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ON REGULAR WEAK PROJECTION HYPERSUBSTITUTIONS

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Abstract

In this paper, we publish a continuity of our work [4]. We give sufficient and necessary conditions for a weak projection hypersubstitution $\sigma_{x_1,b}$ of type $(2,2)$ where the term b contains the variables x_1 and x_2 to be regular.

1 Introduction

Semigroup properties of hypersubstitutions have been studied (see [1], [2], [3], [4] and [5]). Let $\tau = \{(f_i, n_i) : i \in I\}$ be a type. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite alphabet of variables such that the sequence of the operation symbols $(f_i)_{i \in I}$ is disjoint with X, and let $X_n = \{x_1, x_2, \ldots, x_n\}$ be an n-element alphabet where $n \in \mathbb{N}$. Here f_i is n_i-ary for a natural number $n_i \geq 1$. An n-ary $(n \geq 1)$ *term* of type τ is inductively defined as follows:

(i) every variable $x_i \in X_n$ is an n-ary term,

(ii) if $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

Let $W_\tau(X_n)$ be the set containing $x_1, x_2, \ldots, x_n$ and being closed under finite application of (ii). The set of all terms of type τ over the alphabet X is defined by $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$.

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Any mapping $\sigma : \{f_i : i \in I\} \to W_\tau(X)$ is called a *hypersubstitution* of type τ if $\sigma(f_i)$ is an n_i-ary term of type τ for every $i \in I$. Any hypersubstitution σ of type τ can be uniquely extended to a map $\hat{\sigma}$ on $W_\tau(X)$ as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$,

(ii) $\hat{\sigma}[t]{:=}\sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \ldots, t_{n_i})$.

A binary operation is defined on the set $Hyp(\tau)$ of all hypersubstitutions of type τ, by

$$(\sigma_1\sigma_2)(f_i) := \hat{\sigma}_1[\sigma_2(f_i)]$$

for all n_i-ary operation symbols f_i. Together with this binary associative operation $Hyp(\tau)$ forms a monoid since the identity hypersubstitution σ_{id} which maps every f_i to $f_i(x_1, \ldots, x_{n_i})$ is an identity element.

From now on, let $\tau = (2, 2)$ be a type with two binary operation symmbols f and g. For $a, b \in W_{(2,2)}(X_2)$, the hypersubstitution which maps the operation symbol f to the term a and the operation symbol g to the term b is denote by $\sigma_{a,b}$, that is $\sigma_{a,b}(f) = a, \sigma_{a,b}(g) = b$. A hypersubstitution $\sigma_{a,b}$ such that $a \in X_2$ or $b \in X_2$ is called a *weak projection hypersubstitution*.

For $t \in W_{(2,2)}(X_2)$, the first variable (from the left) which occurs in t, the last variable which occurs in t, the set of all variables occurring in t, the first operation symbols occurring in t and the number of occurrence of all operation symbols in t are denoted by $leftmost(t)$, $rightmost(t)$, $var(t)$, $firstop(t)$ and $op(t)$, respectively. For an example, if $t = f(f(x_2, x_1),\ g(g(x_2, x_2),\ f(x_2, x_1)))$, then $leftmost(t) = x_2$, $rightmost(t) = x_1$ $var(t) = \{x_1, x_2\}, firstop(t) = f$ and $op(t) = 5$.

The following result proved in [4].

Theorem 1.1. *If $\sigma_{a,b} \in Hyp(2, 2)$ such that $op(a) \leq 1$ and $op(b) \leq 1$, then $\sigma_{a,b}$ is regular in $Hyp(2, 2)$.*

Notation. Let $t \in W_{(2,2)}(X_2)$. Since t can be represented by a tree, we address each node in the usual way: the root is labeled by 0, the first node on the left branch starting from the root is labeled by 00, the first node on the right branch starting from the root is labeled by 01, etc. Then we label the different occurrences of the operation symbols f and g as follow: for instant, f_{011111} means the operation symbol position at 011111 is f. We abbreviate $01110\underbrace{11\ldots1}_{k}\underbrace{0\ldots0}_{l}$ by $01^301^k0^l$.

For an example see Figure 1:

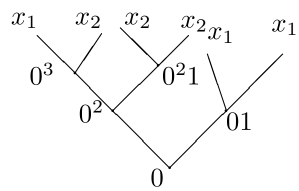

Figure 1.

