

**LINEAR FRACTIONAL  
TRANSFORMATIONS OF CONTINUED  
FRACTIONS WITH BOUNDED PARTIAL  
QUOTIENTS IN THE FIELD OF FORMAL  
SERIES**

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**Abstract**

Let  $\theta$  be an irrational element in the field of formal series. Using a modification of the 1997 technique due to Lagarias and Shallit in the real numbers case, it is shown that if the continued fraction expansion of  $\theta$  has bounded partial quotients, so does its linear fractional transformation.

## 1. Introduction

Let  $\alpha$  be an irrational real number whose simple continued fraction is  $[b_0, b_1, b_2, \dots]$ . We say that  $\alpha$  has bounded partial quotients if  $\sup_{i \geq 1} b_i < \infty$ . Lagarias and Shallit in [4] proved, using the so-called Lagrange constant through a result of

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Cusick and Mendès France in [2], that if  $\alpha$  has bounded partial quotients, so does its linear fractional transformation. We show here that this is also the case in the field of formal series.

Let  $\mathbf{F} := \mathbb{F}((x^{-1}))$  be the field of formal series over a field  $\mathbb{F}$ , equipped with the usual degree valuation  $|\cdot|$ , so normalized that  $|P(x)| = 2^{\deg P(x)}$  ( $P \in \mathbb{F}[x] \setminus \{0\}$ ). It is well-known, see e.g. [1, Chapter 1], that every element  $\xi \in \mathbf{F} \setminus \{0\}$  can be uniquely written as

$$\xi := \sum_{n=r}^{\infty} w_n x^{-n},$$

where  $r \in \mathbb{Z}$ ,  $w_n \in \mathbb{F}$  ( $n \geq r$ ) and  $w_r \neq 0$ , so that  $|\xi| = 2^{-r}$ . Define the head part of  $\xi$  by

$$[\xi] = \begin{cases} \sum_{n=r}^0 w_n x^{-n} & \text{if } r \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the distance to the head part as

$$\|\xi\| := |\xi - [\xi]|.$$

In  $\mathbf{F}$ , there is a continued fraction algorithm similar to the case of real numbers which we briefly recall now; for details, see [6]. Each element  $\xi \in \mathbf{F} \setminus \{0\}$  can be uniquely represented as a continued fraction of the form

$$\xi = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \dots}} := [b_0, b_1, b_2, \dots],$$

where  $b_0 \in \mathbb{F}[x]$  and  $b_i \in \mathbb{F}[x] \setminus \mathbb{F}$  ( $i \geq 1$ ) are called partial quotients. Such continued fraction of  $\xi$  is finite if and only if  $\xi \in \mathbb{F}(x)$ .

Let  $\theta$  be an irrational in  $\mathbf{F}$  whose infinite continued fraction expansion is

$$\theta = [a_0, a_1, a_2, \dots].$$

Define the  $n^{\text{th}}$  complete quotient and the  $n^{\text{th}}$  convergent, respectively, of the continued fraction of  $\theta$  as

$$\theta_n = [a_n, a_{n+1}, a_{n+2}, \dots], \quad \frac{A_n}{B_n} = [a_0, a_1, a_2, \dots, a_n].$$

The partial numerators,  $A_n$ , and partial denominators,  $B_n$ , satisfy the recursions

$$A_{-1} = 1, \quad A_0 = a_0, \quad A_{n+1} = a_{n+1}A_n + A_{n-1} \quad (n \geq 0)$$

and

$$B_{-1} = 0, \quad B_0 = 1, \quad B_{n+1} = a_{n+1}B_n + B_{n-1} \quad (n \geq 0).$$

Define

$$K(\theta) := \sup_{i \geq 1} |a_i|, \quad K_{\infty}(\theta) := \limsup_{i \geq 1} |a_i|.$$

We say that  $\theta$  has **bounded partial quotients** if  $K(\theta)$  is finite. Clearly,  $K_\infty(\theta) \leq K(\theta)$  and  $K(\theta)$  is finite if and only if  $K_\infty(\theta)$  is finite.

