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DIAMOND PRODUCT OF TWO COMMON COMPLETE BIPARTITE GRAPHS

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Abstract

A homomorphism of a graph $G = (V, E)$ into a graph $H = (V', E')$ is a mapping $f : V \longrightarrow V'$ which preserves edges: for all $x, y \in V$, if $\{x, y\} \in E$, then $\{f(x), f(y)\} \in E'$. Let $Hom(G, H)$ be the class of all homomorphisms from graph G into graph H. The diamond product of a graph $G = (V, E)$ with a graph $H = (V', E')$ (denoted by $G \diamond H$) is a graph defined by the vertex set $V(G \diamond H) = Hom(G, H)$ and the edge set $E(G \diamond H) = \{\{f, g\} \subset Hom(G, H) | \{f(x), g(x)\} \in E' \text{ for all } x \in V\}$. Let $K_{m,n}$ be a complete bipartite graph on $m+n$ vertices. This research aims to study the diamond product of two common complete bipartite graphs $K_{m,n}$. We find that the resulting graph is also a complete bipartite graph on $m^m n^n + n^m m^n$ vertices with diameter equal to two.

1. Introduction

In graph theory [2, 4], graph $G = (V, E)$ consists of a finite nonempty set V of objects called vertices, and a set E of 2-element subsets of V called edges. There are a vocabulary and classification of graphs needed in this paper as shown in the following.

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- P_n denotes a path on $n+1$ vertices if one can label vertices so that $E = \{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_n, v_{n+1}\}\}$, where $V = \{v_1, v_2, ..., v_{n+1}\}$.
- C_n denotes a cycle on n vertices if one can label vertices so that $E = \{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$, where $V = \{v_1, v_2, ..., v_n\}$.
- K_n denotes a complete graph on n vertices if every two distinct vertices of G are adjacent.
- A graph G is called a bipartite graph if $V(G)$ can be partitioned into two subsets U and W, called partite sets, such that every edge of G joins a vertex of U and a vertex of W.
- $K_{m,n}$ denotes a complete bipartite graph if one can partition V into two subsets U and W such that every edge of G joins a vertex of U and a vertex of W. Also every vertex of U is adjacent to every vertex of W.
- a $u-v$ walk in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent.
- a $u-v$ path in G is a $u-v$ walk in which no vertices are repeated.
- A graph G is called connected if G contains a $u-v$ path for every pair u, v of distinct vertices in G.
- A regular graph is a graph where each vertex has the same number of neighbors, i.e. every vertex has the same degree or valency. A regular graph with vertices of degree k is called a k-regular graph or regular graph of degree k.
- The distance between two vertices u and v in a graph (denoted by $d(u, v)$) is the number of edges in a shortest path connecting them. This is also known as the geodesic distance because it is the length of the graph geodesic between those two vertices. If there is no path connecting the two vertices, i.e., if they belong to different connected components, then conventionally the distance is defined as infinite.
- The diameter of a graph (denoted by $\text{diam}(G)$) is the maximum eccentricity of any vertex in the graph, i.e. the greatest distance between any two vertices.

Definition 1.1 [1] *A homomorphism of a graph $G = (V, E)$ into a graph $H = (V', E')$ is a mapping $f : V \longrightarrow V'$ which preserves edges: for all $x, y \in V$, if $\{x, y\} \in E$, then $\{f(x), f(y)\} \in E'$. Let $Hom(G, H)$ be the class of all homomorphisms from graph G into graph H.*

From this definition, one can easily see that $Hom(G, H)$ may or may not exist. For example, $Hom(P_1, C_3)$ consists of 6 homomorphisms, while $Hom(C_3, P_1)$ is an empty set.

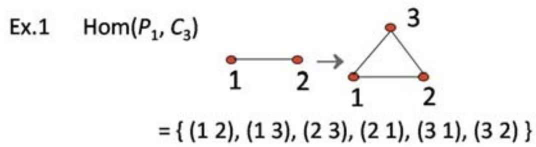

Figure 1: Hom(P_1, C_3)

Definition 1.2 [1] *The diamond product of a graph* $G = (V, E)$ *with a graph* $H = (V', E')$ *(denoted by* $G \diamond H$*) is a graph defined by the vertex set* $V(G \diamond H) = Hom(G, H)$ *and the edge set* $E(G \diamond H) = \{\{f, g\} \subset Hom(G, H) | \{f(x), g(x)\} \in E'$ *for all* $x \in V\}$.