Assume that $firstop(t) = f$. If g_{01^k} exists for some $k > 1$, we let $Rp'_g(t)$ and $Rp''_g(t)$ denote the parts:

$$f_0 f_{01} f_{01^2} \dots f_{01^{k-1}} g_{01^k} F_{01^k 0} F_{01^k 01} \dots F_{01^k 01^m} \dots$$

for some $m \geq 1$ and

$$f_0 f_{01} f_{01^2} \dots f_{01^{k-1}} g_{01^k} F_{01^k 0} F_{01^k 0^2} \dots,$$

respectively, $F \in \{f, g\}$. Dually, assume that $firstop(t) = g$. If f_{01^j} exists for some $j > 1$, we let $Lp'_f(t)$ and $Lp''_f(t)$ denote the parts:

$$g_0 g_{01} g_{01^2} \dots g_{01^{j-1}} f_{01^j} F_{01^j 0} F_{01^j 01} \dots F_{01^j 01^n} \dots$$

for some $n \geq 1$ and

$$g_0 g_{01} g_{01^2} \dots g_{01^{j-1}} f_{01^j} F_{01^j 0} F_{01^j 0^2} \dots,$$

respectively.

Similarly, assume that $firstop(t) = f$. If g_{0^k} exists for some $k > 1$, we let $Lp'_g(t)$ and $Lp''_g(t)$ denote the parts:

$$f_0 f_{0^2} f_{0^3} \dots f_{0^{k-1}} g_{0^k} F_{0^k 1} F_{0^k 10} \dots F_{0^k 10^q} \dots$$

for some $q \geq 1$ and

$$f_0 f_{0^2} f_{0^3} \dots f_{0^{k-1}} g_{0^k} F_{0^k 1} F_{0^k 11} \dots F_{0^k 1^q} \dots$$

respectively, $F \in \{f, g\}$. Assume that $firstop(t) = g$. If f_{0^j} exists for some $j > 1$, we let $Lp'_f(t)$ and $Lp''_f(t)$ denote the parts:

$$g_0 g_{0^2} g_{0^3} \dots g_{0^{j-1}} f_{0^j} F_{0^j 1} F_{0^j 10} \dots F_{0^j 10^r} \dots$$

for some $r \geq 1$ and

$$g_0 g_{0^2} g_{0^3} \dots g_{0^{j-1}} f_{0^j} F_{0^j 1} F_{0^j 11} \dots F_{0^j 1^r} \dots$$

respectively.

Let $FRp'_g(t)$ and $GRp'_g(t)$ denote the number of occurrences of f in $Rp'_g(t)$ and the number of occurrences of g in $Rp'_g(t)$, respectively. $FRp''_g(t)$, $GRp''_g(t)$, $FLp'_g(t)$, $GLp'_g(t)$, $FLp''_g(t)$ and $GLp''_g(t)$ are similarly defined. We define

$$rightmost'_f(t) := rightmost(Rp'_f(t)),\ rightmost''_f(t) := rightmost(Rp''_f(t)),$$

and $leftmost(Lp'_f(t))$, $leftmost(Lp''_f(t))$, $rightmost(Rp'_g(t))$, $leftmost(Lp'_g(t))$, $rightmost(Rp''_g(t))$, $leftmost(Lp''_g(t))$ are similarly defined. For an example, consider the following term

$$t = f_0(g_{00}(x_1, x_2), f_{01}(x_1, g_{01^2}(f_{01^20}(g_{01^20^2}(x_2, x_1), f_{01^201}(x_2, x_1)), x_2)))$$

given by the tree diagram in Figure 2:

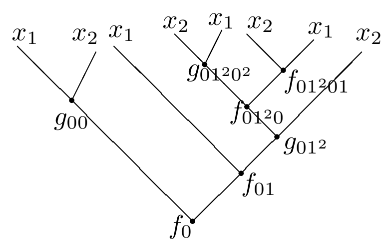

Figure 2.

We have $Rp'_g(t) := f_0f_{01}g_{01^2}f_{01^20}f_{01^201}$ and $Rp''_g(t) := f_0f_{01}g_{01^2}f_{01^20}g_{01^20^2}$. Then $GRp''_g(t) = 2, FRp''_g(t) = 3$, $rightmost'_g(t) = x_1, rightmost''_g(t) = x_2$ and $leftmost'_g(t) = x_2 = leftmost''_g(t)$.

The following proved in [4].

