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A GENERALIZATION OF THE ZARISKI TOPOLOGY OF ARBITRARY RINGS FOR MODULES

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Abstract

Let M be a left R-module. The set of all prime submodules of M is called the spectrum of M and denoted by $\text{Spec}({}_RM)$, and that of all prime ideals of R is denoted by $\text{Spec}(R)$. For each $\mathcal{P} \in \text{Spec}(R)$, we define $\text{Spec}_{\mathcal{P}}({}_RM) = \{P \in Spec({}_RM) : Ann_\ell(M/P) = \mathcal{P}\}$. If $\text{Spec}_{\mathcal{P}}({}_RM) \neq \emptyset$, then $P_{\mathcal{P}} := \bigcap_{P \in Spec_{\mathcal{P}}({}_RM)} P$ is a prime submodule of M and $P \in \text{Spec}_{\mathcal{P}}({}_RM)$. A prime submodule Q of M is called a lower prime submodule provided $Q = P_{\mathcal{P}}$ for some $\mathcal{P} \in \text{Spec}(R)$. We write $\ell.\text{Spec}({}_RM)$ for the set of all lower prime submodules of M and call it *lower spectrum* of M. In this article, we study the relationships among various module-theoretic properties of M and the topological conditions on $\ell.\text{Spec}({}_RM)$ (with the Zariski topology). Also, we topologies $\ell.\text{Spec}({}_RM)$ with the patch topology, and show that for every Noetherian left R-module M, $\ell.\text{Spec}({}_RM)$ with the patch topology is a compact, Hausdorff, totally disconnected space. Finally, by applying Hochster's characterization of a spectral space, we show that if M is a Noetherian left R-module, then $\ell.\text{Spec}({}_RM)$ with the Zariski topology is a spectral space, i.e., $\ell.\text{Spec}({}_RM)$ is homeomorphic to $\text{Spec}(S)$ for some commutative ring S. Also, as an application we show that for any ring R with ACC on ideals $\text{Spec}(R)$ is a spectral space.

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0. Introduction.

Throughout, all rings are associative rings with identity elements, and all modules are unital left modules. The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. If N is a submodule (respectively proper submodule) of M we write $N \leq M$ (respectively $N \lneqq M$). We denote the left annihilator of a factor module M/N of M by $(N : M)$. We call M faithful if $(0 : M) = 0$.

Let Spec(R) denote the set of prime ideals of a ring R. The Zariski topology for Spec(R) is defined by letting $C \subseteq$Spec(R) be closed if and only if there exists an ideal I of R such that $C = \{\mathcal{P} \in Spec(R) \mid I \subseteq \mathcal{P}\}$ (see for example [1], [6] and [9]). A topological space is called spectral if it is homeomorphic to the prime spectrum of a commutative ring equipped with Zariski topology. Hachster [10], has characterized spectral spaces as follows:
A space X is spectral if and only if the following axioms hold:

(i) X is a T_0-space;
(ii) X is quasi-compact;
(iii) the quasi-compact open subsets of X are closed under finite intersection and form an open base;
(iv) each nonempty irreducible closed subset F of X has a generic point (i.e., F is the closure of a unique point).

Let M be a left R-module. A proper submodule P of M is called *prime* if $aRm \subseteq P$, for $a \in R$ and $m \in M$, implies $m \in P$ or $a \in (P : M)$ (see for example [7] and [20]). If M is a nonzero left R-module and $(0 : M) = (N : M)$ for all nonzero submodule N of M then M is called a prime module (see for example [9] and [21]). If M is a prime module, then $(0 : M) = \mathcal{P}$ is a prime ideal and it is called the *affiliated prime* of ${}_RM$. P is a prime submodule of M if and only if M/P is a prime module. Clearly, a two-sided ideal $\mathcal{P}$ of R is a prime ideal of R if and only if $\mathcal{P}$ is a prime submodule of ${}_RR$ (for more information about this and others related topics, see, for instance, [2], [3], [4], [5], [11], [13] and [16]). We define Spec$({}_RM)$ for the set of all prime submodules of M and define Spec$_{\mathcal{P}}({}_RM) = \{P \in Spec({}_RM)$: the affiliated prime ideal of M/P is $\mathcal{P}\}$. Hence Spec$_{\mathcal{P}}({}_RM) = \{P \in Spec({}_RM) : (P : M) = \mathcal{P}\}$. If Spec$_{\mathcal{P}}({}_RM) \neq \emptyset$, then $P_{\mathcal{P}} := \bigcap_{P \in Spec_{\mathcal{P}}({}_RM)} P$ is a prime submodule of M and $P \in$ Spec$_{\mathcal{P}}({}_RM)$ (see Proposition 1.10 in McCasland and Smith [20]). A prime submodule Q of M is called a lower prime submodule provided $Q = P_{\mathcal{P}}$ for some $\mathcal{P} \in$ Spec(R). Clearly, a left ideal $\mathcal{P}$ of any ring R is a lower prime submodule (left ideal) if and only if $\mathcal{P}$ is a prime two-sided ideal of R, and hence this notion of lower prime submodule is a natural generalization of the notion of prime two-sided ideal of rings to modules. We write ℓ.Spec$({}_RM)$ for the set of all lower prime submodules of M and call it *lower spectrum* of M. Clearly, if P, $Q \in \ell$.Spec$({}_RM)$, then $P = Q$ if and only if $(P : M) = (Q : M)$ and for any ring R we have ℓ.Spec$({}_RR)$ =Spec(R). A module M over a commutative ring R is called a *multiplication module* if each submodule of M is of the form IM, where I is an ideal of R (see EL-Bast and Smith [8], for more details). It is clear that each prime submodule of a multiplication module M is of the form $\mathcal{P}M$ for some prime ideal $\mathcal{P}$ of R, and hence $\ell.\text{Spec}({}_RM)$ =$\text{Spec}({}_RM)$. Also if R is a commutative ring, then for each R-module M and each prime ideal $\mathcal{P}$ of R such that $\mathcal{P}M \neq M$, $M(\mathcal{P}) := \{m \in M \mid Am \subseteq \mathcal{P}M \textit{ for some ideal } A \nsubseteq \mathcal{P}\}$ is a submodule of M (see McCasland and Smith [20]). It is shown that for any free left R-module M, $\ell.Spec(M) = \{\mathcal{P}M \ : \ \mathcal{P} \in Spec(R),\ \mathcal{P}M \neq M\ \}$ and for any finitely generated faithful module M over a commutative ring R, $\ell.Spec(M) = \{M(\mathcal{P}) \ : \ \mathcal{P} \in Spec(R),\ \mathcal{P}M \neq M\ \}$ (see Proposition 1.3).