An example of graph $P_1 \diamond C_3$ is shown below.

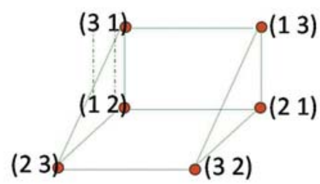

Figure 2: Graph $P_1 \diamond C_3$

With this definition, there are some interesting results as follows:

Theorem 1.1 [3] *The graph* $P_m \diamond P_n$ *are always connected for all positive integers* m, n *and* $diam(P_m \diamond P_n) = n$.

Theorem 1.2 [3] *The graph* $P_m \diamond C_n$ *and* $C_n \diamond P_m$ *are always connected.* $diam(P_m \diamond C_n) \leq m + n$ *and* $diam(C_n \diamond P_m) = n$.

Theorem 1.3 [3] *If* G *is a connected graph, then the graph* $P_m \diamond G$ *is always connected, and* $diam(P_m \diamond G) = diam(G) + 2m$.

2. Preliminaries

In this paper, we study the diamond product of two common complete bipartite graphs $K_{m,n}$, where each consists of $m + n$ vertices.

- Denote $V(K_{m,n}) = \{1, 2, 3, ..., m, m+1, m+2, ..., m+n\}$, where $V_m = \{x \in V(K_{m,n}) \mid x \leq m\}$, and $V_n = \{x \in V(K_{m,n}) \mid m+1 \leq x \leq m+n\}$.
  Since $K_{m,n}$ is a complete bipartite graph, each vertex of V_m is adjacent to all vertices of V_n. Every edge joins a vertex of V_m and a vertex of V_n. We can define a function $h : V(K_{m,n}) \to \{0, 1\}$ such that

$$h(x) = \begin{cases} 0 & \text{if } x \in V_m \\ 1 & \text{if } x \in V_n \end{cases}$$

  By the definition of a complete bipartite graph, we obtain $\forall x, y \in V(K_{m,n})$, $\{x, y\} \in E(K_{m,n})$ if and only if $\mid h(x) - h(y) \mid = 1$.

- Let $f : V(K_{m,n}) \to V(K_{m,n})$ be a homomorphism.
  Then $f \in V(K_{m,n} \diamond K_{m,n})$ if and only if

$$h(f(i)) = \begin{cases} 0 & \text{if } i \in V_m \\ 1 & \text{if } i \in V_n \end{cases}$$

  or

$$h(f(i)) = \begin{cases} 1 & \text{if } i \in V_m \\ 0 & \text{if } i \in V_n \end{cases}$$

  For example, let's take a look at $K_{2,2} \diamond K_{2,2}$.

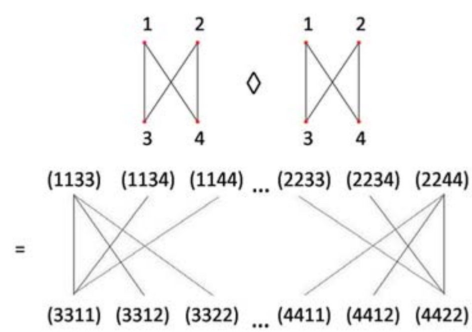

Figure 3: Graph $K_{2,2} \diamond K_{2,2}$

- Define a norm

$$\|f - g\| = \max_{\forall i \in V(K_{m,n})} |h(f(i)) - h(g(i))|$$

3. Results

Lemma 3.1 *For $f, g \in V(K_{m,n} \diamond K_{m,n})$, $\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$ if and only if $\|f - g\| = 1$.*

Proof. ($\Rightarrow$) Let $\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$.
We have $\{f(i), g(i)\} \in E(K_{m,n}), \forall i \in V(K_{m,n})$.
Thus $|h(f(i)) - h(g(i))| = 1, \forall i \in V(K_{m,n})$. That means $\|f - g\| = 1$.

