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Cohomological classification of Ann-functors

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Abstract

Regular Ann-functor classification problem has been solved with Shukla cohomology. In this paper, we would like to present a solution to the above problem in the general case and in the case of strong Ann-functors with, respectively, Mac Lane cohomology and Hochschild cohomology.

1 Introduction

The definition of *Ann-categories* was presented by N.T.Quang [7] in 1988, which is regarded as a categorization of the ring structure. Each Ann-category $\mathcal{A}$ is Ann-equivalent to its reduced Ann-category. This Ann-category is of the type (R, M, h), where R is a ring of congruence classes of objects of $\mathcal{A}$, $M = Aut(0)$ is the R-bimodule and h is a 3-cocycle in $Z^3_{MaL}(R, M)$ (due to Mac Lane [6]). Then, there exists a bijection between the set of congruence classes of Ann-categories of the type (R, M) and the cohomology group $H^3_{MaL}(R, M)$ (see [11]). For *regular Ann-categories* (whose the commutativity constraint satisfies $c_{X,X} = id$), in the above bijection, the group $H^3_{MaL}(R, M)$ is replaced with the Shukla cohomology group $H^3_{Sh}(R, M)$ [12].

In [4], M.Jibladze and T. Pirashvili presented the definition of *categorical rings* as a slightly modified version of the definition of Ann-categories and classified them by Mac Lane ring cohomology. However, in [9] authors have showed that, the fact that has not been proved the R-bimodule structure on M can be deduced from the axiomatics of categorical rings.

Key words: Ann-category, Ann-functor, classification, Mac Lane ring cohomology, Hochschild cohomology.
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The Ann-functor classification problem has been solved for regular Ann-categories with Shukla cohomology [1,8]. In this paper, we present a solution for this problem in the general case via low-dimensioned cohomology groups of Mac Lane ring cohomology. In particular, Hochschild algebra cohomology has been used to classify strong Ann-functors.

In this paper, for convenience, sometimes we denote by XY the tensor product of the two objects X and Y, instead of $X \otimes Y$.

2 Preliminaries

2.1 The basic concepts

The definition of Ann-categories was presented in [7, 9, 11]. We always suppose that $\mathcal{A}$ is an Ann-category with a collection of constraints:

$$(a^+, c, (O, g, d), a, (I, l, r), \mathcal{L}, \mathcal{R}).$$

Definition 1. *Let $\mathcal{A}$ and $\mathcal{A}'$ be Ann-categories. An Ann-functor from $\mathcal{A}$ to $\mathcal{A}'$ is a triple $(F, \breve{F}, \widetilde{F})$, where $(F, \breve{F})$ is a symmetric monoidal functor respect to the operation $\oplus$, $(F, \widetilde{F})$ is a monoidal functor respect to the operation $\otimes$, satisfies the two following commutative diagrams:*

$$\begin{array}{ccccc}
F(X(Y \oplus Z)) & \xleftarrow{\widetilde{F}} & FX.F(Y \oplus Z) & \xleftarrow{id \otimes \breve{F}} & FX(FY \oplus FZ) \\
\downarrow{\scriptstyle F(L)} & & & & \downarrow{\scriptstyle L'} \\
F(XY \oplus XZ) & \xleftarrow{\breve{F}} & F(XY) \oplus F(XZ) & \xleftarrow{\widetilde{F} \oplus \widetilde{F}} & FX.FY \oplus FX.FZ
\end{array}$$

$$\begin{array}{ccccc}
F((X \oplus Y)Z) & \xleftarrow{\widetilde{F}} & F(X \oplus Y).FZ & \xleftarrow{\breve{F} \otimes id} & (FX \oplus FY).FZ \\
\downarrow{\scriptstyle F(R)} & & & & \downarrow{\scriptstyle R'} \\
F(XZ \oplus YZ) & \xleftarrow{\breve{F}} & F(XZ) \oplus F(YZ) & \xleftarrow{\widetilde{F} \oplus \widetilde{F}} & FX.FZ \oplus FY.FZ
\end{array}$$

The commutation of the above diagrams is called the compatibility of the functor F with the distributivity constraints of the two Ann-categories $\mathcal{A}, \mathcal{A}'$.

We call $\varphi : F \to G$ an Ann-morphism *between two Ann-functors $(F, \breve{F}, \widetilde{F})$ and $(G, \breve{G}, \widetilde{G})$ if it is an $\oplus$-morphism as well as an $\otimes$-morphism.*

An Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ is called an Ann-equivalence *if there exists an Ann-functor $(F', \breve{F}', \widetilde{F}') : \mathcal{A}' \to \mathcal{A}$ and natural isomorphisms $\alpha : F \circ F' \cong id_{\mathcal{A}'}$, $\quad \alpha' : F' \circ F \cong id_{\mathcal{A}}$.*

An Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ is an Ann-equivalence iff F is a categorical equivalence.

We have the following note: in the Definition 1, it is only required that $(F, \breve{F})$ is an AC-functor (i.e. an $\oplus$-functor which is compatible with the associativity and commutativity constraints). Indeed, since $(\mathcal{A}, \oplus), (\mathcal{A}', \oplus)$ are $Gr-$categories, each A-functor is compatible with the unitivity constraints.

