

FURTHER RESULTS ON THE NEUTRIX COMPOSITION OF THE DELTA FUNCTION

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Abstract

Let F be a distribution in \mathcal{D}' and let f be a locally summable function. The composition $F(f(x))$ of F and f is said to exist and be equal to the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta. It is proved that the neutrix composition $\delta^{(s)}[\ln^r(1 + x_+^{1/r})]$ exists and is given by

$$\sum_{k=0}^s \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{(r+1)k+r+s+i-1} s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!} \delta^{(k)}(x)$$

for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$. Further results are also proved.

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1. Introduction

In the theory of distributions, many arguments show that no meaning can be generally given to expressions of the form $F(f(x))$, where F is a distribution and f is a locally summable function.

Using the concepts of a neutrix and neutrix limit due to van der Corput [1], the third author gave a general principle for the discarding of unwanted infinite quantities from asymptotic expansions. This has been exploited in the context of distributions, particularly in connection with the composition of distributions, see [2, 3]. With Fisher's definition, Koh and Li gave a meaning to δ^r and $(\delta')^r$ for $r = 2, 3, \dots$, see [12], and the more general form $(\delta^{(s)}(x))^r$ was considered by Kou and Fisher in [13]. More recently the r -th powers of the Dirac function $\delta(x)$ and the Heaviside function $H(x)$ for negative integers have been defined in [14] and [15] respectively.

In the following, we let \mathcal{D} be the space of infinitely differentiable functions with compact support, let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$ and let \mathcal{D}' be the space of distributions defined on \mathcal{D} .

Now let $\rho(x)$ be a function in \mathcal{D} having the following properties:

- (i) $\rho(x) = 0$ for $|x| \geq 1$,
- (ii) $\rho(x) \geq 0$,
- (iii) $\rho(x) = \rho(-x)$,
- (iv) $\int_{-1}^1 \rho(x) dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is an arbitrary distribution in \mathcal{D}' and $F_n(x) = F(x) * \delta_n(x) = \langle F(x-t), \varphi(t) \rangle$, then $\{F_n(x)\}$ is a regular sequence converging to $F(x)$.

If f is an infinitely differentiable function having a single simple zero at the point $x = x_0$, then the distribution $\delta^{(r)}(f(x))$ is defined by

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|} \left[\frac{1}{|f'(x)|} \frac{d}{dx} \right]^r \delta(x - x_0) \quad (1)$$

for $r = 0, 1, 2, \dots$, see [11].

The third author generalized equation (1) in [2] as follows:

Definition 1. *Let f be an infinitely differentiable function. We say that the neutrix composition $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a, b) , with $-\infty < a < b < \infty$, if*

$$\text{N-lim}_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x)) \varphi(x) dx = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}[a, b]$, where N is the neutrix, see [1], having domain N' the positive and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$.

Definition 1 was later generalized with the following definition in [3] and was originally called the neutrix composition of distributions.

Definition 2. Let F be a distribution in \mathcal{D}' and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) , with $-\infty < a < b < \infty$, if

$$N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}[a, b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$. In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x) \rangle$$

for all φ in $\mathcal{D}[a, b]$.

The following theorem was proved in [4].

Theorem 1. The neutrix composition $\delta^{(s)}(\operatorname{sgn} x|x|^\lambda)$ exists and

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = 0$$

for $s = 0, 1, 2, \dots$ and $(s + 1)\lambda = 1, 3, \dots$ and

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{\lambda[(s + 1)\lambda - 1]!} \delta^{((s+1)\lambda-1)}(x)$$

for $s = 0, 1, 2, \dots$ and $(s + 1)\lambda = 2, 4, \dots$

Next two theorems were proved in [5].

Theorem 2. The compositions $\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s})$ and $\delta^{(s-1)}(|x|^{1/s})$ exist and

$$\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s}) = \frac{(2s)!}{2} \delta'(x), \quad \delta^{(s-1)}(|x|^{1/s}) = (-1)^{s-1} \delta(x)$$

for $s = 1, 2, \dots$

Theorem 3. *The neutrix composition $\delta^{(s)}[\ln(1 + |x|^{1/r})]$ exists on the interval $(-1, 1)$ and*

$$\begin{aligned} &\delta^{(s)}[\ln(1 + |x|^{1/r})] \\ &= \sum_{k=0}^m \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{s+i-1} [1 + (-1)^k]^r (i+1)^s}{2(k!)} \delta^{(k)}(x) \end{aligned} \quad (2)$$

for $r = 2, 3, \dots$ and $s = 0, 1, 2, \dots$, where m denotes the largest integer less than or equal $(s + 1)/r - 1$.