For any submodule N of a module M, define

$$V_\ell(N) = \{P_{\mathcal{P}} \in \ell.Spec(M) \mid \mathcal{P} \supseteq (N : M)\}.$$

Then:

(i) $V_\ell(0) = \ell.\text{Spec}({}_RM)$ and $V_\ell(M) = \emptyset$,
(ii) $\bigcap_{i\in\Lambda} V_\ell(N_i) = V_\ell((\Sigma_{i\in\Lambda}(N_i : M))M)$ for any index set Λ
(iii) $V_\ell(N) \bigcup V_\ell(L) = V_\ell(N \bigcap L)$,

Also, for each submodule N of M we denote the complement of $V_\ell(N)$ in $\ell.\text{Spec}({}_RM)$ by $U_\ell(N)$ (i.e., $U_\ell(N) = \{P_{\mathcal{P}} \in \ell.Spec(M) \mid \mathcal{P} \nsupseteq (N : M)\}$). From (i), (ii) and (iii) above, the family $\mathcal{T}_\ell(M) = \{U_\ell(N) | N \leq M\}$ is closed under finite intersections and arbitrary unions. Moreover, we have $U_\ell(M) = \ell.\text{Spec}({}_RM)$ and $U_\ell(0) = \emptyset$. Therefore, $\mathcal{T}_\ell(M)$ is the family of open sets for a topology on $\ell.\text{Spec}({}_RM)$ and call it the *lower Zariski topology* of M. This notion of lower Zariski topology of a module is analogous to that of the usual Zariski topology of a ring. In fact, for any ring R, the lower Zariski topology of ${}_RR$ and the usual Zariski topology of the ring R considered in [9], coincide. Also, the lower Zariski topology and the Zariski topology considered in [11], agree for multiplication modules (see also [11], [12], [17], [18] and [19]).

In this article, we study the relationships among various module-theoretic properties of M and the topological conditions on $\ell.\text{Spec}({}_RM)$ (with the lower Zariski topology). Modules whose lower Zariski topology is T_1 are studied in Section 1. For example we show that for each R-module M, $\ell.\text{Spec}({}_RM)$ is a T_1-space if and only if $Cl.K.dim(M) \leq 0$ (see [2], for the definition of classical Krull dimension of modules). This yields that if M is semisimple, R is a PI-ring and M is an Artinian R-module, or R is a commutative ring and M is co-semisimple, then $\ell.\text{Spec}({}_RM)$ is a T_1-space.

In Section 2, we topologies $\ell.\text{Spec}({}_RM)$ with the patch topology, and show that for every Noetherian left R-module M, $\ell.\text{Spec}({}_RM)$ with the patch topology is a compact, Hausdorff, totally disconnected space. In the final section, comparing with $\text{Spec}(R)$, R commutative, we investigate the lower Zariski topology of modules form the point of view of spectral spaces. In fact, by applying Hochster's characterization of a spectral space (see Hochster [10]), we show that if M is a Noetherian left R-module, then $\ell.\text{Spec}({}_RM)$ with the Zariski topology is a spectral space, i.e., $\ell.\mathrm{Spec}({}_RM)$ is homeomorphic to $\mathrm{Spec}(S)$ for some commutative ring S. As an application we conclude that for any ring R with ACC on ideals $\mathrm{Spec}(R)$ with the usual Zariski topology is a spectral space.

1. Some remarks about the lower Zariski topology of modules

Let X be a topological space and let x and y be points in X. We say that x and y can be separated if each lies in an open set which does not contain the other point. X is a T_1- space if any two distinct points in X can be separated. A topological space X is a T_1-space if and only if all points of X are closed in X (i.e., given any x in X, the singleton set $\{x\}$ is a closed set). (Note: for other terminology on a topological space not defined here we refer to Mankres [14].)

In the literature, there are two different generalizations of the classical Krull dimension for modules via prime dimension. In fact, the notion of prime dimension of a module M over a commutative ring R [denoted by $D(M)$ or $dim(M)$], was introduced by Marcelo and Masqué [15], as the maximum length of the chains of prime submodules of M (see [2] and [13]) for some known results about the prime dimension of modules). Also, the classical Krull dimension of rings has been extended to modules ${}_RM$ in [2], as the maximum length of the strong chains of prime submodules of M (allowing infinite ordinal values) and denoted by $Cl.K.dim(M)$ (see also [5] for another generalization of the classical Krull dimension of rings to modules). (Note: the chain $N_1 \subset_s N_2 \subset_s N_3 \subset_s \cdots$ of submodules of M is called a strong ascending chain if for each $i \in \mathbb{N}$, $N_i \subsetneqq N_{i+1}$ and also $(N_i : M) \subsetneqq (N_{i+1} : M)$; (see [2], for definition of the strong descending chain condition).

In the following result we give a characterization for the lower prime spectrum of finitely generated faithful modules over a commutative ring.

Theorem 1.1. *Let M be a left R-module. Then the following statements are equivalent:*

(1) *$\ell.Spec({}_RM)$ with the lower Zariski topology is a T_1-space.*
(2) *$Cl.K.dim(M) \leq 0$.*

Proof. (1) $\Rightarrow$ (2). Assume that $\ell.Spec(M)$ with lower Zariski topology is a T_1-space. If $\ell.\mathrm{Spec}({}_RM) = \emptyset$, then $Cl.K.dim(M) = -1$. Let $\ell.\mathrm{Spec}({}_RM) \neq \emptyset$ and $P_1 \in \ell.\mathrm{Spec}({}_RM)$. Then $\{P_1\}$ is a closed set in $\ell.\mathrm{Spec}({}_RM)$. We claim that every prime submodule of M is a virtually maximal prime submodule, for if not, we assume that $P_1 \subset_s P_2$, where P_1, P_2 are lower prime submodules of M. Since $\{P_1\}$ is a closed set, $\{P_1\} = V_\ell(N)$, where $N \leq M$ and then $(N : M) \subseteq (P_1 : M)$. Thus we conclude that $P_1 \in V_\ell(N)$. Since $P_1 \subset_s P_2$, $(P_1 : M) \subset (P_2 : M)$. Therefore $P_2 \in V_\ell(N)$. Thus $P_2 \in \{P_1\}$, a contradiction. Thus every lower prime submodule of M is a virtually maximal prime submodule and then Cl.K.dim$(M) \leq 0$.

(2) $\Rightarrow$ (1). Suppose that Cl.K.dim$(M) \leq 0$. If Cl.K.dim$(M) = -1$, then ℓ.Spec$(_RM) = \emptyset$, i.e., ℓ.Spec$(_RM)$ is trivial space and so it is a T_1-space. Now let Cl.K.dim$(M) = 0$, i.e., ℓ.Spec$(_RM) \neq \emptyset$ and every prime submodule of M is a virtually maximal prime submodule. Thus for each lower prime submodule P of M, $V_\ell(P) = \{P\}$, and so $\{P\}$ is a closed set in ℓ.Spec$(_RM)$ i.e., ℓ.Spec$(_RM)$ is a T_1-space. □

The following corollary gives a wider class of modules M for which ℓ.Spec$(_RM)$ with the lower Zariski topology is a T_1-space.

Corollary 1.2. *Let M be a left R-module. Then:*

(a) *If M is semisimple, then ℓ.Spec$(_RM)$ with the lower Zariski topology is a T_1- space.*
(b) *If R is a PI-ring and M is Artinian, then ℓ.Spec$(_RM)$ with the lower zariski topology is a T_1-space.*
(c) *If R is commutative and M is co-semisimple, then ℓ.Spec$(_RM)$ with the lower zariski topology is a T_1- space.*

Proof. (a). By [5, Theorem 1.7] and Theorem 1.1.
(b). By [2, Theorem 1.10] and Theorem 1.1.
(c). By [2, Proposition 1.11] and Theorem 1.1. □

In the next proposition we give a characterization for ℓ.*Spec*(M) when M is a free module or R is a commutative ring and M is a finitely generated faithful R-module.