2.2 The third Mac Lane ring cohomology group $H^3_{MaL}(R, M)$

Let R be a ring and M be an R-bimodule. From the definition of Mac Lane ring cohomology [6], we may obtain the description of the elements of the cohomology group $H^3_{MaL}(R, M)$.

The group $Z^3_{MaL}(R, M)$ of 3-cochains of the ring R, with coefficients in the R-bimodules M, consisting of quadruples $(\sigma, \alpha, \lambda, \rho)$, the functions:

$$\alpha, \lambda, \rho : R^3 \to M$$

and $\sigma : R^4 \to M$ satisfy the following relations:

M1. $x\alpha(y, z, t) - \alpha(xy, z, t) + \alpha(x, yz, t) - \alpha(x, y, zt) + \alpha(x, y, z)t = 0$

M2. $\alpha(x, z, t)+\alpha(y, z, t)-\alpha(x+y, z, t)+\rho(xz, yz, t)-\rho(x, y, zt)+\rho(x, y, z)t = 0$

M3. $-\alpha(x, y, t) - \alpha(x, z, t) + \alpha(x, y + z, t) + x\rho(y, z, t) - \rho(xy, xz, t)$
$- \lambda(x, yt, zt) + \lambda(x, y, z)t = 0$

M4. $\alpha(x, y, z)+\alpha(x, y, t)-\alpha(x, y, z+t)+x\lambda(y, z, t)-\lambda(xy, z, t)+\lambda(x, yz, yt) = 0$

M5. $\lambda(x, z, t) + \lambda(y, z, t) - \lambda(x + y, z, t) + \rho(x, y, z) + \rho(x, y, t) - \rho(x, y, z + t)$
$+ \sigma(xz, xt, yz, yt) = 0$

M6. $\lambda(x, a, b) + \lambda(x, c, d) - \lambda(x, a + c, b + d) - \lambda(x, a, c) - \lambda(x, b, d)$
$+ \lambda(x, a + b, c + d) - x\sigma(a, b, c, d) + \sigma(xa, xb, xc, xd) = 0$

M7. $\rho(a, b, x) + \rho(c, d, x) - \rho(a + c, b + d, x) - \rho(a, c, x) - \rho(b, d, x)$
$+ \rho(a + b, c + d, x) - \sigma(ax, bx, cx, dx) + \sigma(a, b, c, d)x = 0$

M8. $\sigma(a, b, c, d)+\sigma(x, y, z, t)-\sigma(a+x, b+y, c+z, d+t)+\sigma(a, b, x, y)+\sigma(c, d, z, t)$
$-\sigma(a+c, b+d, x+z, y+t)+\sigma(a, c, x, z)+\sigma(b, d, y, t)-\sigma(a+b, c+d, x+y, z+t) = 0$

M9. $\alpha(0, y, z) = \alpha(x, 0, z) = \alpha(x, y, 0) = 0$

M10. $\sigma(0, 0, z, t) = \sigma(x, y, 0, 0) = \sigma(0, y, 0, t) = \sigma(x, 0, z, 0) = \sigma(x, 0, 0, t) = 0.$

The subgroup $B^3_{MaL}(R, M) \subset Z^3_{MaL}(R, M)$ of 3-coboundaries consists of the quadruples $(\sigma, \alpha, \lambda, \rho)$ such that there exist the maps $\mu, \nu : R^2 \to M$ satisfying:

M11. $\sigma(x, y, z, t) = -\mu(x, y) - \mu(z, t) + \mu(x + z, y + t) + \mu(x, z) + \mu(y, t)$
$-\mu(x + y, z + t)$

M12. $\alpha(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z$

M13. $\lambda(x, y, z) = \nu(x, y) + \nu(x, z) - \nu(x, y + z) + x\mu(y, z) - \mu(xy, xz)$

M14. $\rho(x,y,z) = -\nu(x,z) - \nu(y,z) + \nu(x+y,z) + \mu(xz,yz) - \mu(x,y)z.$

Finally, $H^3_{MaL}(R,M) = Z^3_{MaL}(R,M)/B^3_{MaL}(R,M)$.

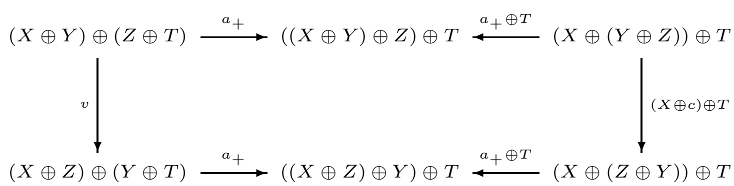
Each Ann-category $\mathcal{I}$ of the type (R,M) has the *structure* f is a collection $f = (\xi,\eta,\alpha,\lambda,\rho)$, where $\xi,\alpha,\lambda,\rho : R^3 \to M$ and $\eta : R^2 \to M$ are functions satisfying 17 the equations (see Proposition 5.8 [11]). Now, we define a function $\sigma : R^4 \to M$, given by:

$$\sigma(x,y,z,t) = \xi(x+y,z,t) - \xi(x,y,z) + \eta(y,z) + \xi(x,z,y) - \xi(x+z,y,t).$$