Our main result reads:

**Theorem 1** *Let  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}[x])$ , the group of all invertible  $2 \times 2$  matrices with entries from  $\mathbb{F}[x]$ . If the continued fraction of an irrational element  $\theta \in \mathbf{F}$  has bounded partial quotients, then*

$$\frac{1}{|\det M|} K_\infty(\theta) \leq K_\infty\left(\frac{a\theta + b}{c\theta + d}\right) \leq |\det M| K_\infty(\theta), \quad (1)$$

$$K\left(\frac{a\theta + b}{c\theta + d}\right) \leq \max\{|\det M| K(\theta), |c(c\theta + d)|\}. \quad (2)$$

## 2. Auxiliary results

The first lemma collects basic properties of continued fractions whose straightforward proof is omitted.

**Lemma 2** *Let  $\theta = [a_0, a_1, a_2, \dots]$  be an irrational element in  $\mathbf{F}$ ,  $A_n/B_n$  its  $n^{\text{th}}$  convergent and  $\theta_n$  its  $n^{\text{th}}$  complete quotient. Let  $\zeta \in \mathbf{F} \setminus \{0\}$ . We have, for  $n \geq 0$ ,*

$$(i) |B_{n+1}| = |a_{n+1}B_n| > |B_n|, \quad |\theta_n| = |a_n|;$$

$$(ii) A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1}, \text{ so that } \gcd(A_n, B_n) = 1;$$

$$(iii) \theta - \frac{A_n}{B_n} = \frac{(-1)^n}{B_n(\theta_{n+1}B_n + B_{n-1})};$$

$$(iv) \frac{\zeta A_n + A_{n-1}}{\zeta B_n + B_{n-1}} = [a_0, a_1, a_2, \dots, a_n, \zeta];$$

$$(v) A_n \text{ is the head part of } B_n \theta.$$

From Lemma 2 (v), we have  $\|B_n \theta\| = |B_n \theta - A_n|$ , and so Lemma 2 (i) and (iii) together yield

$$|B_n| \|B_n \theta\| = \frac{1}{|\theta_{n+1} + B_{n-1}/B_n|} = \frac{1}{|a_{n+1}|}. \quad (3)$$

The result in the next lemma is known as the best approximation property, cf. Theorem 7.13 in [5] for the real case.

**Lemma 3** *Let  $\theta$  be an irrational element in  $\mathbf{F}$  and  $A_n/B_n$  its  $n^{\text{th}}$  convergent. If  $u, v (\neq 0) \in \mathbb{F}[x]$  satisfy, for some  $n \geq 0$ ,*

$$|v\theta - u| < |B_n \theta - A_n|, \quad (4)$$

*then  $|v| \geq |B_{n+1}|$ .*

*Proof.* Suppose that

$$|v| < |B_{n+1}|. \quad (5)$$

Consider the system of linear equations (in  $y, z$ )

$$yB_n + zB_{n+1} = v \quad (6)$$

$$yA_n + zA_{n+1} = u. \quad (7)$$

By Lemma 2 (ii),  $\det \begin{pmatrix} B_n & B_{n+1} \\ A_n & A_{n+1} \end{pmatrix} = (-1)^n$ , and so

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} (-1)^n A_{n+1} & (-1)^{n+1} B_{n+1} \\ (-1)^{n+1} A_n & (-1)^n B_n \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix},$$

implying that  $y$  and  $z$  are in  $\mathbb{F}[x]$ .

We claim that neither  $y$  nor  $z$  is zero. If  $y = 0$ , then  $0 \neq v = zB_{n+1}$ , and so  $|v| \geq |B_{n+1}|$ , which contradicts (5). Assume then that  $y \neq 0$ . If  $z = 0$ , then  $u = yA_n$  and  $v = yB_n$ . Since  $|y| \geq 1$ , we have  $|v\theta - u| = |y(B_n\theta - A_n)| \geq |B_n\theta - A_n|$ , contradicting (4).