In particular, the composition $\delta^{(s)}[\ln(1 + |x|^{1/r})]$ exists and

$$\delta^{(s)}[\ln(1 + |x|^{1/r})] = 0 \quad (3)$$

for $r = 2, 3, \dots$ and $s = 0, 1, 2, \dots, r - 2$, and

$$\delta^{(r-1)}[\ln(1 + |x|^{1/r})] = (-1)^{r-1} r! \delta(x) \quad (4)$$

for $r = 2, 3, \dots$

2. Main Results

We now prove the following theorem.

Theorem 4. *The neutrix composition $\delta^{(s)}[\ln^r(1 + x_+^{1/r})]$ exists and*

$$\begin{aligned} &\delta^{(s)}[\ln^r(1 + x_+^{1/r})] \\ &= \sum_{k=0}^s \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{(r+1)k+r+s+i-1} s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!} \delta^{(k)}(x) \end{aligned} \quad (5)$$

for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$

Proof. To prove equation (5), we will first of all evaluate

$$\text{N-}\lim_{n \rightarrow \infty} \langle \delta_n^{(s)}[\ln^r(1 + x_+^{1/r})], \varphi(x) \rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[a, 1]$, where $a < 0$.

By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^s \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^{s+1}}{(s+1)!} \varphi^{(s+1)}(\xi x),$$

where $0 < \xi < 1$. Then if $\varphi(x)$ in $\mathcal{D}[a, 1]$, we have

$$\begin{aligned} & \text{N-lim}_{n \rightarrow \infty} \langle \delta_n^{(s)}[\ln^r(1 + x_+^{1/r})], \varphi(x) \rangle \\ &= \text{N-lim}_{n \rightarrow \infty} \sum_{k=0}^s \frac{\varphi^{(k)}(0)}{k!} \int_a^1 \delta_n^{(s)}[\ln^r(1 + x_+^{1/r})] x^k dx \\ &+ \text{N-lim}_{n \rightarrow \infty} \frac{1}{(s+1)!} \int_a^1 \delta_n^{(s)}[\ln^r(1 + x_+^{1/r})] x^{s+1} \varphi^{(s+1)}(\xi x) dx. \quad (6) \end{aligned}$$

For large enough n , we have

$$\begin{aligned} & \int_a^1 \delta_n^{(s)}[\ln^r(1 + x_+^{1/r})] x^k dx \\ &= n^{s+1} \int_a^1 \rho^{(s)}[n \ln^r(1 + x_+^{1/r})] x^k dx \\ &= n^{s+1} \int_a^0 \rho^{(s)}[n \ln^r(1 + x_+^{1/r})] x^k dx + n^{s+1} \int_0^1 \rho^{(s)}[n \ln^r(1 + x_+^{1/r})] x^k dx \\ &= n^{s+1} \rho^{(s)}(0) \int_a^0 x^k dx + n^{s+1} \int_0^1 \rho^{(s)}[n \ln^r(1 + x^{1/r})] x^k dx \\ &= -\frac{n^{s+1} a^{k+1} \rho^{(s)}(0)}{k+1} + n^{s+1} \int_0^1 \rho^{(s)}[n \ln^r(1 + x^{1/r})] x^k dx \\ &= E_1 + E_2. \quad (7) \end{aligned}$$

It follows immediately that

$$\text{N-lim}_{n \rightarrow \infty} E_1 = 0. \quad (8)$$

Making the substitution $t = n \ln^r(1 + x^{1/r})$, we have

$$\begin{aligned} E_2 &= n^{s+1} \int_0^1 \rho^{(s)}[n \ln^r(1 + x^{1/r})] x^k dx \\ &= n^{s+1-1/r} \int_0^1 t^{1/r-1} \{\exp[(t/n)^{1/r}] - 1\}^{kr+r-1} \exp[(t/n)^{1/r}] \rho^{(s)}(t) dt \\ &= n^{s+1-1/r} \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} (-1)^{kr+r+i-1} \\ &\quad \times \int_0^1 t^{1/r-1} \exp[(i+1)(t/n)^{1/r}] \rho^{(s)}(t) dt, \quad (9) \end{aligned}$$