This function is respect to the associativity-commutativity constraint v in the Ann-category $\mathcal{A}$, where

$$v = v_{X,Y,Z,T} : (X \oplus Y) \oplus (Z \oplus T) \longrightarrow (X \oplus Z) \oplus (Y \oplus T)$$

is given by the following commutative diagram:

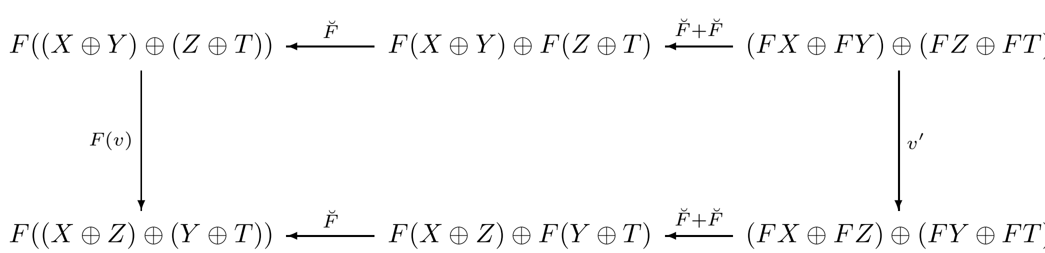


The quadruple $h = (\sigma,\alpha,\lambda,\rho)$ is a 3-cocycle of the ring R with coefficients in the R-bimodule M due to Mac Lane (Theorem 7.2 [11]) and therefore each reduced Ann-category is of the form (R,M,h).

3 An equivalence criterion of an Ann-functor

Firstly, we will show a characterized property of Ann-functors, which is related to the associativity-commutativity constraint v.

Definition 2. *Let $\mathcal{A}$, $\mathcal{A}'$ be symmetric monoidal $\oplus$-categories. Then, the $\oplus$-functor $(F,\breve{F}) : \mathcal{A} \to \mathcal{A}'$ is called compatible with the constraints v, v' if the following diagram commutes for all $X,Y,Z,T \in \mathcal{A}$*



(1)

Then

Lemma 3.1. *Let $\oplus-$functor $(F, \breve{F}) : \mathcal{A} \to \mathcal{A}'$ be compatible with the unitivity constraints. Then $(F, \breve{F})$ is an $AC-$functor iff it is compatible with the constraints $v, v^{'}$.*

Proof. The nescessary condition was presented by D. B. A. Epstein (Lemma 1.5 [2]).

Now, assume that the diagram (1) commutes. To prove that the pair $(F, \breve{F})$ is compatible with the commutativity constraints, we consider the following Diagram 1.

In the Diagram 1, the region (I) commutes thanks to the naturality of the morphism v, the regions (II) and (IV) commute since $(F, \breve{F})$ is compatible with the unitivity constraints, the regions (III) and (VII) commute by the coherence theorem in a symmetric monoidal category, the regions (VI) and (VIII) commute thanks to the naturality of $\breve{F}$, the outside region commutes by the diagram (1). Hence, the region (V) commutes. So $(F, \breve{F})$ is compatible with the commutativity constraints.

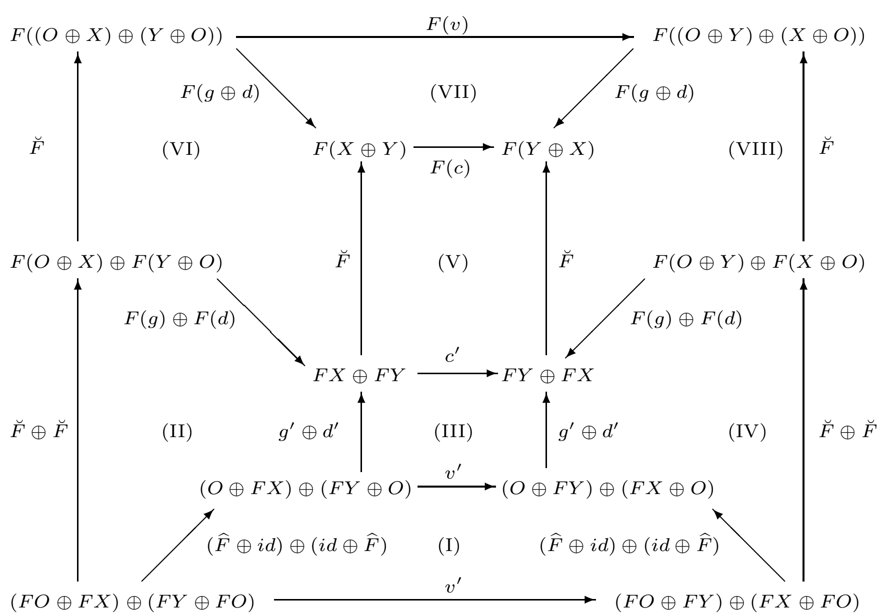


Diagram 1

Next, we consider the following Diagram 2.