Proposition 1.3. *Let M be a nonzero left R-module. Then:*

(a) *If M is a free R-module, then for each prime ideal $\mathcal{P}$ of R, $\mathcal{P}M$ is a prime submodule of M such that $(\mathcal{P}M : M) = \mathcal{P}$. Moreover,*

$$\ell.Spec(_RM) = \{\ \mathcal{P}M \mid for\ each\ ideal\ \mathcal{P}\ of\ R\ \}.$$

(b) *If R is commutative and M is a finitely generated faithful R-module, then*

$$\ell.Spec(M) = \{M(\mathcal{P})\ :\ \mathcal{P} \in Spec(R),\ \mathcal{P}M \neq M\ \}.$$

Proof. (a). Since M is free R-module, then $M = \bigoplus R^{(I)}$, for index set I. One can easily see that for each prime ideal $\mathcal{P}$ of R, $\mathcal{P}M = \bigoplus \mathcal{P}^{(I)}$ is a prime submodule of M. On the other hand for each prime submodule P of M such that $(P : M) = \mathcal{P}$, $\mathcal{P}M \subseteq P$. Therefore

$$\mathcal{P}M \subseteq \bigcap_{P \in Spec_{\mathcal{P}}({}_RM)} P,$$

and hence,

$$\mathcal{P}M = \bigcap_{P \in Spec_{\mathcal{P}}({}_RM)} P.$$

(b). Let $\mathcal{P}$ be a prime ideal of R. By Lemma 2.1 in [5], $(M(\mathcal{P}) : M) = \mathcal{P}$. Since $\mathcal{P}M \subseteq M(\mathcal{P})$, $(\mathcal{P}M : M) = \mathcal{P}$. Thus by Proposition 1.8 in [20], $M(\mathcal{P})$ is a prime submodule of M. Then by Lemma 1.6 in [20], for each prime submodule K of M such that $(K : M) = \mathcal{P}$, $M(\mathcal{P}) \subseteq K$. Therefore

$$\ell.Spec({}_RM) = \{M(\mathcal{P}) \ : \ \mathcal{P} \in Spec(R), \ \mathcal{P}M \neq M \ \}. \ \square$$

We need the following evident lemma.

Lemma 1.4. *Let M be a left R-module. The for each submodule N of M, $V_\ell(N) = V_\ell(IM)$, where $I = (N : M)$. Consequently,*

$$\mathcal{T}_\ell(M) = \{U_\ell(IM) \mid I \text{ is an ideal of } R\}.$$

Let M be a left R-module and $\overline{R} = R/Ann(M)$. From the definition of the lower Zariski topology on ℓ.Spec$({}_RM)$, it is evident that the topological space $\ell.Spec(M)$ is closely related to Spec$(\overline{R})$, particularly, under the correspondence $\psi : \ell.Spec(M) \longrightarrow Spec(\overline{R})$ defined by $\psi(P) = (\overline{P : M})$ for every $P \in \ell$.Spec(M).

Proposition 1.5. *For any left R-module M, the natural map ψ is continuous map. More precisely, $\psi^{-1}(V(\overline{I})) = V_\ell(IM)$ for every ideal I of R containing $Ann(M)$.*

Proof. Suppose that $P \in \ell$.Spec(M) such that $P \in V_\ell(IM)$, then $I \subseteq (P : M)$ and so $(\overline{P : M}) \in V(\overline{I})$. Therefore $\psi(P) = (\overline{P : M}) \in V(\overline{I})$. Thus $P \in \psi^{-1}(V(\overline{I}))$. Conversely, if $P \in \psi^{-1}(V(\overline{I}))$, then $\psi(P) = (\overline{P : M}) \in V(\overline{I})$. Therefore $I \subseteq (P : M)$ and then $P \in V_\ell(IM)$. $\square$

Lemma 1.6. *For any left R-module M, the natural map $\psi : \ell.Spec(M) \longrightarrow Spec(\overline{R})$ is injective.*

Proof. Evident. $\square$

Lemma 1.7. *Let M be a left R-module. If the natural map ψ is surjective, then ψ is closed.*

Proof. By Proposition 1.5, ψ is a continuous map and $\psi^{-1}(V(\overline{I})) = V_\ell(IM)$, for each ideal I of R containing $Ann(M)$. Let $N \leq M$. Then $\psi^{-1}(V(\overline{(N:M)})) = V_\ell((N:M)M) = V_\ell(N)$. It follows that $\psi(V_\ell(N)) = \psi o \psi^{-1}(V(\overline{(N:M)})) = V(\overline{(N:M)})$ as ψ is bijective. □

Corollary 1.8. *Let M be a left R-module. If the natural map ψ is surjective, then ψ is homeomorphic.*

Proof. By Proposition 1.5, Lemma 1.6 and Lemma 1.7. □

Corollary 1.9. *Let M be a left R-module. Then the natural map ψ is homeomorphic in each of the following cases:*

(1) *M is a finitely generated nonzero module over commutative ring R;*
(2) *M is a faithfully flat nonzero module over commutative ring R;*
(3) *M is a free nonzero module over any ring R.*

Proof. (1) and (2) follow from [12, p. 3746, Theorem 2] and (3) follows from Proposition 1.3(a). □

2. Patch topologies associated to the lower spectrum of a module

We need to recall the patch topology (see [9] and [10], for definition and more details). Let X be a topological space. By the patch topology on X, we mean the topology which has as a sub-basis for its closed sets the closed sets and compact open sets of the original space. By a patch we mean a set closed in the patch topology. The patch topology associated to a spectral space is compact and Hausdorff (see Hachster [10]). Also, the patch topology associated to the Zariski topology of a ring R (not necessarily commutative) with ACC on ideals is compact and Hausdorff (see [9, Proposition 16.1]).

Definition 2.1. Let M be a left R-module, and let $\mathcal{V}_\ell(M)$ be the family of all subsets of $\ell.\mathrm{Spec}(_RM)$ of the form $V_\ell(N)\bigcup U_\ell(K)$ where $V_\ell(N)$ is any lower Zariski-closed subset of $\ell.\mathrm{Spec}(_RM)$ and $U_\ell(K)$ is a lower Zariski-quasi-compact subset of $\ell.\mathrm{Spec}(_RM)$. Clearly $\mathcal{V}_\ell(M)$ is closed under finite unions and contains $\ell.\mathrm{Spec}(_RM)$ and the empty set, since $\ell.\mathrm{Spec}(_RM)$ equals $V_\ell(0)\bigcup U_\ell(0)$ and the empty set equals $V_\ell(M)\bigcup U_\ell(0)$. Therefore $\mathcal{V}_\ell(M)$ is basis for the family of closed sets of a topology on $\ell.\mathrm{Spec}(_RM)$, and call it *lower patch topology* (or *lower constructible topology*) of M. Thus

$$\mathcal{V}_\ell(M) = \Big\{V_\ell(N)\bigcup U_\ell(K) \mid N, K \leq M,\ U_\ell(K)\ \textit{is lower Zariski--quasi--compact}\Big\},$$

and hence we obtain the family

$$\mathcal{U}_\ell(M) = \Big\{V_\ell(K)\bigcap U_\ell(N) \mid N, K \leq M,\ U_\ell(K)\ \textit{is lower Zariski--quasi--compact}\ \Big\},$$

which is a basis for the *open* sets of the lower patch topology, i.e., the patch-open subsets of $\ell.\mathrm{Spec}({}_RM)$ are precisely the unions of sets from $\mathcal{U}_\ell(M)$. We denote the patch-topology of $\ell.\mathrm{Spec}({}_RM)$ by $\mathcal{T}_{\ell p}(M)$.