Theorem 1.2. *Let $b \in W_{(2,2)}(X_2)$ be such that $op(b) > 1$ and $var(b) = \{x_1\}$. Then $\sigma_{x_1,b}$ is regular (in $Hyp(2,2)$) if and only if*

(i) *$FLp(b) = 1$; or*

(ii) *$FLp(b) > 1$ and $GLp(b) = 1$; or*

(iii) *If $FLp(b) = 0$, then $GLp(b) = 1$ or $GRp(b) = 1$ or $FRp(b) = 1$ or $FRp'_f(b) = 1$ or $GRp''_f(b) = 1$; or*

(iv) *If $FLp(b) > 1$ and $GLp(b) = 0$, then $GRp(b) = 1$ or $FRp(b) = 1$ or $FRp''_g(b) = 1$; or*

(v) *If $FLp(b) > 1$ and $GLp(b) > 1$, then we have one of the following cases: $GRp(b) = 1$, $FRp(b) = 1$, $GLp'_g(b) = 1$, $FLp'_f(b) = 1$, $FLp''_g(b) = 1$, $GRp'_g(b) = 1$, $FRp''_g(b) = 1$, $GLp''_f(b) = 1$.*

2 Main Results

The following lemma is required.

Lemma 2.1. *Let $b, t \in W_{(2,2)}(X_2)$ be such that $op(b) > 1$ and $var(b) = \{x_1, x_2\}$. Then $\hat{\sigma}_{x_1,b}[t] = b$ if and only if the following hold:*

$$GLp(t) = 1, GLp'_g(t) = 1, leftmost(t) = x_1, leftmost'_g(t) = x_2.$$

Proof. Assume that $\hat{\sigma}_{x_1,b}[t] = b$. Since $b \notin X_2$, $GLp(t) \neq 0$. If $GLp(t) > 1$, then $b = \hat{\sigma}_{x_1,b}[t] = b(b(r_1, r_2), b(r_1, r_2))$ for some $r_1, r_2 \in W_{(2,2)}(X_2)$, a contradiction. Thus $GLp(t) = 1$. If $GLp'_g(t) > 1$, then $\hat{\sigma}_{x_1,b}[t] = b(b(s, b(s_1, s'_1))$ for some $s, s_1, s'_1 \in W_{(2,2)}(X_2)$. Since $var(b) = \{x_1, x_2\}$ we have $op(\hat{\sigma}_{x_1,b}[t]) = op(b(b(s, b(s_1, s'_1)))) > op(b)$, a contradiction. Then $GLp'_g(t) = 1$. Since $var(b) = \{x_1, x_2\}$ and $b(x_1, x_2) = b = \hat{\sigma}_{x_1,b}[t] = b(leftmost(t),\ leftmost'(t))$ we have $leftmost(t) = x_1$ and $leftmost'_g(t) = x_2$. The opposite direction is easy to see. □

Assume that $\sigma_{x_1,b}$ is regular. Then there exists $\sigma_{u,v} \in Hyp(2, 2)$ such that $\sigma_{x_1,b}\sigma_{u,v}\sigma_{x_1,b} = \sigma_{x_1,b}$. Thus

$$b \quad = \quad \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]. \tag{2.1}$$

By Lemma 2.1, we have

$$\begin{gathered} GLp(\hat{\sigma}_{u,v}[b]) = 1, GLp'_g(\hat{\sigma}_{u,v}[b]) = 1, \\ leftmost(\hat{\sigma}_{u,v}[b]) = x_1, leftmost'_g(\hat{\sigma}_{u,v}[b]) = x_2. \end{gathered} \tag{2.2}$$

Hereafter, we will refer to the notations above when we have $\sigma_{x_1,b}$ is regular.

Next, we collect the cases that $\sigma_{x_1,b}$ is not regular.

Lemma 2.2. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0$, $GLp(b) = 1$, $GRp(b) = 1, FRp(b) = 1, leftmost(b) = x_i = rightmost(b)$, $FRp'_f(b) = 1$ and $rightmost'_f(b) = x_i$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1, FRp(b) = 1$ and $leftmost(b) = x_i = rightmost(b)$, we set $b = g(x_i, f((t, x_i))$ for some $t \in W_{(2,2)}(X_2)$. We consider two cases.

Case 1: $t \in X_2$. Since $rightmost'_f(b) = x_i$ we obtain $t = x_i$ so $b = g(x_i, f((x_i, x_i))$. By (2.1), $var(b) = \{x_i\}$, a contradiction.

Case 2: $t \notin X_2$. Then $b = g(x_i, f(g(t_1, t_2), x_i))$ for some $t_1, t_2 \in W_{(2,2)}(X_2)$. By (2.1),

$$b = \hat{\sigma}_{x_1,b}[v](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), x_i)).$$

There are five cases to consider.

Case 2.1: $\hat{\sigma}_{x_1,b}[v] = x_1$. Then $b \in X_2$, a contradiction.

Case 2.2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$. Then $b \in X_2$, a contradiction.

Case 2.3: $\hat{\sigma}_{x_1,b}[v] = x_2, \hat{\sigma}_{x_1,b}[u] \notin X_2$ and $var(\hat{\sigma}_{x_1,b}[u]) = \{x_i\}$. We have $var(b) = \{x_i\}$, a contradiction.