In the Diagram 2, the region (I) commutes thanks to the naturality of the morphism v; the first component of the region (II) commutes since $(F, \breve{F})$ is compatible with the unitivity constraints, the second one commutes thanks to the composition of morphisms, so the region (II) commutes; the regions (III) and (X) commute thanks to the coherence theorem in a symmetric monoidal category; the first component of the region (IV) commutes thanks to the composition of morphisms, the second one commutes since $(F, \breve{F})$ is compatible with the unitivity constraints, so the region (IV) commutes; the region (V) and (VII) commute thanks to the composition of morphisms; the regions (VIII) and (IX) commute thanks to the naturality of $\breve{F}$; the outside region commutes thanks to the diagram (1). Therefore, the region (V) commutes, i.e., the pair $(F, \breve{F})$ is compatible with the associtivity constraints.

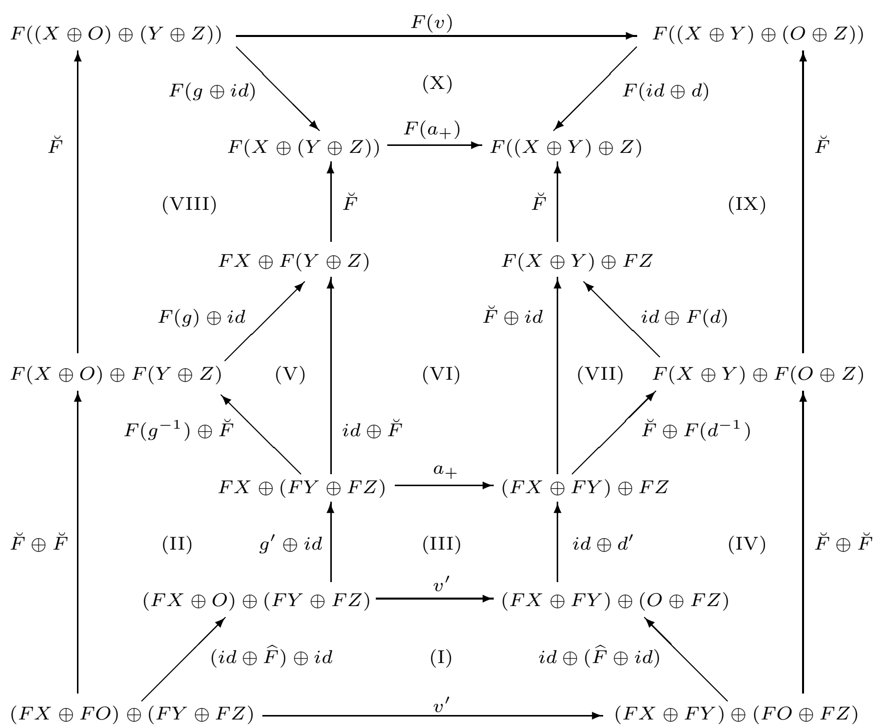


Diagram 2

□

Proposition 3.2. *In the definition of Ann-functors, the condition that $(F, \breve{F})$ is a symmetric monoidal $\oplus$-functor is equivalent to the two following conditions:*

(a) $(F, \breve{F})$ *is compatible with the unitivity constraints respect to the operation* $\oplus$,

(b) $(F, \breve{F})$ *is compatible with the constraints* $v, v^{'}$.

Proof. Directly deduced from Lemma 3.1. □

4 Ann-functors and the low-dimensioned cohomology groups of rings due to Mac Lane

4.1 Ann-functors of the type (p, q)

Now, we will show that each Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}^{'}$ induces an Ann-functor $\overline{F}$ on their reduced Ann-categories, and this correspondence is 1-1. Firstly, we have

Theorem 4.1 (Theorem 4.6 [11]). *Let $\mathcal{A}$ and $\mathcal{A}'$ be Ann-categories. Then, each Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ induces the ring homomorphism:*

$$\begin{array}{rccc} F_0 : & \Pi_0(\mathcal{A}) & \to & \Pi_0(\mathcal{A}') \\ & clsX & \mapsto & clsFX \end{array}$$

and the group homomorphism

$$\begin{array}{lcc} F_1 : \Pi_1(\mathcal{A}) & \to & \Pi_1(\mathcal{A}') \\ \quad u & \mapsto & \gamma_{F0}^{-1}(Fu) \end{array}$$

satisfying

$$F_1(su) = F_0(s)F_1(u); \qquad F_1(us) = F_1(u)F_0(s).$$

Furthermore, F is a equivalence iff F_0, F_1 are isomorphisms.

The pair (F_0, F_1) is called *the pair of induced homomorphisms* of the $Ann-$functor $(F, \breve{F}, \widetilde{F})$. If $\mathcal{S}, \mathcal{S}'$ are, respectively, the reduced $Ann-$categories of $\mathcal{A}, \mathcal{A}'$, then the functor $\overline{F} : \mathcal{S} \to \mathcal{S}'$ given by

$$\overline{F}(s) = F_0(s), \ \overline{F}(s, u) = (F_0 s, F_1 u)$$

is called the *reduced functor* of $(F, \breve{F}, \widetilde{F})$ on reduced $Ann-$categories.

Proposition 4.2. *Let $\overline{F}$ be the induced functor of the $Ann-$functor $(F, \breve{F}, \widetilde{F})$: $\mathcal{A} \to \mathcal{A}'$. Then the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ {\scriptstyle H}\uparrow & & \downarrow{\scriptstyle G'} \\ \mathcal{S} & \xrightarrow{\overline{F}} & \mathcal{S}' \end{array}$$

commutes, where H, G' are canonical Ann-equivalences (see Proposition 5.4[11]), and therefore $\overline{F}$ induces an Ann-functor.