Next we show that

$$|y(B_n\theta - A_n)| \neq |z(B_{n+1}\theta - A_{n+1})|. \quad (8)$$

Suppose  $|y(B_n\theta - A_n)| = |z(B_{n+1}\theta - A_{n+1})|$ . By Lemma 2 (i) and (iii), we have

$$|B_i\theta - A_i| = \frac{1}{|\theta_{i+1}B_i + B_{i-1}|} = \frac{1}{|B_{i+1}|} \quad (i \geq 0),$$

and so  $|yB_{n+2}| = |zB_{n+1}|$ . Since  $|yB_n| < |yB_{n+2}|$ , the ultrametric inequality and (6) yield  $|zB_{n+1}| = |v|$  implying that  $|B_{n+1}| \leq |v|$ , contradicting (5). Thus, (8) holds.

Finally, consider  $|v\theta - u| = |y(B_n\theta - A_n) + z(B_{n+1}\theta - A_{n+1})|$ . Using (8), the ultrametric inequality and  $y \in \mathbb{F}[x] \setminus \{0\}$ , we have

$$|v\theta - u| = \max\{|y(B_n\theta - A_n)|, |z(B_{n+1}\theta - A_{n+1})|\} \geq |y(B_n\theta - A_n)| \geq |B_n\theta - A_n|,$$

which contradicts (4), and the lemma follows.  $\square$

For irrational  $\theta \in \mathbf{F}$ , define its **type** and its **Lagrange constant**, respectively, by

$$L(\theta) = \sup_{|B| \geq 1} (|B| \|B\theta\|)^{-1}, \quad L_\infty(\theta) = \limsup_{|B| \geq 1} (|B| \|B\theta\|)^{-1}.$$

To determine the type and Lagrange constant, it suffices to use the partial denominators as we show now.

**Lemma 4** *We have*

$$L(\theta) = \sup_{i \geq 0} (|B_i| \|B_i \theta\|)^{-1}, \quad L_\infty(\theta) = \limsup_{i \geq 0} (|B_i| \|B_i \theta\|)^{-1}. \quad (9)$$

*Proof.* Let  $B \in \mathbb{F}[x] \setminus \{0\}$ . Since the continued fraction of any irrational is infinite, there exists  $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$  such that  $|B_m| \leq |B| < |B_{m+1}|$ . By Lemma 3,

$$\frac{1}{|B| \|B\theta\|} \leq \frac{1}{|B| \|B_m \theta\|} \leq \frac{1}{|B_m| \|B_m \theta\|},$$

and the result follows.  $\square$

**Corollary 5** *A) For irrational  $\theta \in \mathbf{F}$ , we have*

$$K(\theta) = L(\theta), \quad K_\infty(\theta) = L_\infty(\theta). \quad (10)$$

*B) Let  $\phi = [d_0, d_1, d_2, \dots]$ ,  $\gamma = [e_0, e_1, e_2, \dots]$  be two irrational elements in  $\mathbf{F}$ . If there exist  $s_1, s_2 \in \mathbb{N}_0$  such that  $|d_{s_1+i}| = |e_{s_2+i}|$  ( $i \geq 0$ ), then*

$$K_\infty(\phi) = K_\infty(\gamma), \quad L_\infty(\phi) = L_\infty(\gamma).$$

*Proof.* Part A) follows immediately from the definition of  $K(\theta)$ ,  $K_\infty(\theta)$ , (3) and Lemma 4. Part B) follows from at once the definition of  $K_\infty$ , Lemma 4 and (10).  $\square$

The next lemma is proved by modifying the proofs of Theorems 172 and 175 of [3] in the real to the formal series case.