We need the following definition (a slightly different notion of lower patch topology).

Definition 2.2. Let M be a left R-module, and let $\widetilde{\mathcal{U}}_\ell(M)$ be the family of all subsets of $\ell.\mathrm{Spec}({}_RM)$ of the form $V_\ell(N)\bigcap U_\ell(K)$ where $N,\ K \leq M$. Clearly $\widetilde{\mathcal{U}}_\ell(M)$ contains $\ell.\mathrm{Spec}({}_RM)$ and the empty set, since $\ell.\mathrm{Spec}({}_RM)$ equals $V_\ell(0)\bigcap U_\ell(M)$ and the empty set equals $V_\ell(M)\bigcap U_\ell(0)$. Let $\widetilde{\mathcal{T}}_{\ell p}(M)$ to be the collection $\widetilde{U}$ of all unions of elements of $\widetilde{\mathcal{U}}_\ell(M)$. Then $\widetilde{\mathcal{T}}_{\ell p}(M)$ is a topology on $\ell.\mathrm{Spec}({}_RM)$ and it is called the *finer lower patch topology* or the *finer lower constructible topology* (in fact, $\widetilde{\mathcal{U}}_l(M)$ is a basis for the finer lower patch topology of M).

Lemma 2.3. *Let M be an R-module and $P \in \ell.Spec({}_RM)$. Then for each finer lower patch-neighborhood $\mathcal{U}_\ell$ of P, there exists a submodule L of M such that $(P : M) \subset (L : M)$ and $P \in V_\ell(P)\bigcap U_\ell(L) \subseteq \mathcal{U}_\ell$.*

Proof . Since $P \in \mathcal{U}_\ell$, there exists a neighborhood of the form $V_\ell(K)\bigcap U_\ell(N) \subseteq \mathcal{U}_\ell$ such that $P \in V_\ell(K)\bigcap U_\ell(N)$ where $(P : M) \supseteq (K : M)$ and $(P : M) \not\supseteq (N : M)$. Since $P \in V_\ell(P)$ and $V_\ell(P) \subseteq V_\ell(K)$, we may replace $V_\ell(K)$ by $V_\ell(P)$. Now we claim that $V_\ell(P)\bigcap U_\ell(N) = V_\ell(P)\bigcap U_\ell((I + \mathcal{P})M)$, where $\mathcal{P} = (P : M)$ and $I = (N : M)$. Since $U_\ell(IM) \subseteq U_\ell((I + \mathcal{P})M)$,

$$V_\ell(P)\bigcap U_\ell(N) = V_\ell(P)\bigcap U_\ell(IM) \subseteq V_\ell(P)\bigcap U_\ell((I + \mathcal{P})M).$$

Suppose that $Q \in V_\ell(P)\bigcap U_\ell((I + \mathcal{P})M)$, then $Q \notin U_\ell(P)$. On the other hand $Q \in U_\ell((I + \mathcal{P})M) = U_\ell(N)\bigcup U_\ell(P)$. This follows that $Q \in U_\ell(N)$. Thus $V_\ell(P)\bigcap U_\ell(N) = V_\ell(P)\bigcap U_\ell((I + \mathcal{P})M)$. Now let $L = (I + \mathcal{P})M$. Then $\mathcal{P} \subset I + \mathcal{P} \subseteq (L : M)$ and $P \in V_\ell(P)\bigcap U_\ell(L) \subseteq \mathcal{U}_\ell$. □

Let X be a topological space. Then for each subset Y of $\ell.\mathrm{Spec}(M)$, we will denote the closure of Y in $\ell.\mathrm{Spec}(M)$ with finer lower patch topology by $\overline{Y}$.

Proposition 2.4. *Let M be an R-module and $Y \subseteq \ell.Spec({}_RM)$. If $Q \in \overline{Y}$ with finer lower patch topology, then there exists $\mathcal{A} \subseteq Y$ such that $V_\ell(Q) = V_\ell(\bigcap_{P\in\mathcal{A}} P)$.*

Proof. Let $Q \in \overline{Y}$. If $Q \in Y$, then we are thorough. Thus we can assume that $Q \notin Y$. Let $\mathcal{A} = \{P \in Y \mid (Q : M) \subset (P : M)\}$. Since $Q \in U_\ell(M)\bigcap V_\ell(Q)$, there exists $P' \in Y$ such that $P' \in U_\ell(M)\bigcap V_\ell(Q)$. Since $Q \notin Y$, $(Q : M) \subset (P' : M)$ and hence $\mathcal{A} \neq \emptyset$. Since $(Q : M) \subseteq (P : M)$ for each $P \in \mathcal{A}$,

$$(Q : M) \subseteq \bigcap_{P\in\mathcal{A}} (P : M) = (\bigcap_{P\in\mathcal{A}} P : M).$$

If $\bigcap_{P\in\mathcal{A}} (P : M) \nsubseteq (Q : M)$, then

$$Q \in U_\ell(\bigcap_{P\in\mathcal{A}} P)\bigcap V_\ell(Q).$$

Since $Q \in \overline{Y}$, there exists $P'' \in Y$ such that

$$P'' \in U_\ell(\bigcap_{P\in\mathcal{A}} P)\bigcap V_\ell(Q).$$

Therefore $P'' \in V_\ell(Q)$ and hence $P'' \in \mathcal{A}$. But

$$(\bigcap_{P\in\mathcal{A}} P : M) = \bigcap_{P\in\mathcal{A}} (P : M) \subseteq (P'' : M).$$

Thus $P'' \notin U_\ell(\bigcap_{P\in\mathcal{A}} P)$. a contradiction. Thus $\bigcap_{P\in\mathcal{A}} (P : M) \subseteq (Q : M)$, and hence

$$V_\ell(Q) = V_\ell(\bigcap_{P\in\mathcal{A}} P) = V_\ell(\bigcap_{P\in\mathcal{A}} (P : M)M). \square$$

Proposition 2.5. *Let M be a left R-module. Then $\ell.Spec(M)$ with the finer lower patch topology is Hausdorff. Moreover, $\ell.Spec({}_RM)$ with this topology is totally disconnected.*