Proof. This Proposition is naturally extended from Proposition 2 [10]. □

Definition 3. *Let $\mathcal{S} = (R, M, h)$, $\mathcal{S}' = (R', M', h')$ be Ann−categories. A functor $F : \mathcal{S} \to \mathcal{S}'$ is called a functor of the type (p, q) if*

$$F(x) = p(x), \qquad F(x, a) = (p(x), q(a)),$$

where $p : R \to R'$ is a ring homomorphism and $q : M \to M'$ is a group homomorphism satisfying

$$q(xa) = p(x)q(a), q(ax) = q(a)p(x),$$

for $x \in R, a \in M$.

Proposition 4.3. *Let $\mathcal{A} = (R, M, h)$, $\mathcal{A}' = (R', M', h')$ be Ann-categories and $(F, \breve{F}, \widetilde{F})$ is an Ann-functor from $\mathcal{A}$ to $\mathcal{A}'$. Then, $(F, \breve{F}, \widetilde{F})$ is a functor of the type (p, q).*

Proof. For $x, y \in R$,

$$\breve{F}_{x,y} : F(x) \oplus F(y) \to F(x \oplus y), \widetilde{F}_{x,y} : F(x) \otimes F(y) \to F(x \otimes y)$$

are morphisms in the Ann-category $\mathcal{A}'$. Hence, $F(x) + F(y) = F(x + y)$ and $F(x).F(y) = F(xy)$, so the map $p : R \to R'$ given by $p(x) = F(x)$ is a ring homomorphism.

Assume that $F(x, a) = (p(x), q_x(a))$. Since $(F, \breve{F})$ is a Gr-functor, according to Theorem 5 [10], $q_x = q$ for all $x \in R$. Moreover, q is a group homomorphism:

$$q(a + b) = q(a) + q(b) \tag{2}$$

for all $a, b \in M$.

Since $(F, \widetilde{F})$ is an $\otimes$-functor, the following diagram

$$\begin{array}{ccc} Fx \otimes Fy & \xrightarrow{\widetilde{F}} & F(x \otimes y) \\ {\scriptstyle F((x,a))\otimes F((y,b))}\downarrow & & \downarrow{\scriptstyle F((x,a)\otimes(y,b))} \\ Fx \otimes Fy & \xrightarrow{\widetilde{F}} & F(x \otimes y) \end{array}$$

commutes, for all morphisms $(x, a), (y, b)$. So, we have:

$$\begin{aligned} &F((x, a) \otimes (y, b)) = F(x, a) \otimes F(y, b) \\ \Leftrightarrow\ &q_{xy}(ay + xb) = q_x(a)F(y) + F(x)q_y(b) \end{aligned} \tag{3}$$

Applying $q_x = q_y = q_{xy} = q$ to the relation (3), we have:

$$q(ay + xb) = q(a)F(y) + F(x)q(b) \tag{4}$$

Applying $x = 1$ to (4), we have:

$$q(ay) = q(a)F(y) = q(a)p(y) \tag{5}$$

Applying $y = 1$ to (4), we have:

$$q(xb) = F(x)q(b) = p(x)q(b) \tag{6}$$

If R'-bimodule M' is regarded as an R-bimodule thanks to the actions $xa' = p(x).a', a'x = a'p(x)$, from the equations (2), (5), (6) we may show that $q : M \to M'$ is a homomorphism between R-bimodules. □

4.2 Classification of Ann-functors

The existence problem of Ann-functors between Ann-categories has been solved for the regular Ann-categories (Theorem 5.1 [8], Theorem 4.2 [1]) thanks to Shukla cohomology. In this section, we will solve that problem in the general case.

Definition 4. *If $F : (R, M, h) \to (R', M', h')$ is a functor of the type (p, q), then F induces 3-cocycles $h_* = q_*h = q(h),\ h'^* = p^*h' = h'p$, for example*

$$\begin{aligned} \sigma'^*(x, y, z, t) &= \sigma'(p(x), p(y), p(z), p(t)) \\ \sigma_*(x, y, z, t) &= q(\sigma(x, y, z, t)). \end{aligned}$$

The function $k = p^*h' - q_*h$ is called an obstruction of the functor of the type (p, q).

Then we have

Theorem 4.4. *The functor $F : (R, M, h) \to (R', M', h')$ of the type (p, q) is an Ann−functor iff the obstruction $\overline{k} = 0$ in $H^3_{MaL}(R, M')$.*

Proof. Let $(F, \breve{F}, \widetilde{F}) : (R, M, h) \to (R', M', h')$ be an *Ann*−functor of the type (p, q). Since $\breve{F}_{x,y} = (\bullet, \mu(x, y))$, $\widetilde{F}_{x,y} = (\bullet, \nu(x, y))$ where $\mu, \nu : R^2 \to M'$, we may identify $\breve{F}, \widetilde{F}$ with μ, ν and call μ, ν the *pair of associated functions* with $\breve{F}, \widetilde{F}$. According to Lemma 3.1, $(F, \breve{F}, \widetilde{F})$ is compatible with the pair of constraints (v, v'), i.e., the diagram (1) commutes, so we have:

7. $\sigma'^*(x, y, z, t) - \sigma_*(x, y, z, t) = \mu(x, y) + \mu(z, t) - \mu(x + z, y + t) - \mu(x, z)$
$\qquad - \mu(y, t) + \mu(x + y, z + t)$

Since F is compatible with the associativity constraint of multiplication, the distributivity constraints of the Ann-categories $\mathcal{A}$ and $\mathcal{A}'$, we have:

8. $\alpha'^*(x, y, z) - \alpha_*(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z$

9. $\lambda'^*(x,y,z) - \lambda_*(x,y,z) = \nu(x,y+z) - \nu(x,y) - \nu(x,z) + x\mu(y,z) - \mu(xy,xz)$

10. $\rho'^*(x,y,z) - \rho_*(x,y,z) = \nu(x+y,z) - \nu(x,z) - \nu(y,z) + \mu(x,y)z - \mu(xz,yz)$

From the equations $(7)-(10)$, we have:

$$h'^* - h_* = \delta g \tag{11}$$

where $g = (-\mu, \nu)$. Hence the obstruction of the functor F vanishes in the cohomology group $H^3_{MaL}(R, M)$.

Conversely, assume that the obstruction of the functor F vanishes in the cohomology group $H^3_{MaL}(R, M')$. Then there exists a 2-cochain $g = (\mu, \nu)$ such that $h'^* - h_* = \delta g$. Take $\breve{F}, \widetilde{F}$ be functor morphisms associated with the functions $-\mu, \nu$, we can verify that $(F, \breve{F}, \widetilde{F})$ is an Ann-functor. □

Theorem 4.5. *If there exists an Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$, of the type (p,q) then:*

(a) There exists a bijection between the set of congruence classes of Ann-functors of the type (p,q) and the cohomology group $H^2_{MaL}(R, M')$ of the ring R with coefficients in R-bimodule M'.

(b) There exists a bijection

$$Aut(F) \to Z^1_{MaL}(R, M')$$

between the group of automorphisms of the Ann-functor F and the group $Z^1_{MaL}(R, M')$.

Proof. (a) Suppose that there exists $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$, which is an Ann-functor of the type (p,q). According to Theorem 4.4, we have

$$\overline{h'^* - h_*} = 0.$$

Hence, there exists a 2-cochain k such that

$$h'^* - h_* = \delta k.$$

Let the 2-cochain k be fixed. Now, we assume that

$$(G, \breve{G}, \widetilde{G}) : (R, M, h) \to (R', M', h')$$

is an Ann-functor of the type (p,q). Then, from the proof of the Theorem 4.4, we have

$$h'^* - h_* = \delta g.$$

Hence, $k - g$ is a 2-cocycle. Consider the correspondence:

$$\Phi : class(G) \mapsto class(k - g)$$

from the set of the congruence classes of Ann-functors of the type (p, q) to the group $H^2_{MaL}(R, M')$.

Firstly, we prove that the above correspondence is a map. Indeed, suppose that

$$(G', \breve{G}', \widetilde{G}') : (R, M, h) \to (R', M', h')$$

is also an Ann-functor of the type (p, q) and $u : G \to G'$ is an Ann-morphism. Since u is an $\oplus$-morphism as well as an $\otimes$-morphism, we have:

$$g' = g - \delta(u) \tag{12}$$

So

$$k - g' = k - g + \delta(u).$$

Thus $\overline{k - g} = \overline{k - g'} \in H^2_{MaL}(R, M')$.

Now, we prove that Φ is an injection. Assume that

$$(G, \breve{G}, \widetilde{G}), (G', \breve{G}', \widetilde{G}') : (R, M, h) \to (R', M', h')$$

are Ann-functors of the type (p, q) and satisfying

$$\overline{k - g} = \overline{k - g'} \in H^2_{MaL}(R, M').$$

Then, there exists an 1-cochain u such that

$$k - g = k - g' - \delta(u)$$

That means

$$g' = g - \delta(u).$$

Hence, the following diagrams:

$$\begin{array}{ccc} G(x) \oplus G(y) & \xrightarrow{\breve{G}} & G(x \oplus y) \\ \downarrow{\scriptstyle u_x \oplus u_y} & & \downarrow{\scriptstyle u_{x \oplus y}} \\ G'(x) \oplus G'(y) & \xrightarrow{\breve{G}'} & G'(x \oplus y) \end{array} \qquad \begin{array}{ccc} G(x) \otimes G(y) & \xrightarrow{\widetilde{G}} & G(x \otimes y) \\ \downarrow{\scriptstyle u_x \otimes u_y} & & \downarrow{\scriptstyle u_{x \otimes y}} \\ G'(x) \otimes G'(y) & \xrightarrow{\widetilde{G}'} & G'(x \otimes y) \end{array}$$

commute, it means that $u : G \to G'$ is an *Ann*-morphism. Therefore,

$$class(G) = class(G').$$

Finally, we must prove that the correspondence Φ is a surjection. Indeed, assume that g is an arbitrary 2-cocycle. We have:

$$\delta(k-g) = \delta k - \delta g = \delta k = h'^{*} - h_{*}.$$

Then, according to Theorem 4.4, there exists an Ann-functor

$$(G, \breve{G}, \widetilde{G}) : (R, M, h) \to (R', M', h')$$

of the type (p, q), and the pair of isomorphisms $\breve{G}, \widetilde{G}$ associated with the 2-cochain $k - g$.