where

$$\begin{aligned} n^{s+1-1/r} \int_0^1 t^{1/r-1} \exp[(i+1)(t/n)^{1/r}] \rho^{(s)}(t) dt \\ = \sum_{j=0}^{sr+r-2} \int_0^1 \frac{(i+1)^j t^{(j+1)/r-1}}{j! n^{(j+1)/r-s-1}} \rho^{(s)}(t) dt \\ + \frac{1}{(sr+r-1)!} \int_0^1 (i+1)^{sr+r-1} t^s \rho^{(s)}(t) dt + O(n^{-1/r}). \end{aligned}$$

It follows that

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} n^{s+1-1/r} \int_0^1 t^{1/r-1} \exp[(i+1)(t/n)^{1/r}] \rho^{(s)}(t) dt \\ = \frac{(-1)^s s! (i+1)^{sr+r-1}}{2(sr+r-1)!} \end{aligned} \tag{10}$$

for $i = 0, 1, 2, \dots, kr+r-1$ and it now follows from equations (9) and (10) that

$$N\text{-}\lim_{n \rightarrow \infty} E_2 = \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{kr+r+s+i-1} s! (i+1)^{sr+r-1}}{2(sr+r-1)!}. \tag{11}$$

Then using equations (7), (8) and (11), we see that

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} \int_a^1 \delta_n^{(s)}[\ln^r(1+x_+^{1/r})] x^k dx \\ = \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{kr+r+s+i-1} s! (i+1)^{sr+r-1}}{2(sr+r-1)!}, \end{aligned} \tag{12}$$

for $k = 0, 1, 2, \dots, s$.

When $k = s+1$, we have

$$\begin{aligned} \int_0^1 \left| \delta_n^{(s)}[\ln^r(1+x_+^{1/r})] x^{s+1} \right| dx \\ \leq n^{s+1-1/r} \int_0^1 t^{1/r-1} \{ \exp[(t/n)^{1/r}] - 1 \}^{sr+2r-1} \exp[(t/n)^{1/r}] |\rho^{(s)}(t)| dt \\ = n^{s+1-1/r} \int_0^1 t^{1/r-1} [(t/n)^{1/r} + O(n^{-2/r})]^{sr+2r-1} [1 + O(n^{-1/r})] |\rho^{(s)}(t)| dt \\ = n^{s+1-1/r} \int_0^1 t^{1/r-1} [(t/n)^{s+2-1/r} + O(n^{-(s+2)})] |\rho^{(s)}(t)| dt \\ = O(n^{-1}) \end{aligned} \tag{13}$$

and so if ψ is an arbitrary function in $\mathcal{D}[a, 1]$, we have

$$\lim_{n \rightarrow \infty} \int_0^1 \left| \delta_n^{(s)}[\ln^r(1 + x_+^{1/r})] x^{s+1} \psi(x) \right| dx = 0. \tag{14}$$

It then follows from equations (6), (12) and (14) that

$$\begin{aligned} & \text{N-}\lim_{n \rightarrow \infty} \langle \delta_n^{(s)}[\ln^r(1 + x_+^{1/r})], \varphi(x) \rangle \\ &= \sum_{k=0}^s \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{kr+r+s+i-1} s!(i+1)^{sr+r-1} \varphi^{(k)}(0)}{2(sr+r-1)!k!} \\ &= \sum_{k=0}^s \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \\ &\quad \times \frac{(-1)^{(r+1)k+r+s+i-1} s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!} \langle \delta^{(k)}(x), \varphi(x) \rangle, \end{aligned}$$

proving equation (5) on the interval $[a, 1]$.

Since $\delta_n^{(s)}[\ln^r(1 + x_+^{1/r})] = 0$ for $x > 0$, it follows that equation (5) holds for $x > a$ and since $a < 0$ is arbitrary, it follows that equation (5) holds on the real line, completing the proof of the theorem.

Theorem 5. *The neutrix composition $\delta^{(s)}[\ln^r(1 + |x|^{1/r})]$ exists and*

$$\begin{aligned} & \delta^{(s)}[\ln^r(1 + |x|^{1/r})] \\ &= \sum_{k=0}^s \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \\ &\quad \times \frac{(-1)^{r+s+k+i-1} [1 + (-1)^k] s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!} \delta^{(k)}(x), \tag{15} \end{aligned}$$

for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$.