Case 2.4: $\hat{\sigma}_{x_1,b}[v] = x_2, \hat{\sigma}_{x_1,b}[u] \notin X_2$ and $var(\hat{\sigma}_{x_1,b}[u]) = \{x_1, x_2\}$. Then $GLp(u) \geq 1$. Since $FRp'_f(b) = 1$, $FRp(t_2) = 0$. Then $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]) = x_i$ and so $var(b) = \{x_i\}$, a contradiction.

Case 2.5: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. There are two cases to consider.

Case 2.5.1: $x_2 \notin var(\hat{\sigma}_{x_1,b}[v])$. Then $var(b) = \{x_i\}$, a contradiction.

Case 2.5.2: $x_2 \in var(\hat{\sigma}_{x_1,b}[v])$. If $x_1 \in var(\hat{\sigma}_{x_1,b}[u])$, then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. If $x_1 \notin var(\hat{\sigma}_{x_1,b}[u])$, then $var(b) = \{x_i\}$, a contradiction. □

Lemma 2.3. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0$, $GLp(b) = 1$, $GRp(b) = 1$, $FRp(b) > 1$, $FRp'_f(b) = 1$, $leftmost(b) = x_i = rightmost(b)$, $GRp'_f(b) = 1$ and $rightmost''_f(b) = x_i$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1,$ $FRp'_f(b) = 1, leftmost(b) = x_i = rightmost(b), GRp'_f(b) = 1$ and $rightmost''_f(b) = x_i$, we let

$$b = g(x_i, f(x_i, f(t_1, t_2))) \text{ where } t_1, t_2 \in W_{(2,2)}(X_2).$$

Then

$$\begin{aligned} \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] &= \hat{\sigma}_{x_1,b}[v](x_i, \hat{\sigma}_{x_1,b}[u](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \\ &\quad \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]))). \end{aligned} \tag{2.3}$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (2.3), $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$. Then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. By (2.3),

$$\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = \hat{\sigma}_{x_1,b}[u](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]])). \tag{2.4}$$

We consider three cases.

Case 3.1: $var(\hat{\sigma}_{x_1,b}[u]) = \{x_1\}$. By (2.4), $var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_1\}$, a contradiction.

Case 3.2: $var(\hat{\sigma}_{x_1,b}[u]) = \{x_2\}$. By (2.4),

$$\begin{aligned} \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] &= \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), \\ &\quad \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]])). \end{aligned}$$

If $GLp(u) = 0$, then $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$ (because $GRp(t_2) = 0$), a contradiction. If $GLp(u) \geq 1$, then $\hat{\sigma}_{x_1,b}[u] = b(s_1, s_2)$ for some $s_1, s_2 \in W_{(2,2)}(X_2)$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 3.3: $var(\hat{\sigma}_{x_1,b}[u]) = \{x_1, x_2\}$. Then $GLp(u) \geq 1$. Thus $\hat{\sigma}_{x_1,b}[u] = b(s_1, s_2)$ for some $s_1, s_2 \in W_{(2,2)}(X_2)$. By (2.4), $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider three cases.

Case 4.1: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1\}$. By (2.3), $var(b) = \{x_i\}$, a contradiction.

Case 4.2: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_2\}$. There are two cases to consider.

Case 4.2.1: $GLp(v) = 0$. Then $\hat{\sigma}_{x_1,b}[v] = x_2$. By (2.3),

$$\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = \hat{\sigma}_{x_1,b}[u](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]))$$

If $GLp(u) = 0$, then $\hat{\sigma}_{x_1,b}[u] \in X_2$. Since $GRp(t_2) = 0$, $b \in X_2$, a contradiction. If $GLp(u) \geq 1$, then $\hat{\sigma}_{x_1,b}[u] = b(s_1, s_2)$ for some $s_1, s_2 \in W_{(2,2)}(X_2)$. If $x_2 \in var(\hat{\sigma}_{x_1,b}[u])$, then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. If $x_2 \notin var(\hat{\sigma}_{x_1,b}[u])$, then $var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_l\}$, a contradiction.

Case 4.2.2: $GLp(v) \geq 1$. Then $\hat{\sigma}_{x_1,b}[v] = b(s_3, s_4)$ for some $s_3, s_4 \in W_{(2,2)}(X_2)$. Then $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] =$

$$b(s_3, s_4)(x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]])))).$$

If $GLp(u) = 0$, then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction. If $GLp(u) \geq 1$, then $\hat{\sigma}_{x_1,b}[u] = b(s_1, s_2)$ for some $s_1, s_2 \in W_{(2,2)}(X_2)$. If $x_2 \in var(\hat{\sigma}_{x_1,b}[u])$, then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. If $x_2 \notin var(\hat{\sigma}_{x_1,b}[u])$, then $var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_l\}$, a contradiction.