Clearly, $\Phi(G) = \overline{g}$. So Φ is a surjection.

(b) Assume that $F = (F, \breve{F}, \widetilde{F}) : (R, M, h) \to (R', M', h')$ is an Ann-functor of the type (p, q) and $u \in Aut(F)$. Then, from the equation (12) with $g' = g$, we have $\delta(u) = 0$, i.e., $u \in Z^1_{MaL}(R, M')$. □

5 Ann-functors and Hochschild cohomology

In this section, we will consider special Ann-functors which are related to the low-dimensioned Hochschild groups [3].

Following, we will find a condition for the existence of Ann-functors

$$F = (F, id, \widetilde{F}) : (R, M, h) \to (R', M', h')$$

of the type $(p, 0)$, where $p : R \to R'$ is a ring homomorphism.

Suppose that there exists an Ann-functor

$$F = (F, id, \widetilde{F} = \nu) : (R, M, h) \to (R', M', h')$$

of the type $(p, 0)$. Then, the equations (7) - (10) turn into:

13. $\sigma'^{*}(x, y, z, t) = 0$
14. $\alpha'^{*}(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z$
15. $\lambda'^{*}(x, y, z) = \nu(x, y + z) - \nu(x, y) - \nu(x, z)$
16. $\rho'^{*}(x, y, z) = \nu(x + y, z) - \nu(x, z) - \nu(y, z)$

and the Theorem 4.4 turns into

Corollary 5.1. *Let $p : R \to R'$ be a ring homomorphism. There exists an Ann-functor $(F, id, \widetilde{F})$ from (R, M, h) to (R', M', h') of the type $(p, 0)$ iff $\overline{h'^{*}} = 0 \in H^3_{MaL}(R, M')$.*

Each cocycle of $\mathbb{Z}$-algebras due to Hochschild is a multi-linear function. This suggests us the following definition:

Definition 5. *An Ann-functor*

$$(F, id, \widetilde{F}) : (R, M, h) \to (R', M', h')$$

of the type $(p, 0)$ *is called a* **strong** *Ann-functor if the function* $\nu : R^2 \to M'$ *corresponding to* $\widetilde{F}$ *is bi-additive.*

If ν is a normal bi-additive function, then ν is a 2-cocycle of the $\mathbb{Z}$-algebra R with coefficients in R-bimodule M' due to Hochschild. Then, in the equations (13)-(16), α'^* is a normal multi-linear function and other functions are equal to 0. So, we have

$$h'^* \equiv \alpha'^* = \delta(\nu),$$

in which $\delta(\nu)$ is a 3-coboundary of the ring R with coefficients in R-bimodule M' due to Hochschild. Then, we have the following proposition, as a direct corollary of Theorem 4.4.

Proposition 5.2. *Let* $F : (R, M, h) \to (R', M', h')$ *be a functor of the type* $(p, 0)$*. There exists a strong Ann-functor* $(F, id, \widetilde{F})$ *iff its cohomology class* $\overline{h'^*} = 0$ *in the cohomology group* $H^3_{Hochs}(R, M')$*.*

Theorem 5.3. *If there exists a strong Ann-functor* $(F, id, \widetilde{F}) : (R, M, h) \to (R', M', h')$ *of the type* $(p, 0)$*, then:*

(a) There exists a bijection between the set of congruence classes of strong Ann-functors of the type $(p, 0)$ *and the cohomology group* $H^2_{Hochs}(R, M')$ *of the ring* R *with coefficients in the* R*-bimodule* M'*.*

(b) There exists a bijection

$$Aut(F) \to Z^1_{Hochs}(R, M')$$

between the group of automorphisms of the Ann-functor F *and the group* $Z^1_{Hochs}(R, M')$*.*

Proof. (a) The restriction Φ_H of the map Φ, referred in Theorem 4.5, on the set of congruence classes of strong Ann-functors gives us an injection to the group $H^2_{Hochs}(R, M')$. Moreover, it is easy to see that Φ_H is also a surjection.

(b) Assume that $F = (F, id, \widetilde{F}) : (R, M, h) \to (R', M', h')$ is a strong $Ann-$functor of the type $(p, 0)$ and $u \in Aut(F)$. Then u is bi-linear respect to the $\oplus$. So $u \in Z^1_{Hochs}(R, M')$. The converse also holds. □

6 Application

In this section, as an application, we show the relation between the ring extension problem and the obstruction theory of Ann-functors.

We have known that, for a ring A (not necessary to have the unit) determined a *ring of bimultiplication* M_A. A bimultiplication σ of the ring A is a pair of maps $a \to \sigma a,\ a \to a\sigma$ from A to A, which satisfy:

$$\begin{array}{lclclcl} \sigma(a+b) & = & \sigma a + \sigma b & ; & (a+b)\sigma & = & a\sigma + b\sigma \\ \sigma(ab) & = & (\sigma a)b & ; & (ab)\sigma & = & a(b\sigma) \quad ; \quad a(\sigma b) = (a\sigma)b \end{array}$$

for all $a, b \in A$. The sum and the product of two bimultiplication are, respectly, the addition and the composition of maps .