Proof. To prove equation (15), we now have to evaluate

$$\text{N-}\lim_{n \rightarrow \infty} \langle \delta_n^{(s)}[\ln^r(1 + |x|^{1/r})], \varphi(x) \rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1, 1]$. By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^s \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^{s+1}}{(s+1)!} \varphi^{(s+1)}(\xi x),$$

where $0 < \xi < 1$. Then if φ is in $\mathcal{D}[-1, 1]$, we have

$$\begin{aligned} & \text{N-lim}_{n \rightarrow \infty} \langle \delta_n^{(s)}[\ln^r(1 + |x|^{1/r})], \varphi(x) \rangle \\ &= \text{N-lim}_{n \rightarrow \infty} \sum_{k=0}^s \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^1 \delta_n^{(s)}[\ln^r(1 + x^{1/r})] x^k dx \\ & \quad + \text{N-lim}_{n \rightarrow \infty} \frac{1}{(s+1)!} \int_{-1}^1 \delta_n^{(s)}[\ln^r(1 + |x|^{1/r})] x^{s+1} \varphi^{(s+1)}(\xi x) dx. \end{aligned} \quad (16)$$

Since

$$\begin{aligned} \int_{-1}^0 \delta_n^{(s)}[\ln^r(1 + |x|^{1/r})] x^k dx &= (-1)^k \int_0^1 \delta_n^{(s)}[\ln^r(1 + |x|^{1/r})] x^k dx \\ &= (-1)^k n^{s+1} \int_0^1 \rho^{(s)}[n \ln^r(1 + x^{1/r})] x^k dx, \end{aligned} \quad (17)$$

it follows from equations (9) and (11) that

$$\begin{aligned} & \text{N-lim}_{n \rightarrow \infty} \int_{-1}^1 \delta_n^{(s)}[\ln^r(1 + |x|^{1/r})] x^k dx \\ &= \frac{[1 + (-1)^k]}{2} \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{r+s+i-1} s!(i+1)^{sr+r-1}}{(sr+r-1)!} \end{aligned} \quad (18)$$

for $k = 0, 1, 2, \dots, s$.

When $k = s + 1$, we have as in the proof of equation (14),

$$\lim_{n \rightarrow \infty} \int_{-1}^1 \left| \delta_n^{(s)}[\ln^r(1 + |x|^{1/r})] x^{s+1} \psi(x) \right| dx = 0, \quad (19)$$

for an arbitrary continuous function ψ . It then follows from equations (16), (18) and (19) that

$$\begin{aligned} & \text{N-lim}_{n \rightarrow \infty} \langle \delta_n^{(s)}[\ln^r(1 + x^{1/r})], \varphi(x) \rangle \\ &= \text{N-lim}_{n \rightarrow \infty} \sum_{k=0}^s \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^1 \delta_n^{(s)}[\ln^r(1 + |x|^{1/r})] x^k dx \\ & \quad + \lim_{n \rightarrow \infty} \frac{1}{(s+1)!} \int_{-1}^1 \delta_n^{(s)}[\ln^r(1 + |x|^{1/r})] x^{s+1} \varphi^{(s+1)}(\xi x) dx \\ &= \sum_{k=0}^s \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{r+s+i-1} [1 + (-1)^k] s!(i+1)^{sr+r-1} \varphi^{(k)}(0)}{2(sr+r-1)! k!} \\ &= \sum_{k=0}^s \sum_{i=0}^{kr+r-1} \binom{kr+r-1}{i} \frac{(-1)^{r+s+k+i-1} [1 + (-1)^k] s!(i+1)^{sr+r-1}}{2(sr+r-1)! k!} \langle \delta^{(k)}(x), \varphi(x) \rangle, \end{aligned}$$

proving equation (15) on the interval $[-1, 1]$. However, it is clear that outside this interval, $\delta_n^{(s)}[\ln^r(1 + |x|^{1/r})] = 0$, and so equation (15) is proved on the real line.