Case 4.3: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1, x_2\}$. Then $GLp(v) \geq 1$. Then $\hat{\sigma}_{x_1,b}[v] = b(s_3, s_4)$ for some $s_3, s_4 \in W_{(2,2)}(X_2)$. By (2.3),

$$\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = b(s_3, s_4)(x_i, \hat{\sigma}_{x_1,b}[u](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]])))).$$

If $GLp(u) = 0$, then $\hat{\sigma}_{x_1,b}[u] = x_i$ because $rightmost''_f(b) = x_i$. Thus $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction. If $GLp(u) \geq 1$, then $\hat{\sigma}_{x_1,b}[u] = b(s_5, s_6)$ for some $s_5, s_6 \in W_{(2,2)}(X_2)$. Since $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1, x_2\}$, we get $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.4. *If $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1, FRp(b) > 1, FRp'_f(b) = 1, GRp'_f(b) > 1$ and $leftmost(b) = x_i = rightmost(b)$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1, FRp(b) > 1, FRp'_f(b) = 1, GRp'_f(b) > 1$ and $leftmost(b) = x_i = rightmost(b)$, we let

$$b = g(x_i, f(g(t_1, t_2), f(t_3, t_4))) \text{ where } t_1, t_2, t_3, t_4 \in W_{(2,2)}(X_2).$$

Then

$$\begin{aligned} b \;&=\; \hat{\sigma}_{x_1,b}[v](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), \\ &\quad \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_4]]))). \end{aligned} \tag{2.5}$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (2.3), $b = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$. Clearly, $b \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. Then $GLp(u) \geq 1$. We have

$$b = \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]], \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_4]])).$$

We consider two cases.

Case 3.1: $x_2 \notin var(\hat{\sigma}_{x_1,b}[u])$. Since $FRp(t_2) = 0$, $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]] \in X_2$. $var(b) = \{\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]\}$, a contradiction.

Case 3.2: $x_2 \in var(\hat{\sigma}_{x_1,b}[u]) = \{x_2\}$. Since $GLp(u) \geq 1$, $\hat{\sigma}_{x_1,b}[u] = b(s_1, s_2)$ for some $s_1, s_2 \in W_{(2,2)}(X_2)$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider three cases.

Case 4.1: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1\}$. By (2.5), $var(b) = \{x_i\}$, a contradiction.

Case 4.2: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_2\}$. If $GLp(u) = 0$, then $var(b) = \{x_i\}$, a contradiction. If $GLp(u) \geq 1$, $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4.3: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1, x_2\}$. Then $GLp(v) \geq 1$. If $GLp(u) = 0$, then $var(b) = \{x_i\}$, a contradiction. If $GLp(u) \geq 1$, $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.5. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1, FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) = 1, GRp''_f(b) = 1, rightmost''_f(b) = x_i$, then $\sigma_{x_1,b}$ is not regular.*

Proof. It can be proved similarly as Lemma 2.4. □

Lemma 2.6. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1, FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) = 1, GRp''_f(b) > 1$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1, FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) = 1, GRp''_f(b) > 1$, we let

$$b = g(x_i, f(f(t_1, t_2), f(t_3, t_4)) \text{ where } t_1, t_2, t_3, t_4 \in W_{(2,2)}(X_2) \text{ and } GRp(t_1) \geq 1.$$

Then

$$\begin{aligned} b = {} & \hat{\sigma}_{x_1,b}[v](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), \\ & \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_4]]))). \end{aligned} \quad (2.6)$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (2.6), $b = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$. Clearly, $b \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. Then $GLp(u) \geq 1$. By (2.6), $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider two cases.

Case 4.1: $x_2 \notin var(\hat{\sigma}_{x_1,b}[v])$. We have $var(b) = \{x_i\}$, a contradiction.

Case 4.2: $x_2 \in var(\hat{\sigma}_{x_1,b}[v])$. If $GLp(u) = 0$, then $\hat{\sigma}_{x_1,b}[u] \in X_2$. If $\hat{\sigma}_{x_1,b}[u] = x_1$, then $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$ because $GLp(t_1) \geq 1$. This is a contradiction. If $\hat{\sigma}_{x_1,b}[u] = x_2$, then $var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_i\}$, a contradiction. If $GLp(u) \geq 1$, clearly, $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.7. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1, FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) > 1, GRp''_f(b) = 1, rightmost''_f(b) = x_i$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1, FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) > 1, GRp''_f(b) = 1$, we let

$$b = g(x_i, f(f(t_1, t_2), f(t_3, t_4))) \text{ where } t_1, t_2, t_3, t_4 \in W_{(2,2)}(X_2).$$

Then

$$\begin{aligned} b \;=\; & \hat{\sigma}_{x_1,b}[v](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), \\ & \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_4]])))). \end{aligned} \tag{2.7}$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (**??**), $b = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$. Clearly, $b \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. Then $GLp(u) \geq 1$. By (2.7), $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider two cases.