**Lemma 6** *Let  $\theta = [a_0, a_1, a_2, \dots]$  be an irrational element in  $\mathbf{F}$  with  $|\theta| > 1$ , and let  $\psi = \frac{a\theta+b}{c\theta+d}$ , where  $a, b, c, d \in \mathbb{F}[x]$  are such that  $|ad - bc| = 1$ .*

- 1) *If  $|c| > |d| > 0$ , then  $b/d$  and  $a/c$  are two consecutive convergents of the continued fraction of  $\psi$ .*
- 2) *If  $b/d$  and  $a/c$  are the  $(n-1)^{\text{th}}$  and  $n^{\text{th}}$  convergents of the continued fraction of  $\psi$ , respectively, then the  $(n+1)^{\text{th}}$  complete quotient is of the form  $\delta\theta$  for some  $\delta \in \mathbb{F}^* := \mathbb{F} \setminus \{0\}$ .*
- 3) *If the continued fraction of  $\psi$  is  $[c_0, c_1, c_2, \dots]$ , then there exist  $k, n \in \mathbb{N}_0$  such that*

$$|a_{k+i}| = |c_{n+i}| \quad (i \geq 0).$$

*Proof.* We first prove parts 1) and 2) simultaneously. Denote the finite continued fraction expansion of  $a/c$  by  $[c_0, c_1, \dots, c_n]$  and let  $A_n/B_n$  be its  $n^{\text{th}}$  (last) convergent. Since  $|ad - bc| = 1$ , we have  $\gcd(a, c) = 1 = \gcd(A_n, B_n)$ . Thus,

$$|A_n d - B_n b| = |ad - bc| = 1 = |A_n B_{n-1} - A_{n-1} B_n|,$$

yielding  $A_n d - B_n b = \delta'(A_n B_{n-1} - A_{n-1} B_n)$  for some  $\delta' \in \mathbb{F}^*$ , and so

$$A_n(d - \delta' B_{n-1}) = B_n(b - \delta' A_{n-1}). \quad (11)$$

Since  $\gcd(A_n, B_n) = 1$ , the relation (11) gives

$$B_n | (d - \delta' B_{n-1}). \quad (12)$$

From  $|B_n| = |c| > |d| > 0$ , and  $|B_n| > |B_{n-1}| \geq 0$ , we get  $|d - \delta' B_{n-1}| < |B_n|$ , which is consistent with (12) only when  $d - \delta' B_{n-1} = 0$ , i.e., when  $d = \delta' B_{n-1}$ ,  $b = \delta' A_{n-1}$ . Consequently,  $\psi = \frac{A_n \delta \theta + A_{n-1}}{B_n \delta \theta + B_{n-1}}$  for some  $\delta \in \mathbb{F}^*$ , and so by Lemma 2 (iv),

$$\psi = [c_0, c_1, \dots, c_n, \delta \theta].$$

If we develop  $\delta \theta$  as a continued fraction, we obtain  $\delta \theta = [c_{n+1}, c_{n+2}, \dots]$ , with  $|c_{n+1}| > 1$ . Hence,  $\psi = [c_0, c_1, \dots, c_n, c_{n+1}, c_{n+2}, \dots]$ .

To prove part 3), from Lemma 2 (iv), we have

$$\theta = [a_0, a_1, \dots, a_{k-1}, \theta_k] = \frac{A_{k-1} \theta_k + A_{k-2}}{B_{k-1} \theta_k + B_{k-2}},$$

which implies

$$\psi = \frac{P \theta_k + R}{Q \theta_k + S},$$

where

$$P = aA_{k-1} + bB_{k-1}, \quad R = aA_{k-2} + bB_{k-2}, \quad Q = cA_{k-1} + dB_{k-1}, \quad S = cA_{k-2} + dB_{k-2}$$

are in  $\mathbb{F}[x]$  with  $|PS - QR| = |(ad - bc)(A_{k-1}B_{k-2} - A_{k-2}B_{k-1})| = 1$ . From Lemma 2 (iii), we have  $|\theta - \frac{A_i}{B_i}| = \frac{1}{|B_i(\theta_{i+1}B_i + B_{i-1})|} < \frac{1}{|B_i^2|}$  ( $i \geq 0$ ), and so

$$A_{k-1} = \theta B_{k-1} + \frac{\beta_1}{B_{k-1}}, \quad A_{k-2} = \theta B_{k-2} + \frac{\beta_2}{B_{k-2}},$$

where  $|\beta_1| < 1$ ,  $|\beta_2| < 1$ . Thus,

$$Q = (c\theta + d)B_{k-1} + \frac{c\beta_1}{B_{k-1}}, \quad S = (c\theta + d)B_{k-2} + \frac{c\beta_2}{B_{k-2}}.$$