($\Leftarrow$) Let $\|f - g\| = 1$, where $f, g \in V(K_{m,n} \diamond K_{m,n})$.
From the definition of norm, $\exists\, i_0 \in V(K_{m,n})$ such that $|h(f(i_0)) - h(g(i_0))| = 1$
Without loss of generality, assume $h(f(i_0)) = 0,\ h(g(i_0)) = 1$.
If $i_0 \in V_m$, then we obtain

$$h(f(i)) = \begin{cases} 0 & \text{if } i \in V_m \\ 1 & \text{if } i \in V_n \end{cases}$$

and

$$h(g(i)) = \begin{cases} 1 & \text{if } i \in V_m \\ 0 & \text{if } i \in V_n \end{cases}$$

So $|h(f(i)) - h(g(i))| = 1,\ \forall i \in V(K_{m,n})$.
If $i_0 \in V_n$, then we obtain

$$h(f(i)) = \begin{cases} 1 & \text{if } i \in V_m \\ 0 & \text{if } i \in V_n \end{cases}$$

and

$$h(g(i)) = \begin{cases} 0 & \text{if } i \in V_m \\ 1 & \text{if } i \in V_n \end{cases}$$

So $|h(f(i)) - h(g(i))| = 1,\ \forall i \in V(K_{m,n})$.
From both cases, we obtain $|h(f(i)) - h(g(i))| = 1,\ \forall i \in V(K_{m,n})$.
By the definitions of function h and diamond product, $\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$.
$\square$

Lemma 3.2 *$K_{m,n} \diamond K_{m,n}$ is a bipartite graph.*

Proof. First let's define ${V_m}^{\diamond} = \left\{ f \in V(K_{m,n} \diamond K_{m,n}) \mid h(f(i)) = \begin{cases} 0 & \text{if } i \in V_m \\ 1 & \text{if } i \in V_n \end{cases} \right\}$,
and ${V_n}^{\diamond} = \left\{ f \in V(K_{m,n} \diamond K_{m,n}) \mid h(f(i)) = \begin{cases} 1 & \text{if } i \in V_m \\ 0 & \text{if } i \in V_n \end{cases} \right\}$.
Obviously, $V(K_{m,n} \diamond K_{m,n}) = {V_m}^{\diamond} \cup {V_n}^{\diamond}$, and ${V_m}^{\diamond} \cap {V_n}^{\diamond} = \emptyset$.

To show that the graph of $K_{m,n} \diamond K_{m,n}$ is bipartite, we need to prove that

$\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$ if and only if f and g belong to different sets of vertices ${V_m}^{\diamond}$ and ${V_n}^{\diamond}$.
($\Rightarrow$) Proof by contrapositive.
Let f and g belong to the same set of vertices.
Without loss of generality: let $f, g \in {V_m}^{\diamond}$.
We have

$$\|f - g\| = \max_{\forall i \in V(K_{m,n})} |h(f(i)) - h(g(i))|.$$

If $i \in V_m$, then $h(f(i)) = 0, h(g(i)) = 0$

$$\max_{\forall i \in V_m} |h(f(i)) - h(g(i))| = \max |0 - 0| = 0.$$

If $i \in V_n$, then $h(f(i)) = 1, h(g(i)) = 1$

$$\max_{\forall i \in V_n} |h(f(i)) - h(g(i))| = \max |1 - 1| = 0.$$

Therefore $\|f - g\| = 0$ implies that $f, g \notin E(K_{m,n} \diamond K_{m,n})$ by Lemma 3.1. We then conclude that if f and g belong to the same sets of vertices, there is no edge $\{f, g\}$ in the graph $K_{m,n} \diamond K_{m,n}$.

($\Leftarrow$) Without loss of generality: let $f \in {V_m}^{\diamond}$ and $g \in {V_n}^{\diamond}$.
We have

$$\|f - g\| = \max_{\forall i \in V(K_{m,n})} |h(f(i)) - h(g(i))|$$

If $i \in V_m$, then $h(f(i)) = 0, h(g(i)) = 1$

$$\max_{\forall i \in V_m} |h(f(i)) - h(g(i))| = \max |0 - 1| = 1$$

If $i \in V_n$, then $h(f(i)) = 1, h(g(i)) = 0$

$$\max_{\forall i \in V_n} |h(f(i)) - h(g(i))| = \max |1 - 0| = 1$$

Then $|h(f(i)) - h(g(i))| = 1,\ \forall i \in V(K_{m,n})$, and $\|f - g\| = 1$.
Therefore $\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$. □