Case 4.1: $x_2 \notin var(\hat{\sigma}_{x_1,b}[v])$. We have $var(b) = \{x_i\}$, a contradiction.

Case 4.2: $x_2 \in var(\hat{\sigma}_{x_1,b}[v])$. If $GLp(u) = 0$, then $var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_i\}$, a contradiction. If $GLp(u) \geq 1$. By (2.7), $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.8. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1, FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) > 1, GRp''_f(b) > 1$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) = 1, GRp(b) = 1, FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) > 1, GRp''_f(b) > 1$, we let

$$b = g(x_i, f(f(t_1, t_2), f(t_3, t_4)) \text{ or } b = g(x_i, f(g(t_1, t_2), f(t_3, t_4))$$

where $t_1, t_2, t_3, t_4 \in W_{(2,2)}(X_2)$ and $GRp(t_1) \geq 1$. Assume that $b = g(x_i, f(f(t_1, t_2), f(t_3, t_4))$. By Lemma 2.6, we get a contradiction. Assume that $b = g(x_i, f(g(t_1, t_2), f(t_3, t_4))$. It can be proved similarly as Lemma 2.6 and we get a contradiction. □

Using the lemmas above we have the following theorem.

Theorem 2.1. *Let $b \in W_{(2,2)}(X_2)$ be the such that $op(b) > 1, var(b) = \{x_1, x_2\}$, $FLp(b) = 0$, $GLp(b) = 1$ and $GRp(b) = 1$. Then $\sigma_{x_1,b}$ is regular if and only if*

(i) $FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j$ *with* $i \neq j$.

(ii) $FRp(b) = 1$, $leftmost(b) = x_i = rightmost(b), FRp'_f(b) = 1, rightmost'_f(b) = x_k, k \neq i$,

(iii) $FRp(b) > 1, leftmost(b) = x_i$, $rightmost(b) = x_j$ *with* $i \neq j$,

(iv) *if* $FRp(b) > 1, leftmost(b) = x_i = rightmost(b)$ *then*

(iv.1) $FRp'_f(b) = 1$, $GRp'_f(b) = 1$, $rightmost''_f(b) = x_l$, $l \neq i$ *or*

(iv.2) $FRp'_f(b) > 1$, $GRp'_f(b) = 1$, $GRp''_f(b) = 1$, $rightmost''_f(b) = x_k$, $k \neq i$ *or*

(iv.3) $FRp'_f(b) > 1$, $GRp'_f(b) > 1$, $GRp''_f(b) = 1$, $rightmost''_f(b) = x_k$, $k \neq i$.

Proof. Assume that $\sigma_{x_1,b}$ is regular. Then there exists $\sigma_{u,v} \in Hyp(2,2)$ such that $\sigma_{x_1,b}\sigma_{u,v}\sigma_{x_1,b} = \sigma_{x_1,b}$. Using Lemma 2.1,

$$\begin{aligned} &GLp(\hat{\sigma}_{u,v}[b]) = 1, GLp'_g(\hat{\sigma}_{u,v}[b]) = 1, \\ &leftmost(\hat{\sigma}_{u,v}[b]) = x_1, leftmost'(\hat{\sigma}_{u,v}[b]) = x_2. \end{aligned} \tag{2.8}$$

If $FRp(b) = 0$, then $op(b) = 1$. This contradicts to $op(b) > 1$. There are thirteen cases to consider.

(1) $FRp(b) = 1, leftmost(b) = x_i$ and $rightmost(b) = x_j$ with $i \neq j$.

(2) $FRp(b) = 1, leftmost(b) = x_i = rightmost(b)$, $FRp'_f(b) = 1, rightmost'_f(b) = x_i$.

(3) $FRp(b) = 1, leftmost(b) = x_i = rightmost(b)$, $FRp'_f(b) = 1, rightmost'_f(b) = x_k$ with $k \neq i$.

(4) $FRp(b) > 1, leftmost(b) = x_i$ and $rightmost(b) = x_j$ with $i \neq j$.

(5) $FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) = 1$, $GRp'_f(b) = 1$ and $rightmost''_f(b) = x_l$, $l \neq i$.

(6) $FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) = 1$, $GRp'_f(b) = 1$ and $rightmost''_f(b) = x_i$.