Proof. Suppose that P, $Q \in \ell.Spec(M)$ are distinct points. Since $P \neq Q$, $(P : M) \neq (Q : M)$. Therefore either $(P : M) \nsubseteq (Q : M)$ or $(Q : M) \nsubseteq (P : M)$. Assume that $(P : M) \nsubseteq (Q : M)$. By Definition 2.2, $U_1 := U_\ell(M)\bigcap V_\ell(P)$ is a finer lower patch-neighborhood of P and since $(P : M) \nsubseteq (Q : M)$, $U_2 := U_\ell(P)\bigcap V_\ell(Q)$ is a finer lower patch-neighborhood of Q. Clearly $U_\ell(P)\bigcap V_\ell(P) = \emptyset$ and hence $U_1 \cap U_2 = \emptyset$. Thus $\ell.\mathrm{Spec}({}_RM)$ is a Hausdorff space. On the other hand for every submodule N of M, observer that the sets $U_\ell(N)$ and $V_\ell(N)$ are open in finer lower patch topology, since $V_\ell(N) = U_\ell(M) \bigcap V_\ell(N)$ and $U_\ell(N) = U_\ell(N) \bigcap V_\ell(0)$. Since $U_\ell(N)$ and $V_\ell(N)$ are complement of each other, they are both finer lower both-closed as well. Therefore, the finer patch topology on $\ell.\text{Spec}({}_RM)$ has a basis of open sets which are also closed, and hence $\ell.\text{Spec}({}_RM)$ is totally disconnected in this topology. □

An R-module M will be called *weakly Noetherian* if, for every element a in R and element m in M, the submodule $RaRm$ is finitely generated (see [20]). For any ring R, every Noetherian module is weakly Noetherian. If R is a commutative ring, then every R-module is weakly Noetherian.

Definition 2.6. An R-module M is called p^*-module if for each prime ideal $\mathcal{P}$ of R such that $(\mathcal{P}M : M) = \mathcal{P}$, there exists a prime submodule P of M such that $(P : M) = \mathcal{P}$.

For example for each ring R, ${}_RR$ is a P^*-module. By Proposition 1.3 (a), every finitely generated faithful module over a commutative ring R is a P^*-module. Also every torsion free divisible module over any domain is a P^*-module. Now we show that every Noetherian left R-module M is also a P^*-module.

Lemma 2.7. *Let M be a Notherian left R-module. Then M is p^*-module.*

Proof. Let M be a Notherian left R-module. Then M is finitely generated and weakly Notherian. By [20, Proposition 1.8], for each prime ideal $\mathcal{P}$ of R, $M(\mathcal{P})$ is a prime submodule of M such that $(\mathcal{P}M : M) = \mathcal{P}$. □

Theorem 2.8. *Let R be a ring and M be a p^*-module such that $R/Ann(M)$ has ACC on ideals. Then $\ell.Spec({}_RM)$ with the finer lower patch topology is a compact space.*

Proof. Suppose M is a p^*-module such that $R/Ann(M)$ has ACC on ideals. Let $\mathcal{A}$ be a family of finer lower patch-open sets covering $\ell.\text{Spec}({}_RM)$ and suppose that no finite subfamily of $\mathcal{A}$ covers $\ell.\text{Spec}({}_RM)$. Let

$$S = \{L \mid L \text{ is an ideal of } R \text{ such that } Ann(M) \subseteq L \text{ and no finite subfamily of } \mathcal{A} \textit{ covers } V_\ell(LM)\}.$$

Since $V_\ell(Ann(M)M) = V_\ell(0) = \ell.\text{Spec}(M)$, $S \neq \emptyset$. We may use the ACC on ideals of $R/Ann(M)$ to choose an ideal $\mathcal{Q}$ of R maximal with respect to the property that no finite subfamily of $\mathcal{A}$ covers $V_\ell(\mathcal{Q}M)$ (i.e., $\mathcal{Q}$ is a maximal element of S). It is clear that $\mathcal{Q}M \neq M$. We claim that $\mathcal{Q}$ is a prime ideal of R, for if not, suppose that I and J are two ideals of R properly containing

$\mathcal{Q}$ and $IJ \subseteq \mathcal{Q}$. Then $V_\ell(IM)$ and $V_\ell(JM)$ covered by finite subfamily of $\mathcal{A}$. Suppose $P \in V_\ell(IJM)$, then $IJ \subseteq \mathcal{P} := (P : M)$. Since $\mathcal{P}$ is prime, either $I \subseteq \mathcal{P}$ or $J \subseteq \mathcal{P}$, and hence either $P \in V_\ell(IM)$ or $P \in V_\ell(JM)$. Thus $V_\ell(IJM)$ covered by a finite subfamily of $\mathcal{A}$. Since $IJ \subset \mathcal{Q}$, then $V_\ell(\mathcal{Q}M) \subseteq V_\ell(IJM)$. Thus $V_\ell(\mathcal{Q}M)$ covered by finite subfamily of $\mathcal{A}$, a contradiction. Thus $\mathcal{Q}$ is a prime ideal of R. We claim that $(\mathcal{Q}M : M) = \mathcal{Q}$, for if not, then there exists an ideal $\mathcal{Q}_1$ of R such that $\mathcal{Q}_1 = (\mathcal{Q}M : M)$ and $\mathcal{Q} \subset \mathcal{Q}_1$. This follows that $\mathcal{Q}M = \mathcal{Q}_1 M$ and so no finite subfamily of $\mathcal{A}$ covers $V_\ell(\mathcal{Q}_1 M)$, contrary to maximality of $\mathcal{Q}$. Therefore $(\mathcal{Q}M : M) = \mathcal{Q}$ and since M is p^*-module, there exists $Q \in \ell.\mathrm{Spec}(M)$ such that $(Q : M) = \mathcal{Q}$. Let $U \in \mathcal{A}$ such that $Q \in U$. By Lemma 2.3, there exists a submodule K of M such that $\mathcal{Q} = (Q : M) \subset (K : M)$ and

$$Q \in U_\ell(K) \bigcap V_\ell(Q) \subseteq U.$$

Let $(K : M) = I$. By Lemma 1.4, we know that $U_\ell(K) = U_\ell(IM)$ and $V_\ell(Q) = V_\ell(\mathcal{Q}M)$, and so $Q \in U_\ell(IM) \bigcap V_\ell(\mathcal{Q}M) \subseteq U$. Since $\mathcal{Q} \subset I$, then $V_\ell(IM)$ can be covered by some finite subfamily $\mathcal{A}'$ of $\mathcal{A}$. But

$$V_\ell(\mathcal{Q}M) \setminus V_\ell(IM) = V_\ell(\mathcal{Q}M) \setminus [U_\ell(IM)]^c = V_\ell(\mathcal{Q}M) \bigcap U_\ell(IM) \subseteq U.$$

and so $V_\ell(\mathcal{Q}M)$ can be covered by $\mathcal{A}' \bigcup \{U\}$, contrary to our choice of Q. Thus there must exist a finite subfamily of $\mathcal{A}$ which covers $\ell.\mathrm{Spec}({}_RM)$. Therefore $\ell.\mathrm{Spec}({}_RM)$ is compact in the finer lower patch topology of M. □

It is well-known that if M is a Noetherian module over a commutative ring R, then $R/Ann(M)$ is a Noetherian ring. Thus by Lemma 2.7 and Theorem 2.8, we conclude that for each Noetherian module M over a commutative ring R, $\ell.\mathrm{Spec}({}_RM)$ with the finer lower patch topology is a compact space. Furthermore, by a similar method that used in the proof of Theorem 2.8, we show that this fact is also true for a Noetherian module over a non-commutative ring.

