & (\text{the composition of two maps}) \\ c \otimes d &= cd + c\lambda + \varphi d, & \text{where } c : \varphi \rightarrow \varphi', d : \lambda \rightarrow \lambda' \end{aligned}$$

With these two operations, \mathcal{M}_A becomes an Ann-category with the constraints naturally determined to be strict.

We regard the ring R as an Ann-category of the type $(R, 0, id)$. We call \mathcal{S} the reduced Ann-category of \mathcal{M}_A . The regular homomorphism $\theta : R \rightarrow P_A$ determines a functor

$$(\theta, 0) : (R, 0, id) \rightarrow \mathcal{S} = (\Pi_0, \Pi_1, k)$$

The obstruction of this functor is the element

$$\overline{k^*} \in H^3_{MacL}(R, \Pi_1), \quad k^* = \theta^*(k).$$

Now we have the following Proposition:

Proposition 6.1. *Let $\mathcal{S} = (\Pi_0, \Pi_1, k)$ be the reduced Ann-category of the strict Ann-category \mathcal{M}_A . Then:*

- (a) $\Pi_0 = P_A = M_A/\mu A$, $\Pi_1 = C_A$,
- (b) $k^* = \theta^*(k)$ belongs to the same cohomology class of h of the homomorphism θ .

Proof. (a) Obviously.

(b) The proof is completely similar to the proof of Proposition 8 [10] with appropriate supplement. We have the normal Ann-equivalence $(H, \check{H}, \tilde{H})$ from the reduced Ann-category \mathcal{S} to the Ann-category \mathcal{M}_A . Since the obstructions \overline{h} of θ are independent of the choice of the function φ , we will choose $\varphi = H \circ \theta : R \rightarrow \mathcal{M}_A$. Then we can take

$$f(x, y) = \check{H}_{\theta x, \theta y}, \quad g(x, y) = \tilde{H}_{\theta x, \theta y}$$

From the compatibility of $(H, \check{H}, \tilde{H})$ with the constraints, $\overline{k^*} = \overline{h}$. □

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