(7) $FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) = 1$, $GRp'_f(b) > 1$.

(8) $FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) = 1, GRp''_f(b) = 1, rightmost''_f(b) = x_k, i \neq k$.

(9) $FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) = 1, GRp''_f(b) = 1, rightmost''_f(b) = x_i$.

(10) $FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1$, $GRp'_f(b) = 1$, $GRp''_f(b) > 1$.

(11) $FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) > 1, GRp''_f(b) = 1, rightmost''_f(b) = x_l, i \neq l$.

(12) $FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) > 1, GRp''_f(b) = 1, rightmost''_f(b) = x_i$.

(13) $FRp(b) > 1, leftmost(b) = x_i = rightmost(b), FRp'_f(b) > 1, GRp'_f(b) > 1, GRp''_f(b) > 1$.

If b satisfies one of the cases (2), (6), (7), (9), (10), (12) and (13), then by Lemma 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8 (respectively) we have $\sigma_{x_1,b}$ is not regular.

Conversely, we have to show that $\sigma_{x_1,b}\sigma_{u,v}\sigma_{x_1,b} = \sigma_{x_1,b}$ for some $u, v \in W_{(2,2)}(X_2)$. If b satisfies (i), let $u = x_2$ and $v = g(x_i, x_j)$; if b satisfies (ii), let $u = g(x_k, x_i)$ and $v = x_2$; if b satisfies (iii), let $u = x_2$ and $v = g(x_i, x_j)$; if b satisfies (iv.1), let $u = x_1$ and $v = g(x_i, x_l)$; if b satisfies (iv.2), let $u = x_1$ and $v = g(x_i, x_k)$; if b satisfies (iv.3), let $u = x_1$ and $v = g(x_i, x_l)$. □

Lemma 2.9. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0$, $GLp(b) = 1$, $GRp(b) > 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) = 1$, $rightmost'_f(b) = x_j$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0$, $GLp(b) = 1$, $GRp(b) > 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) = 1$, $rightmost'_f(b) = x_j$, we let $b = g(x_i, f(t_1, t_2))$ where $t_1, t_2 \in W_{(2,2)}(X_2)$. Then

$$\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \quad = \quad \hat{\sigma}_{x_1,b}[v](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]])). \quad (2.9)$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (2.9), $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$. Then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. By (2.9),

$$\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \quad = \quad \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]). \quad (2.10)$$

We consider three cases.

Case 3.1: $var(\hat{\sigma}_{x_1,b}[u]) = \{x_1\}$. Since $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]] = x_j$, $var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_j\}$. This is a contradiction.

Case 3.2: $var(\hat{\sigma}_{x_1,b}[u]) = \{x_2\}$. Since $FRp(t_2) = 0$, $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]] = x_j$. Then $var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_j\}$, a contradiction.

Case 3.3: $var(\hat{\sigma}_{x_1,b}[u]) = \{x_1, x_2\}$. Then $GLp(u) \geq 1$. Using 3.1 and 3.2, $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = x_j$. Then $var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_j\}$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider two cases.

Case 4.1: $x_2 \notin var(\hat{\sigma}_{x_1,b}[v])$. By (2.9), we have $var(b) = \{x_i\}$, a contradiction.

Case 4.2: $x_2 \in var(\hat{\sigma}_{x_1,b}[v])$. If $GLp(u) = 0$, then $\hat{\sigma}_{x_1,b}[u] \in X_2$. If $\hat{\sigma}_{x_1,b}[u] = x_1$, then $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$ because $GRp(t_1) \geq 1$. This is a contradiction. If $\hat{\sigma}_{x_1,b}[u] = x_2$, then $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$ because $GRp(t_2) \geq 1$. This is a contradiction. Finally, if $GLp(u) \geq 1$, by (2.9), $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.10. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0$, $GLp(b) = 1$, $GRp(b) > 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1, GRp''_f(b) = 1$, $rightmost''_f(b) = x_i$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0$, $GLp(b) = 1$, $GRp(b) > 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1, GRp''_f(b) = 1, rightmost''_f(b) = x_i$, we let $b = g(x_i, f(f(t_1, t_2), t_3))$ where $t_1, t_2, t_3 \in W_{(2,2)}(X_2)$. Then $b =$

$$\hat{\sigma}_{x_1,b}[v](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]])). \quad (2.11)$$

There are five cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (2.11), $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] = x_1$. Since $GLp(t_1) = 0$, $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] = x_2$. Then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$ because $FRp(t_3) = 0$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. Then $GLp(u) \geq 1$. By (2.11),

$$b = \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]]). \quad (2.12)$$

We consider two cases.