Theorem 2.9. *Let M be a Noetherian left R-module. Then $\ell.Spec({}_RM)$ with the finer lower patch topology is a compact space.*

Proof. Let M be a Noetherian left R-module, and let $\mathcal{A}$ be a family of finer lower patch-open sets covering $\ell.\mathrm{Spec}({}_RM)$. Suppose that no finite subfamily of $\mathcal{A}$ covers $\ell.\mathrm{Spec}({}_RM)$. Let

$$T = \{\mathcal{L}M \mid \mathcal{L} \text{ is an ideal of } R \text{ such that no finite subfamily of } \mathcal{A} \textit{ covers } V_\ell(\mathcal{L}M)\}.$$

Since $V_\ell(0M) = V_\ell(0) = \ell.\mathrm{Spec}(M)$, $T \neq \emptyset$. We may use the ACC on submodules of M to choose an ideal $\mathcal{F}$ of R such that $\mathcal{F}M$ maximal with respect to the property that no finite subfamily of $\mathcal{A}$ covers $V_\ell(\mathcal{F}M)$. Let $(\mathcal{F}M : M) = \mathcal{Q}$.

Then $V_\ell(\mathcal{F}M) = V_\ell(\mathcal{Q}M)$. It is clear that $\mathcal{Q}M \neq M$. We claim that $\mathcal{Q}$ is a prime ideal of R, for if not, suppose that I and J are two ideals of R properly containing $\mathcal{Q}$ and $IJ \subseteq \mathcal{Q}$. Then $\mathcal{F}M \subset IM$ and $\mathcal{F}M \subset JM$. Thus $V_\ell(IM)$ and $V_\ell(JM)$ covered by finite subfamily of $\mathcal{A}$. Suppose $P \in V_\ell(IJM)$, then $IJ \subseteq \mathcal{P} := (P : M)$. Since $\mathcal{P}$ is prime, either $I \subseteq \mathcal{P}$ or $J \subseteq \mathcal{P}$, and hence either $P \in V_\ell(IM)$ or $P \in V_\ell(JM)$. Thus $V_\ell(IJM)$ covered by a finite subfamily of $\mathcal{A}$. Since $IJ \subseteq \mathcal{P}$, then $V_\ell(\mathcal{Q}M) \subseteq V_\ell(IJM)$. Thus $V_\ell(\mathcal{Q}M)$ covered by finite subfamily of $\mathcal{A}$, a contradiction. Thus $\mathcal{Q}$ is a prime ideal of R. Now we claim that $(\mathcal{Q}M : M) = \mathcal{Q}$, for if not, then there exists an ideal $\mathcal{Q}_1$ of R such that $\mathcal{Q} \subset \mathcal{Q}_1$ and $\mathcal{Q}_1 = (\mathcal{Q}M : M)$. Therefore $\mathcal{Q}_1 M \subseteq \mathcal{Q}M \subseteq \mathcal{F}M$ and hence $\mathcal{Q}_1 \subseteq (\mathcal{F}M : M) = \mathcal{Q}$, a contradiction. Thus $(\mathcal{Q}M : M) = \mathcal{Q}$. By Lemma 2.7, M is p^*-module and so there exists $Q \in \ell.\text{Spec}(M)$ such that $(Q : M) = \mathcal{Q}$. Let $U \in \mathcal{A}$ such that $Q \in U$. By Lemma 2.3, there exists a submodule K of M such that $\mathcal{Q} = (Q : M) \subset (K : M)$ and

$$Q \in U_\ell(K) \bigcap V_\ell(Q) \subseteq U.$$

Let $(K : M) = I$. By Lemma 1.4, we know that $U_\ell(K) = U_\ell(IM)$ and $V_\ell(Q) = V_\ell(\mathcal{Q}M)$, and so $Q \in U_\ell(IM) \bigcap V_\ell(\mathcal{Q}M) \subseteq U$. Since $\mathcal{Q} \subset I$, then $V_\ell(IM)$ can be covered by some finite subfamily $\mathcal{A}'$ of $\mathcal{A}$. But

$$V_\ell(\mathcal{Q}M) \setminus V_\ell(IM) = V_\ell(\mathcal{Q}M) \setminus [U_\ell(IM)]^c = V_\ell(\mathcal{Q}M) \bigcap U_\ell(IM) \subseteq U.$$

and so $V_\ell(\mathcal{Q}M)$ can be covered by $\mathcal{A}' \bigcup \{U\}$, contrary to our choice of Q. Thus there must exist a finite subfamily of $\mathcal{A}$ which covers $\ell.\text{Spec}({}_RM)$. Therefore $\ell.\text{Spec}({}_RM)$ is compact in the finer lower patch topology of M. □

We need the following evident lemma.

Lemma 2.10. *Assume that τ, τ^* are two topology on X such that $\tau \subseteq \tau^*$. If X is quasi-compact in τ^* then τ is also quasi-compact in τ.*

Theorem 2.11. *Let M be an R-module. If $\ell.Spec({}_RM)$ is compact with the finer lower patch topology, then for each submodule N of M, $U_\ell(N)$ is a quasi-compact subset of $\ell.Spec({}_RM)$ with the lower Zariski topology. Consequently, $\ell.Spec({}_RM)$ with the lower Zariski topology is quasi-compact.*

Proof. By Definition 2.2, for each submodule N of M, $V_\ell(N) = V_\ell(N) \bigcap U_\ell(M)$ is an open subset of $\ell.\text{Spec}({}_RM)$ with finer lower patch topology, and hence, for each submodule N of M, $U_\ell(N)$ is a closed subset in $\ell.\text{Spec}({}_RM)$ with finer lower patch topology. Since every closed subset of a compact space is compact, $U_\ell(N)$ is compact in $\ell.\text{Spec}({}_RM)$ with finer lower patch topology and so by Lemma 2.10, it is quasi-compact in $\ell.\text{Spec}({}_RM)$ with the lower Zariski topology. Now, since $\ell.\text{Spec}({}_RM) = U_\ell(M)$, $\ell.\text{Spec}({}_RM)$ is quasi-compact with lower Zariski topology.□

Corollary 2.12. *Let M be a left R-module. If ℓ.Spec$({}_RM)$ is compact with finer lower patch topology, then the finer lower patch topology and the lower patch topology of M coincide.*

Proof. By Theorem 2.11, for each submodule K of M, $U_\ell(K)$ is quasi-compact. Therefore for each $N, K \leq M$, $V_\ell(N) \bigcap U_\ell(K)$ is an element of the basis $\mathcal{U}_\ell(M)$ of the lower patch topology on ℓ.Spec$({}_RM)$. □

Corollary 2.13. *Let M be left R-module. If M is Noetherian or M is a p^*-module such that $R/Ann(M)$ has ACC on ideals, then the finer lower patch topology and the lower patch topology of M coincide.*

Proof. By Theorem 2.8, Theorem 2.9 and Corollary 2.12. □

We conclude this section with the following corollaries.

Corollary 2.14. *Let M be a p^*-module and $R/Ann(M)$ has ACC on ideals. Then ℓ.Spec$({}_RM)$ with the lower Zariski topology is a Hausdorf, compact, totally disconnected space.*

Proof. By Proposition 2.5, Theorem 2.8, and Corollary 2.13. □

Corollary 2.15. *Let M be a Notherian left R-module. Then ℓ.Spec$({}_RM)$ with the lower Zariski topology is a Hausdorf, compact, totally disconnected space.*

Proof. By Proposition 2.5, Theorem 2.9, and Corollary 2.13. □

3. Modules whose lower Zariski topologies are spectral

Let M be an R-module and let ℓ.*Spec*$({}_RM)$ be endowed with the lower Zariski topology. For each subset Y of ℓ.Spec$({}_RM)$, We will denote the closure of Y in ℓ.Spec$({}_RM)$ by $\overline{Y}$, and intersections of elements of Y by $\Im(Y)$ (note that if $Y = \emptyset$, then $\Im(Y) = M$).