Lemma 3.3 *$K_{m,n} \diamond K_{m,n}$ is a connected graph with diam($K_{m,n} \diamond K_{m,n}$) = 2. In particular, $\forall f, g \in V(K_{m,n} \diamond K_{m,n})$ and $f \neq g$, there exists an $f - g$ path.*

Proof. Let f and g be two distinct vertices in $K_{m,n} \diamond K_{m,n}$. Consider $\|f - g\|$ in two cases.
Case1: If $\|f - g\| = 1$, then $\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$, by Lemma 3.1.
The distance $d(f, g) = 1$.
Case2: If $\|f - g\| = 0$, then $\{f, g\} \notin E(K_{m,n} \diamond K_{m,n})$.

Without loss of generality, we may assume $f, g \in V_m{}^{\diamond}$
Thus,

$$h(f(i)) = h(g(i)) = \begin{cases} 0 & \text{if } i \in V_m \\ 1 & \text{if } i \in V_n \end{cases}$$

By Lemma 3.2, $\exists\ k \in V_n{}^{\diamond}$ such that

$$h(k(i)) = \begin{cases} 1 & \text{if } i \in V_m \\ 0 & \text{if } i \in V_n \end{cases}$$

Therefore,$\|f - k\| = 1$ and $\|k - g\| = 1$. Hence, there is a $f - g$ path in this graph and the distance $d(f, g) = 2$. With both possible cases, we have $\operatorname{diam}(K_{m,n} \diamond K_{m,n}) = 2$. □

Theorem 3.1 *$K_{m,n} \diamond K_{m,n}$ is a complete bipartite graph on $m^m n^n + n^m m^n$ vertices.*

Proof. From all above lemmas, all vertices $f \in V_m{}^{\diamond}$ have the same value of $h(f(i)), \forall i \in V$ and all vertices $g \in V_n{}^{\diamond}$ have the same value of $h(g(i)), \forall i \in V$ such that $\|f - g\| = 1$. It means that each vertex of $V_m{}^{\diamond}$ is adjacent to all vertices of $V_n{}^{\diamond}$, making it a complete bipartite graph.

A number of vertices is counted from the number of homomorphisms from $K_{m,n}$ to itself. We know $K_{m,n} \diamond K_{m,n}$ have two partite sets $V_m{}^{\diamond}$ and $V_n{}^{\diamond}$. From the definition of $V_m{}^{\diamond}$, a homomorphism maps each vertex of V_m to a vertex of V_m giving us m^m choices and maps each vertex of V_n to a vertex of V_n with n^n choices. Thus $|V_m{}^{\diamond}| = m^m n^n$. On the other hand, a homomorphism in $V_n{}^{\diamond}$ maps each vertex of V_m to a vertex of V_n giving us n^m choices and maps each vertex of V_n to a vertex of V_m with m^n choices. Thus $|V_n{}^{\diamond}| = n^m m^n$. Both cases combined, we obtain the number of vertices in the theorem. □

Corollary 3.1 *$K_{m,n} \diamond K_{m,n}$ is a regular graph if and only if $m = n$.*

Proof. Since $K_{m,n} \diamond K_{m,n}$ is a complete bipartite graph, we may pick $f \in V_m{}^{\diamond}$ and $g \in V_n{}^{\diamond}$.
From Theorem 3.1, we have the following.

- $\{f, k\} \in E(K_{m,n} \diamond K_{m,n}),\ \forall k \in V_n{}^{\diamond}$
  Thus $\deg(f) = |V_n{}^{\diamond}| = n^m \cdot m^n,\ \forall f \in V_m{}^{\diamond}$

- $\{g, h\} \in E(K_{m,n} \diamond K_{m,n}),\ \forall h \in V_m{}^{\diamond}$
  Thus $\deg(g) = |V_m{}^{\diamond}| = m^m \cdot n^n,\ \forall g \in V_n{}^{\diamond}$
  Hence, $K_{m,n} \diamond K_{m,n}$ is a regular graph, if and only if

$$\begin{gathered} \deg(f) = \deg(g) \\ n^m \cdot m^n = m^m \cdot n^n \end{gathered}$$

$$\frac{n^m}{n^n}\frac{m^n}{m^m} = 1 \text{ since } m, n \neq 0$$
$$(\frac{m}{n})^{n-m} = 1$$
$$n - m = 0$$
$$m = n.$$

□

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