Theorem 6. *The neutrix composition $\delta^{(r^s-1)}[\ln^{1/r}(1 + |x|)]$ exists and*

$$\begin{aligned} & \delta^{(r^s-1)}[\ln^{1/r}(1 + |x|)] \\ &= \sum_{k=0}^{r^s-1-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{r^s-i-1} [1 + (-1)^k] r(r^s-1)! (i+1)^{r^s-1-1}}{2(r^s-1)! k!} \delta^{(k)}(x) \end{aligned} \quad (20)$$

for $s = 1, 2, \dots$ and $r = 2, 3, \dots$

Proof. This time we must evaluate

$$\text{N-lim}_{n \rightarrow \infty} \langle \delta_n^{(r^s-1)}[\ln^{1/r}(1 + |x|)], \varphi(x) \rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1, 1]$.

By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{r^s-1-1} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^{r^s-1}}{(r^s-1)!} \varphi^{(r^s-1)}(\xi x),$$

where $0 < \xi < 1$. Then if $\varphi(x)$ in $\mathcal{D}[-1, 1]$, we have

$$\begin{aligned} & \text{N-lim}_{n \rightarrow \infty} \langle \delta_n^{(r^s-1)}[\ln^{1/r}(1 + |x|)], \varphi(x) \rangle \\ &= \text{N-lim}_{n \rightarrow \infty} \sum_{k=0}^{r^s-1-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^1 \delta_n^{(r^s-1)}[\ln^{1/r}(1 + |x|)] x^k dx \\ & \quad + \text{N-lim}_{n \rightarrow \infty} \frac{1}{(r^s-1)!} \int_{-1}^1 \delta_n^{(r^s-1)}[\ln^{1/r}(1 + |x|)] x^{r^s-1} \varphi^{(r^s-1)}(\xi x) dx. \end{aligned} \quad (21)$$

For large enough n , we have

$$\begin{aligned} & \int_{-1}^1 \delta_n^{(r^s-1)}[\ln^{1/r}(1 + |x|)] x^k dx = n^{r^s} \int_{-1}^1 \rho^{(r^s-1)}[n \ln^{1/r}(1 + |x|)] x^k dx \\ &= n^{r^s} [1 + (-1)^k] \int_0^1 \rho^{(r^s-1)}[n \ln^{1/r}(1 + x)] x^k dx. \end{aligned} \quad (22)$$

Making the substitution $t = n \ln^{1/r}(1+x)$, we have

$$\begin{aligned} & n^{r^s} \int_0^1 \rho^{(r^s-1)}[n \ln^{1/r}(1+x)] x^k dx \\ &= rn^{r^s-r} \int_0^1 t^{r-1} \{\exp[(t/n)^r] - 1\}^k \exp[(t/n)^r] \rho^{(r^s-1)}(t) dt \\ &= rn^{r^s-r} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \int_0^1 t^{r-1} \exp[(i+1)(t/n)^r] \rho^{(r^s-1)}(t) dt, \end{aligned}$$

where

$$\begin{aligned} rn^{r^s-r} \int_0^1 t^{r-1} \exp[(i+1)(t/n)^r] \rho^{(r^s-1)}(t) dt \\ = \sum_{j=0}^{\infty} \int_0^1 \frac{r(i+1)^j t^{r(j+1)-1}}{j! n^{r(j+1)-r^s}} \rho^{(r^s-1)}(t) dt. \end{aligned}$$

It follows that

$$\begin{aligned} \text{N-}\lim_{n \rightarrow \infty} rn^{r^s-r} \int_0^1 t^{r-1} \exp[(i+1)(t/n)^r] \rho^{(r^s-1)}(t) dt \\ = \int_0^1 \frac{r(i+1)^{r^s-1-1} t^{r^s-1}}{(r^s-1-1)!} \rho^{(r^s-1)}(t) dt \\ = \frac{(-1)^{r^s-1} r(r^s-1)! (i+1)^{r^s-1-1}}{2(r^s-1-1)!} \end{aligned} \quad (23)$$

for $i = 0, 1, 2, \dots, k$ and so

$$\begin{aligned} \text{N-}\lim_{n \rightarrow \infty} n^{r^s} \int_0^1 \rho^{(r^s-1)}[n \ln^{1/r}(1+x)] x^k dx \\ = \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{r^s+k-i-1} r(r^s-1)! (i+1)^{r^s-1-1}}{2(r^s-1-1)!}. \end{aligned} \quad (24)$$

It now follows from equations (22) and (24) that

$$\begin{aligned} \text{N-}\lim_{n \rightarrow \infty} \int_{-1}^1 \delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)] x^k dx \\ = [1 + (-1)^k] \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{r^s+k-i-1} r(r^s-1)! (i+1)^{r^s-1-1}}{2(r^s-1-1)!}, \end{aligned} \quad (25)$$

for $k = 0, 1, 2, \dots, r^s-1-1$.