A topological space X is called *irreducible* if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topology space X is called an *irreducible set* if the subspace Y of X is irreducible. For this to be so, it is necessary and sufficient that, for every pair of sets Y_1, Y_2 which are closed in X and satisfy $Y \subseteq Y_1 \cup Y_2$, $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (see [6, page 94]).

A topological space X is a T_0-space if and only if for any two distinct points in X there exists an open subset of X which contains one of the points but not the other. This characterization should be contrasted with an analogous characterization of T_1 spaces, where one can specify beforehand which points will belong to the open set.

We know that, for any ring R, Spec(R) is always a T_0-space for the usual Zariski topology. This is not true for $Spec({}_RM)$ (see [11, page 429]).

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a *generic point* of Y if $Y = \overline{\{y\}}$. Note that a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space.

Following Hochster [10], we say that a topological space X is a *spectral space* in case X is homeomorphic to Spec(S), with the Zariski topology, for some commutative ring S. Spectral spaces have been characterized by [10, page 52, Proposition 4] as the topological spaces X which satisfy the following conditions:

(i) X is a T_0-space;
(ii) X is quasi-compact;
(iii) the quasi-compact open subsets of X are closed under finite intersection and form an open base;
(iv) each irreducible closed subset of X has a generic point.

For any commutative ring R, Spec(R) is well-known to satisfy these condition (see [6, Chap II, 401-403]).

Corollary 3.1. *Let M ba a module over a commutative ring R. For each following cases $\ell.Spec(M)$ with lower Zariski topology is a spectral space:*

(1) *M is a finitely generated nonzero R-module ;*
(2) *M is a faithfully flat nonzero R-module;*
(3) *M is a free module.*

Proof. By Corollary 1.11 is clear. □

In this section, we will show that if $\ell.\text{Spec}({}_RM)$ with the finer lower patch topology is quasi compact, then $\ell.\text{Spec}({}_RM)$ with the lower Zariski topology is a spectral space.

Proposition 3.2. *Let M be an left R-module and let $Y = \{P_1, P_2, ..., P_k\}$ be a finite subset of $\ell.Spec({}_RM)$ with lower Zariski topology. Then $\overline{Y} = V_\ell(P_1) \cup V_\ell(P_2) \cup .... \cup V_\ell(P_k)$.*

Proof. Clearly, $Y \subseteq V_\ell(P_1) \cup V_\ell(P_2) \cup .... \cup V_\ell(P_k)$. Suppose F be any closed subset of $\ell.Spec(M)$ such that $Y \subseteq F$. Thus $F = V_\ell(N)$, for submodule $N$of M. Let $Q \in V_\ell(P_1) \cup V_\ell(P_2) \cup .... \cup V_\ell(P_k)$. Then there exists j $(1 \leq j \leq k)$ such that $Q \in V_\ell(P_j)$ and so $(P_j : M) \subseteq (Q : M)$. Since $P_j \in F$, $(N : M) \subseteq (P_j : M) \subseteq (Q : M)$, and hence $Q \in F$. Thus $V_\ell(P_1)\cup V_\ell(P_2)\cup....\cup V_\ell(P_k) \subseteq F$. Therefore, $\overline{Y} = V_\ell(P_1)\cup V_\ell(P_2)\cup....\cup V_\ell(P_k)$.□

The above proposition immediately yields that the following interesting result.

Corollary 3.3. *Let M be a left R-module. Then*

(a) $\{\overline{P}\} = V_\ell(P)$, *for all* $P \in \ell.Spec(M)$.
(b) $Q \in \{\overline{P}\}$ *if and only if* $(P : M) \subseteq (Q : M)$ *if and only if* $V_\ell(Q) \subseteq V_\ell(P)$.
(c) *The set* $\{P\}$ *is closed in* $\ell.Spec(_RM)$ *if and only if* P *is a virtually maximal prime submodule of* M.

Proof. By Proposition 3.2 is clear. □

Lemma 3.4. *Let M be a left R-module and $P, Q \in \ell.Spec(_RM)$. If $V_\ell(P) = V_\ell(Q)$, then $P = Q$.*

Proof. If $V_\ell(P) \subseteq V_\ell(Q)$, then $P \in V_\ell(Q)$. Therefore $(Q : M) \subseteq (P : M)$. On the other hand $V_\ell(Q) \subseteq V_\ell(P)$, then $Q \in V_\ell(P)$ and then $(P : M) \subseteq (Q : M)$. Therefore $(P : M) = (Q : M)$ and hence $P = Q$. □

Proposition 3.5. *Let M be a left R-module. Then $\ell.Spec(_RM)$ with the lower Zariski topology is a T_0-space.*

Proof. Let P_1, P_2 be two distinct prime submodules of M. Since $P_1 \neq P_2$, by Lemma 3.3, $V_\ell(P) \neq V_\ell(Q)$. Therefore either $P \notin V_\ell(Q)$ or $Q \notin V_\ell(P)$. We assume that $P \notin V_\ell(Q)$. Thus $P \in U_\ell(Q)$, but $Q \notin U_\ell(Q)$. This means that $\ell.Spec(M)$ is a T_0-space. □

Lemma 3.6. *Let M be a left R-module. Then for each $P \in Spec(_RM)$, $V_\ell(P)$ is irreducible.*

Proof. Let $V_\ell(P) \subseteq Y_1 \bigcup Y_2$, where Y_1 and Y_2 are closed sets. Then there exist $N_1, N_2 \leq M$ such that $Y_1 = V_\ell(N_1)$ and $Y_2 = V_\ell(N_2)$. Suppose that Q is a lower prime submodule of M such that $(Q : M) = (P : M)$. Since $Q \in V_\ell(P)$, either $Q \in Y_1$ or $Q \in Y_2$. Without loss of generality we can assume that $Q \in Y_1 = V_\ell(N_1)$, then $(N_1 : M) \subseteq (Q : M) = (P : M)$. Therefore $V_\ell(P) \subseteq V_\ell(N_1) = Y_1$. Thus $V_\ell(P)$ is irreducible. □

Corollary 3.7. *Let M be a left R-module and P be a prime submodule of M. If $Q \in \ell.Spec(_RM)$ such that $(Q : M) = (P : M)$. Then Q is a generic point* *for the irreducible closed subset $V_\ell(P)$ of ℓ.Spec$({}_RM)$.*

Proof. By Lemma 3.6, $V_\ell(P)$ is irreducible closed subset of $\ell.\mathrm{Spec}({}_RM)$. On the other hand by Corollary 3.3, $\{\overline{Q}\} = V_\ell(Q) = V_\ell(P)$. Thus Q is a generic point of irreducible closed subset $V_\ell(P)$. □

Let M be a left R-module and $Y \subseteq \ell.\mathrm{Spec}({}_RM)$. We will denote the intersection of all elements in Y by $\Im(Y)$ and closure of Y in $\ell.\mathrm{Spec}({}_RM)$ with the lower Zariski topology by $\overline{Y}$.