Each element $c \in A$ induces a bimultiplication μ_c determined by

$$\mu_c a = ca; \qquad a\mu_c = ac, \qquad \forall a \in A$$

and it is called an *inner bimultiplication of* A. The map: $\mu : A \to M_A$ is a ring homomorphism and if A has the unit 1 then $\mu(1) = 1$. Since:

$$\sigma\mu_c = \mu_{\sigma c}; \quad \mu_c\sigma = \mu_{c\sigma}$$

the image μA of this homomorphism is a two-sided ideal of M_A. Let us denote $P_A = M_A/\mu(A)$.

Let R be a ring with the unit $1 \neq 0$. Each ring extension of A by R induces a regular ring homomorphism

$$\theta : R \to P_A$$

i.e., $\theta(1) = 1$ and any two elements of θR are permutable [The bimultiplications σ and τ are called *permutable* if $\varphi(a\psi) = (\varphi a)\psi$ and $\psi(a\varphi) = (\psi a)\varphi$ for all $a \in A$].

Inversely, according to [5], each regular ring homomorphism $\theta : R \to P_A$ induces a R-bimodule structure on *the bicenter* $C_A = \{c \in A | ca = ac = 0,\ \forall a \in A\}$, with the operators

$$xc = (\varphi x)c, cx = c(\varphi x); c \in C_A, x \in A, \varphi x \in \theta x.$$

The *obstruction* $\overline{h} \in H^3_{MacL}(R, C_A)$ of the homomorphism θ is defined as lollows. For each $x \in R$, let us choose $\varphi(x) \in \theta(x)$. Then there exists the factor set $f, g : R^2 \to A$ satisfying:

$$\begin{aligned} \mu_{f(x,y)} &= \sigma(x+y) - \sigma(x) - \sigma(y) \\ \mu_{g(x,y)} &= \sigma(xy) - \sigma(x)\sigma(y), \end{aligned}$$

for $x, y \in R$.

From the association, commutation, distribution on properties of the ring A the collection of functions whose values belong to C_A has the formal type $h = (\xi, \eta, \alpha, \lambda, \rho) = \delta(f, g)$. The collection h is an element in $Z^3_{MacL}(R, C_A)$, and its cohomology class is called *the obstruction of the homomorphism* θ.

Now, we present the ring extension problem by the language of Ann-functors. Consider the category $\mathcal{M}_A$ whose objects are elements of the ring M_A, and if φ, λ are bimultiplications of A, then let us denote:

$$Hom(\varphi, \lambda) = \{c \in A \mid \lambda = \mu_c + \varphi\}.$$

The composition of two arrows is the operation $+$ in A. The operations $\oplus$ and $\otimes$ are given by:

$$\begin{array}{lcll} \varphi \oplus \lambda & = & \varphi + \lambda, & \varphi, \lambda \in \mathcal{M}_A \\ c \oplus d & = & c + d, & c, d \in A \\ \varphi \otimes \lambda & = & \varphi \circ \lambda, & \text{(the composition of two maps)} \\ c \otimes d & = & cd + c\lambda + \varphi d, & \text{where } c : \varphi \to \varphi', d : \lambda \to \lambda' \end{array}$$

With these two operations, $\mathcal{M}_A$ becomes an Ann-category with the constraints naturally determined to be strict.

We regard the ring R as an Ann-category of the type $(R, 0, id)$. We call $\mathcal{S}$ the reduced Ann-category of $\mathcal{M}_A$. The regular homomorphism $\theta : R \to P_A$ determines a functor

$$(\theta, 0) : (R, 0, id) \to \mathcal{S} = (\Pi_0, \Pi_1, k)$$

The obstruction of this functor is the element

$$\overline{k^*} \in H^3_{MacL}(R, \Pi_1),\ k^* = \theta^*(k).$$

Now we have the following Proposition:

Proposition 6.1. *Let* $\mathcal{S} = (\Pi_0, \Pi_1, k)$ *be the reduced Ann-category of the strict Ann-category* $\mathcal{M}_A$. *Then:*

(a) $\Pi_0 = P_A = M_A/\mu A\ ,\ \Pi_1 = C_A$,

(b) $k^* = \theta^*(k)$ *belongs to the same cohomology class of* h *of the homomorphism* θ.

Proof. (a) Obviously.
(b) The proof is completely similar to the proof of Proposition 8 [10] with appropriate supplement. We have the normal Ann-equivalence $(H, \breve{H}, \widetilde{H})$ from the reduced Ann-category $\mathcal{S}$ to the Ann-category $\mathcal{M}_A$. Since the obstructions $\overline{h}$ of θ are independent of the choice of the function φ, we will choose $\varphi = H \circ \theta : R \to \mathcal{M}_A$. Then we can take

$$f(x, y) = \breve{H}_{\theta x, \theta y}\ ,\ g(x, y) = \widetilde{H}_{\theta x, \theta y}$$

From the compatibility of $(H, \breve{H}, \widetilde{H})$ with the constraints, $\overline{k^*} = \overline{h}$. □

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