When $k = r^{s-1}$, we have

$$\begin{aligned} & \int_0^1 \left| \delta_n^{(r^s-1)}[\ln^{1/r}(1+x)] x^{r^{s-1}} \right| dx \\ & \leq rn^{r^s-r} \int_0^1 t^{r-1} \{ \exp[(t/n)^r] - 1 \}^{r^{s-1}} \exp[(t/n)^r] |\rho^{(r^s-1)}(t)| dt \\ & = rn^{r^s-r} \int_0^1 t^{r-1} [(t/n)^r + O(n^{-2r})]^{r^{s-1}} [1 + O(n^{-r})] |\rho^{(r^s-1)}(t)| dt \\ & = rn^{r^s-r} \int_0^1 t^{r-1} [(t/n)^{r^s} + O(n^{-(r^s+r)})] |\rho^{(r^s-1)}(t)| dt \\ & = O(n^{-r}) \end{aligned}$$

and so if ψ is an arbitrary function in $\mathcal{D}[a, 1]$, we have

$$\lim_{n \rightarrow \infty} \int_0^1 \left| \delta_n^{(r^s-1)}[\ln^{1/r}(1+x)] x^{r^{s-1}} \psi(x) \right| dx = 0.$$

Then if φ is an arbitrary function in $\mathcal{D}[-1, 1]$, we have

$$\lim_{n \rightarrow \infty} \int_{-1}^1 \left| \delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)] x^{r^{s-1}} \varphi^{r^{s-1}}(\xi x) \right| dx = 0 \quad (26)$$

and it follows from equations (21), (25) and (26) that

$$\begin{aligned} & \text{N-}\lim_{n \rightarrow \infty} \langle \delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)] x^k, \varphi(x) \rangle \\ & = \sum_{k=0}^{r^{s-1}-1} [1 + (-1)^k] \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{r^s+k-i-1} r(r^s-1)! (i+1)^{r^{s-1}-1} \varphi^{(k)}(0)}{2(r^{s-1}-1)! k!} \\ & = \sum_{k=0}^{r^{s-1}-1} [1 + (-1)^k] \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{r^s-i-1} (r^s-1)! (i+1)^{r^{s-1}-1}}{2(r^{s-1}-1)! k!} \langle \delta^{(k)}(x), \varphi(x) \rangle, \end{aligned}$$

proving equation (20) on the interval $[-1, 1]$. However, it is clear that outside this interval $\delta_n^{(r^s-1)}[\ln^{1/r}(1+|x|)] = 0$, and so equation (20) is proved.

Finally we have

Theorem 7. *The neutrix composition $\delta^{(r^s-1)}(\ln^{1/r} |1+x|)$ exists and*

$$\begin{aligned} & \delta^{(r^s-1)}(\ln^{1/r} |1+x|) \\ & = \sum_{k=0}^{r^{s-1}-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^i r(r^s-1)! (i+1)^{r^{s-1}-1}}{(r^{s-1}-1)! k!} \delta^{(k)}(x) \quad (27) \end{aligned}$$

for $s = 1, 2, \dots$ and $r = 1, 3, 5, \dots$

Proof. This time we must evaluate

$$N\text{-}\lim_{n \rightarrow \infty} \langle \delta_n^{(r^s-1)}(\ln^{1/r} |1+x|), \varphi(x) \rangle,$$

for an arbitrary function $\varphi(x)$ in $\mathcal{D}[-1, 1]$.

For large enough n , we have on making the substitution $t = n \ln^{1/r}(1+x)$,

$$\begin{aligned} & \int_{-1}^1 \delta_n^{(r^s-1)}(\ln^{1/r} |1+x|) x^k dx \\ &= \int_{-1}^1 \delta_n^{(r^s-1)}[n \ln^{1/r}(1+x)] x^k dx \\ &= r n^{r^s-r} \int_{-1}^1 t^{r-1} \{ \exp[(t/n)^r] - 1 \}^k \exp[(t/n)^r] \rho^{(r^s-1)}(t) dt \\ &= r n^{r^s-r} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \int_{-1}^1 t^{r-1} \exp[(i+1)(t/n)^r] \rho^{(r^s-1)}(t) dt, \end{aligned}$$

where

$$\begin{aligned} r n^{r^s-r} \int_{-1}^1 t^{r-1} \exp[(i+1)(t/n)^r] \rho^{(r^s-1)}(t) dt \\ = \sum_{j=0}^{\infty} \int_{-1}^1 \frac{r(i+1)^j t^{r(j+1)-1}}{j! n^{r(j+1)-r^s}} \rho^{(r^s-1)}(t) dt. \end{aligned}$$