Since  $c\theta + d \neq 0$ ,  $|B_{k-1}| > |B_{k-2}| \rightarrow \infty$  ( $k \rightarrow \infty$ ), we have  $|Q| > |S| > 0$  for all large  $k$ . For such  $k$ , part 1) and part 2) ensure that there exists  $\delta \in \mathbb{F}^*$  such that  $\delta \theta_k = \psi_n$  for some  $n$ , i.e.,  $|a_{k+i}| = |c_{n+i}|$  ( $i \geq 0$ ).  $\square$

Lemma 6 and Corollary 5 B) immediately yield:

**Lemma 7** *Let  $\theta$  be an irrational element in  $\mathbf{F}$ ,  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}[x])$ , and  $M(\theta) := \frac{a\theta+b}{c\theta+d}$ . If  $|\det M| = 1$ , then*

$$L_\infty(M(\theta)) = L_\infty(\theta).$$

For transformation with non-unit determinant, we have weaker results.

**Lemma 8** *Let  $\theta$  be an irrational in  $\mathbf{F}$ ;  $h, d_1, d_3 \in \mathbb{F}[x] \setminus \{0\}$  and  $d_2 \in \mathbb{F}[x]$ . Then*

$$L_\infty(h\theta) \leq |h|L_\infty(\theta) \quad (13)$$

$$L_\infty\left(\frac{d_1\theta + d_2}{d_3}\right) \leq |d_1d_3|L_\infty(\theta). \quad (14)$$

*Proof.* If  $\theta$  has unbounded partial quotients, i.e.,  $L_\infty(\theta) = \infty$ , both inequalities are trivial. Now assume  $\theta$  has bounded partial quotients. For  $h \in \mathbb{F}[x] \setminus \{0\}$ ,  $k \in \mathbb{N}_0$ , clearly,

$$\sup_{\deg B \geq k} (|Bh| \|Bh\theta\|)^{-1} \leq \sup_{\deg B \geq k} (|B| \|B\theta\|)^{-1}$$

and

$$\limsup_{|B| \geq 1} (|Bh| \|Bh\theta\|)^{-1} \leq \limsup_{|B| \geq 1} (|B| \|B\theta\|)^{-1}.$$

Consequently,

$$\begin{aligned} L_\infty(h\theta) &= \limsup_{|B| \geq 1} (|B| \|Bh\theta\|)^{-1} = |h| \limsup_{|B| \geq 1} (|Bh| \|Bh\theta\|)^{-1} \\ &\leq |h| \limsup_{|B| \geq 1} (|B| \|B\theta\|)^{-1} = |h|L_\infty(\theta), \end{aligned}$$

which verifies (13).

To verify (14), from Corollary 5 B) and (13), we have

$$\begin{aligned} L_\infty\left(\frac{d_1\theta + d_2}{d_3}\right) &= L_\infty\left(\frac{d_3}{d_1\theta + d_2}\right) \leq |d_3|L_\infty\left(\frac{1}{d_1\theta + d_2}\right) \\ &= |d_3|L_\infty(d_1\theta + d_2) \leq |d_1||d_3|L_\infty(\theta). \quad \square \end{aligned}$$

### 3. Proof of Theorem 1

By Corollary 5, it suffices to prove the two results for  $L_\infty$ ,  $L$  in place of  $K_\infty$ ,  $K$ , respectively. Let  $\psi := \frac{a\theta + b}{c\theta + d} = M(\theta)$ .

We start by showing that there exists  $M_2 \in GL_2(\mathbb{F}[x])$  such that

$$|\det M_2| = 1, \quad M_2M = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \in GL_2(\mathbb{F}[x]), \quad |\alpha\gamma| = |\det M|.$$

Write  $M_2 = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$ . To fulfil the matrix equality, it is required that  $Ga + Hc = 0$ .

If  $a = 0$ , then  $c \neq 0$  and so we must take  $H = 0$ . Now choose  $F \in \mathbb{F}^*$ ,  $G = 1/F$  and arbitrary  $E \in \mathbb{F}[x]$  to fulfil all requirements.