Proposition 3.8. *Let M be a left R-module and $Y \subseteq \ell.Spec({}_RM)$. Then $V_\ell(\Im(Y)) = \bar{Y}$. Hence, Y is closed if and only if $V_\ell(\Im(Y)) = Y$.*

Proof. Clearly, $Y \subseteq V_\ell(\Im(Y))$. Let $V_\ell(N)$ be a closed subset of $\ell.\mathrm{Spec}({}_RM)$ containing Y. Then $(N : M) \subseteq (P : M)$ for every $P \in Y$ so that $(N : M) \subseteq (\Im(Y) : M)$. Hence for every $Q \in V_\ell(\Im(Y))$, $(N : M) \subseteq (\Im(Y) : M) \subseteq (Q : M)$. Therefore $V_\ell(\Im(Y)) \subseteq V_\ell(N)$. Thus $\overline{Y} = V_\ell(\Im(Y))$. □

Now we show that if $Y \subseteq \ell.\mathrm{Spec}({}_RM)$ such that $\Im(Y)$ is a prime submodule of M, then Y is irreducible.

Proposition 3.9. *Let M be a left R-module and $Y \subseteq \ell.Spec({}_RM)$. If $\Im(Y)$ is a prime submodule of M, then Y is irreducible.*

Proof. Suppose that $P := \Im(Y)$ is a prime submodule of M. By Proposition 3.8, $\bar{Y} = V_\ell(P)$. Now let $Y \subseteq Y_1 \cup Y_2$, where Y_1, Y_2 are closes sets. Thus $\bar{Y} \subseteq Y_1 \cup Y_2$. Since $V_\ell(P) \subseteq Y_1 \cup Y_2$ and by Lemma 3.6, $V_\ell(P)$ is irreducible, $V_\ell(P) \subseteq Y_1$ or $V_\ell(P) \subseteq Y_2$. This follows that either $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (since $Y \subseteq V_\ell(P)$). Thus Y is irreducible. □

Let R be a ring and M be a left R-module. For any ideal I of R, $\sqrt{I}$ will denote the *radical* of I, that is

$$\sqrt{I} = \bigcap\{\mathcal{P} : \mathcal{P} \text{ is a prime ideal of } R \text{ and } I \subseteq \mathcal{P}\}$$

Also, for a submodule N of M the *prime radical* $\sqrt{N}$ (or $\mathrm{rad}_M(N)$) is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule then $\sqrt{N}$ is defined to be M. In particular, for any module M, we define $rad_R(M) = \sqrt{(0)}$. This is called *prime radical* of M. Thus, if M has a prime submodule, then $rad_R(M)$ is equal to the intersection of all the prime submodules in M but, if M has no prime submodule, then $rad_R(M) = M$.

Corollary 3.10. *Let M be a left R-module and $N \leq M$. If $\sqrt{N}$ is a prime submodule, then the subset $V_\ell(N)$ of $\ell.Spec({}_RM)$ is irreducible with the* *lower Zariski topology.. Consequently, If $rad_R(M)$ is a prime submodule, then $\ell.Spec(_RM)$ is irreducible.*

Proof. By Proposition 3.9. □

Proposition 3.11. *Let M be a left R-module. If $\ell.Spec(_RM)$ is quasi-compact with the finer lower patch topology, then every irreducible closed subset of $\ell.Spec(_RM)$ with the lower Zariski topology has a generic point.*

Proof. The first we show that $Y = \bigcup_{P\in Y} V_\ell(P)$. Clearly $Y \subseteq \bigcup_{P\in Y} V_\ell(P)$. By Corollary 3.3(a), for each $P \in Y$ we have $V_\ell(P) = \{\overline{P}\} \subseteq \overline{Y}$, and since $\overline{Y} = Y$, $\bigcup_{P\in Y} V_\ell(P) \subseteq Y$. By Definition 2.2, for each $P \in Y$, $V_\ell(P) = V_\ell(P) \bigcap U_\ell(M)$ is an open subset of $\ell.\mathrm{Spec}(_RM)$ with the finer lower patch topology. Since Y is a closed subset in $\ell.\mathrm{Spec}(_RM)$ with the finer lower patch topology and since every closed subset of a compact space is compact, Y is compact in $\ell.\mathrm{Spec}(_RM)$ with the finer lower patch topology. Thus there exists a finite subset Y' of Y such that $Y = \bigcup_{P\in Y'} V_\ell(P)$. Also since Y is irreducible $Y = V_\ell(P)$ for some $P \in Y$ and so Y has a generic point. □

Corollary 3.12. *Let M be a left R-module with $|\ell.Spec(_RM)| < \infty$. Then every irreducible closed subset of $\ell.Spec(_RM)$ with the lower Zariski topology has a generic point.*

Corollary 3.13. *Let M be a p^*-module over a ring R such that $R/Ann(M)$ has ACC on ideals. Then every irreducible closed subset of $\ell.Spec(_RM)$ with the lower Zariski topology has a generic point.*

Corollary 3.14. *Let M be a Notherian left R-module. Then every irreducible closed subset of $\ell.Spec(_RM)$ with the lower Zariski topology has a generic point.*

Theorem 3.15. *Let M be a left R-module. If $\ell.Spec(_RM)$ is quasi-compact with the finer lower patch topology. Then $\ell.Spec(_RM)$ with the lower Zariski topology is a spectral space.*

Proof. By Proposition 3.5, $\ell.\mathrm{Spec}(_RM)$ is a T_0-space and by Theorem 2.12, $\ell.\mathrm{Spec}(_RM)$ is quasi-compact and has a basis of quasi-compact open subsets. Also, by Theorem 2.12, the family of quasi-compact open subsets of $\ell.Spec(M)$ is closed under finite intersections. Finally, by Proposition 3.11, every irreducible closed subset of $\ell.\mathrm{Spec}(_RM)$ has a generic point. Thus by Hochster's characterization of a spectral space, $\ell.\mathrm{Spec}(_RM)$ is a spectral space. □

Corollary 3.16. *Let M be a left R-module such that $|\ell.Spec(_RM)| < \infty$. Then $\ell.Spec(_RM)$ with lower Zariski topology is a spectral space.*

Proof. By Theorem 3.15 is clear. □

Corollary 3.17. *Let M be a p^*-module over ring R such that $R/Ann(M)$ has ACC on ideals. Then $\ell.Spec(_RM)$ with the lower Zariski topology is a spectral space.*

Proof. By Theorem 2.8 and Theorem 3.15 is clear. □

We conclude this article with the following interesting results for Noetherian modules and for arbitrary rings, respectively.

Corollary 3.18. *Let M be a Noetherian left R-module. Then $\ell.Spec(_RM)$ with the lower Zariski topology is a spectral space.*

Proof. By Theorem 2.9. Corollary 2.15 and Theorem 3.15. □

Corollary 3.19. *Let R be a ring (not necessary commutative) with ACC on ideals. Then Spec(R) with the usual Zariski topology is a spectral space i.e., Spec(R) is homeomorphic to Spec(S) for some commutative ring S.*

Proof. By Corollary 2.14 and Theorem 3.15. □

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