Noting that $r^s - 1$ is an even integer when r is an odd integer, and using equation (23), it follows that

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} r n^{r^s-r} \int_{-1}^1 t^{r-1} \exp[(i+1)(t/n)^r] \rho^{(r^s-1)}(t) dt \\ = \int_{-1}^1 \frac{r(i+1)^{r^s-1-1} t^{r^s-1}}{(r^s-1-1)!} \rho^{(r^s-1)}(t) dt = \frac{r(r^s-1)!(i+1)^{r^s-1-1}}{(r^s-1-1)!} \end{aligned}$$

for $i = 0, 1, 2, \dots, k$ and so

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} n^{r^s-r} \int_{-1}^1 \rho^{(r^s-1)}[n \ln^{1/r}(1+x)] x^k dx \\ = \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{k-i} (r^s-1)!(i+1)^{r^s-1-1}}{(r^s-1-1)!}. \end{aligned}$$

Thus,

$$\begin{aligned} \text{N-}\lim_{n \rightarrow \infty} \int_{-1}^1 \delta_n^{(r^s-1)} [\ln^{1/r} |1+x|] x^k dx \\ = \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{k-i} (r^s-1)! (i+1)^{r^s-1-i}}{(r^{s-1}-1)!}, \end{aligned} \quad (28)$$

for $k = 0, 1, 2, \dots, r^{s-1} - 1$.

When $k = r^{s-1}$, it follows that $\int_{-1}^1 \delta_n^{(r^s-1)} [\ln^{1/r} (|1+x|)] x^{r^{s-1}} dx = O(n^{-r})$ and so if φ is an arbitrary function in $\mathcal{D}[-1, 1]$, then

$$\lim_{n \rightarrow \infty} \int_{-1}^1 \delta_n^{(r^s-1)} [\ln^{1/r} (|1+x|)] x^{r^{s-1}} \varphi^{(r^{s-1})}(x) dx = 0. \quad (29)$$

Now let φ be an arbitrary function in $\mathcal{D}[-1, 1]$. By Taylor's Theorem, we have $\varphi(x) = \sum_{k=0}^{r^{s-1}-1} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^{r^{s-1}}}{(r^{s-1})!} \varphi^{(r^{s-1})}(\xi x)$, where $0 < \xi < 1$. Then if φ is in $\mathcal{D}[-1, 1]$ and using equations (28) and (29), we have

$$\begin{aligned} & \text{N-}\lim_{n \rightarrow \infty} \langle \delta_n^{(r^s-1)} (\ln^{1/r} |1+x|), \varphi(x) \rangle \\ &= \text{N-}\lim_{n \rightarrow \infty} \sum_{k=0}^{r^{s-1}-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^1 \delta_n^{(r^s-1)} [\ln^{1/r} |1+x|] x^k dx \\ & \quad + \text{N-}\lim_{n \rightarrow \infty} \frac{1}{(r^{s-1})!} \int_{-1}^1 \delta_n^{(r^s-1)} (\ln^{1/r} (|1+x|)) x^{r^s} \varphi^{(r^{s-1})}(\xi x) dx \\ &= \sum_{k=0}^{r^{s-1}-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{k-i} r (r^s-1)! (i+1)^{r^s-1-i} \varphi^{(k)}(0)}{(r^{s-1}-1)! k!} \\ &= \sum_{k=0}^{r^{s-1}-1} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^i (r^s-1)! (i+1)^{r^s-1-i}}{(r^{s-1}-1)! k!} \langle \delta^{(k)}(x), \varphi(x) \rangle, \end{aligned}$$

proving equation (27) on $[-1, 1]$. However, it is clear that outside this interval $\delta_n^{(r^s-1)} [\ln^{1/r} (|1+x|)] = 0$, and so equation (27) is proved on the real line.

For further results on the neutrix composition of distributions, see [7], [8], [9] and [10].

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