If  $c = 0$ , then  $a \neq 0$  and we must take  $G = 0$ . Now choose  $E \in \mathbb{F}^*$ ,  $H = 1/E$  and arbitrary  $F \in \mathbb{F}[x]$  to fulfil all requirements.

If both  $a \neq 0$  and  $c \neq 0$ , then take  $G = \text{lcm}(a, c)/a$  and  $H = -\text{lcm}(a, c)/c$ . Since  $\text{gcd}(G, H) = 1$ , there are  $\mu, \nu \in \mathbb{F}[x]$  such that  $\mu G + \nu H = 1$ . Taking  $E = \nu$  and  $F = -\mu$ , all the requirements are fulfilled.

Having obtained such  $M_2$ , we apply Lemma 7 to get

$$L_\infty(\psi) = L_\infty(M_2(\psi)) = L_\infty(M_2M(\theta)) = L_\infty\left(\frac{\alpha\theta + \beta}{\gamma}\right),$$

and the second inequality of (1) now follows from the inequality (14) of Lemma 8.

To prove the first inequality of (1), we consider the adjoint matrix

$$M' := \text{adj}(M) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

which has  $M'M = (\det M)I_2$ , and so

$$M'(\psi) = M'(M(\theta)) = M'M(\theta) = \theta.$$

Applying the second inequality of (1) to  $\psi$ , we have

$$L_\infty(\theta) = L_\infty(M'(\psi)) \leq |\det M'| L_\infty(\psi) = |\det M| L_\infty(\psi),$$

and the result follows.

We turn now to the second assertion of Theorem 1. For each  $B \in \mathbb{F}[x] \setminus \{0\}$ , let

$$x_B = |B| \|B\psi\| = |B| \left| B \left( \frac{a\theta + b}{c\theta + d} \right) - A \right| \quad \left( A = \left[ B \left( \frac{a\theta + b}{c\theta + d} \right) \right] \right).$$

If  $c = 0$ , then  $|\det M| = |ad| \neq 0$  and so

$$|ad| x_B = |aB| |aB\theta - (dA - bB)| \geq |aB| \|aB\theta\| \geq 1/L(\theta),$$

yielding

$$L(\psi) = \sup_{|B| \geq 1} (|B| \|B\psi\|)^{-1} \leq |ad| L(\theta),$$

which is the first term in the right hand expression of (2).

If  $c \neq 0$ , then

$$|c\theta + d| x_B = |B| |(Ba - Ac)\theta - (Ad - Bb)|. \quad (15)$$

Since  $\theta$  has bounded partial quotients, both  $K(\theta)$  and  $K_\infty(\theta)$  are finite. The result of the first part shows then that  $K_\infty(\psi)$  is finite and so is  $K(\psi)$ . Corollary



5 in turn shows that  $L(\psi)$  is finite. Thus, there is an infinite sequence of non-zero approximations

$$x_{B^{(i)}} = |B^{(i)}| \left\| B^{(i)} \psi \right\|$$

such that

$$L(\psi) - \frac{1}{2^i} \leq \frac{1}{x_{B^{(i)}}} \leq L(\psi) \quad (i \geq 0). \quad (16)$$

By taking a suitable subsequence, we may reduce to the case where either all of the approximations have  $B^{(i)}a - A^{(i)}c = 0$  or all of them have  $B^{(i)}a - A^{(i)}c \neq 0$ .

We first treat the subcase  $B^{(i)}a - A^{(i)}c = 0$  for all  $i \geq 0$ . Since  $ad - bc = \det M \neq 0$ , we have  $A^{(i)}d - B^{(i)}b \in \mathbb{F}[x] \setminus \{0\}$  and so (15) gives

$$|c\theta + d| x_{B^{(i)}} = |B^{(i)}| |A^{(i)}d - B^{(i)}b| \geq 1.$$

Consequently,

$$L(\psi) - \frac{1}{2^i} \leq \frac{1}{x_{B^{(i)}}} \leq |c\theta + d| \leq |c(c\theta + d)| \quad (i \geq 0).$$

Letting  $i \rightarrow \infty$ , we get the second term in the right hand expression of (2).

Finally, consider the subcase that  $B^{(i)}a - A^{(i)}c \neq 0$  for all  $i \geq 0$ . From (15), we have

$$\begin{aligned} |c\theta + d| \left| \frac{B^{(i)}a - A^{(i)}c}{B^{(i)}} \right| x_{B^{(i)}} &= |B^{(i)}a - A^{(i)}c| |(B^{(i)}a - A^{(i)}c)\theta - (A^{(i)}d - B^{(i)}b)| \\ &\geq |B^{(i)}a - A^{(i)}c| \left\| (B^{(i)}a - A^{(i)}c)\theta \right\| \geq \frac{1}{L(\theta)}. \end{aligned} \quad (17)$$

Using the first inequality in (16) and the inequality (17), we get

$$\begin{aligned} L(\psi) - \frac{1}{2^i} &\leq \frac{1}{x_{B^{(i)}}} \leq |c\theta + d| \left| \frac{B^{(i)}a - A^{(i)}c}{B^{(i)}} \right| L(\theta) \\ &= |c\theta + d| \frac{|c|}{|B^{(i)}|} \left| \frac{B^{(i)}a - A^{(i)}c}{c} \right| L(\theta). \end{aligned} \quad (18)$$

Using the strong triangle inequality, we have

$$\begin{aligned} \left| B^{(i)} \left( \frac{a}{c} \right) - A^{(i)} \right| &\leq \max \left\{ \left| B^{(i)} \left( \frac{a\theta + b}{c\theta + d} \right) - B^{(i)} \left( \frac{a}{c} \right) \right|, \left| B^{(i)} \left( \frac{a\theta + b}{c\theta + d} \right) - A^{(i)} \right| \right\} \\ &= \max \left\{ \frac{|B^{(i)}| |\det(M)|}{|c(c\theta + d)|}, \frac{x_{B^{(i)}}}{|B^{(i)}|} \right\}. \end{aligned} \quad (19)$$

Combining (18) and (19) gives

$$L(\psi) - \frac{1}{2^i} \leq L(\theta) \max \left\{ |\det M|, |c(c\theta + d)| \frac{x_{B^{(i)}}}{|B^{(i)}|^2} \right\}.$$

Using the first inequality in (16), i.e.,  $x_{B^{(i)}} \leq \frac{1}{L(\psi)-1/2^i}$ , we deduce that

$$L(\psi) - \frac{1}{2^i} \leq \max \left\{ |\det M| L(\theta), \frac{|c(c\theta + d)|}{|(B^{(i)})|^2} \cdot \frac{L(\theta)}{L(\psi) - 1/2^i} \right\}. \quad (20)$$

If  $L(\theta) \geq L(\psi)$ , then the inequality (2) holds trivially, using the first term in the right hand expression. If  $L(\theta) < L(\psi)$ , then letting  $i \rightarrow \infty$  in (20), the ratio  $\frac{L(\theta)}{L(\psi)-1/2^i}$  becomes  $\leq 1$  in the limit, and (2) follows.  $\square$

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## References

- [1] J.W.S. Cassels, “Local Fields”, Cambridge University Press, Cambridge, 1986.
- [2] T. W. Cusick and M. Mendès France, *The Lagrange spectrum of a set*, Acta Arith., **34** (1979), 287-293.
- [3] G. H. Hardy and E. M. Wright, “An Introduction to the Theory of Numbers”, Oxford University Press, London, 1971.
- [4] J. C. Lagarias and J. O. Shallit, *Linear fractional transformations of continued fractions with bounded partial quotients*, J. Théor. Nombres Bordeaux **9** (1997), 267-279.
- [5] I. Niven, H. S. Zuckerman and H. L. Montgomery, “An Introduction to the Theory of Numbers”, John Wiley and Sons, New York, 1991.
- [6] W. M. Schmidt, *On continued fractions and Diophantine approximation in power series fields*, Acta. Arith., **95** (2000), 139-166.