Case 4.1: $x_1 \notin var(\hat{\sigma}_{x_1,b}[u])$. Since $FRp(t_3) = 0$, $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = x_j$. Then $var(b) = \{x_j\}$, a contradiction.

Case 4.2: $x_1 \in var(\hat{\sigma}_{x_1,b}[u])$. Since $GLp(u) \geq 1$, $\hat{\sigma}_{x_1,b}[u] = b(s_1, s_2)$ for some s_1, s_2 $W_{(2,2)}(X_2)$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 5: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider two cases.

Case 5.1: $x_2 \notin var(\hat{\sigma}_{x_1,b}[v])$. We have $var(b) = \{x_i\}$, a contradiction.

Case 5.2: $x_2 \in var(\hat{\sigma}_{x_1,b}[v])$. If $GLp(u) = 0$, then $\hat{\sigma}_{x_1,b}[u] \in X_2$. If $\hat{\sigma}_{x_1,b}[u] = x_1$, then $var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_i\}$ because $GLp(t_1) = 0$. This is a contradiction. If $\hat{\sigma}_{x_1,b}[u] = x_2$, then $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$ because $GRp(t_3) \geq 1$. This is a contradiction. □

Lemma 2.11. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0, GLp(b) = 1, GRp(b) > 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1, GRp''_f(b) > 1$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) = 1, GRp(b) > 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1, GRp''_f(b) > 1$, we let

$$b = g(x_i, f(f(t_1, t_2), t_3)) \text{ or } b = g(x_i, f(g(t_1, t_2), t_3))$$

where $t_1, t_2, t_3 \in W_{(2,2)}(X_2)$ and $GRp(t_1) \geq 1$.

Case A. $b = g(x_i, f(f(t_1, t_2), t_3))$. By Lemma 2.10, a contradiction.

Case B. $b = g(x_i, f(g(t_1, t_2), t_3))$. It can be similarly proved as Lemma 2.10. □

Lemma 2.12. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0, GLp(b) = 1, GRp(b) > 1, FRp(b) > 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) = 1$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) = 1, GRp(b) > 1, FRp(b) > 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) = 1$, we let

$$b = g(x_i, f(g(t_1, t_2), t_3)) \text{ where } t_1, t_2, t_3 \in W_{(2,2)}(X_2).$$

Then $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] =$

$$\hat{\sigma}_{x_1,b}[v](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]])). \quad (2.13)$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. Clearly, $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$.

Case 2.1: $\hat{\sigma}_{x_1,b}[u] = x_1$. By (2.13), since $FRp(t_2) = 0$, $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]] \in X_2$, a contradiction.

Case 2.2: $\hat{\sigma}_{x_1,b}[u] = x_2$. Clearly, $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. Then $GLp(u) \geq 1$. We have

$$\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]]).$$

We consider two cases.

Case 3.1: $x_2 \notin var(\hat{\sigma}_{x_1,b}[u])$. Since $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]] \in X_2$, $var(b) = var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_l\}, l \in \{1, 2\}$, a contradiction.

Case 3.2: $x_2 \in var(\hat{\sigma}_{x_1,b}[u])$. Since $FRp(t_3) \geq 1$, $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider three cases.

Case 4.1: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1\}$. By (2.13), $var(b) = \{x_i\}$, a contradiction.

Case 4.2: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_2\}$. If $GLp(u) = 0$, then $\hat{\sigma}_{x_1,b}[u] \in X_2$. If $\hat{\sigma}_{x_1,b}[u] = x_1$, then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. If $\hat{\sigma}_{x_1,b}[u] = x_2$. Since $GRp(t_3) \geq 1$, $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. If $GLp(u) \geq 1$, $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4.3: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1, x_2\}$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.13. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0, GLp(b) = 1, GRp(b) > 1, FRp(b) > 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1, GRp''_f(b) = 1, rightmost''_f(b) = x_i$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) = 1, GRp(b) > 1, FRp(b) > 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1, GRp''_f(b) = 1, rightmost''_f(b) = x_i$, we let

$$b = g(x_i, f(f(t_1, t_2), t_3)) \text{ where } t_1, t_2, t_3 \in W_{(2,2)}(X_2).$$

Then $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] =$

$$\hat{\sigma}_{x_1,b}[v](x_i, \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]])). \quad (2.14)$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. Clearly, $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$.

Case 2.1: $\hat{\sigma}_{x_1,b}[u] = x_1$. By (2.14), since $GLp(t_2) = 0$, $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]] = x_i$, a contradiction.

Case 2.2: $\hat{\sigma}_{x_1,b}[u] = x_2$. Clearly, $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = x_j$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. Then $GLp(u) \geq 1$. We have

$$\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] = \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]])).$$

Since $FLp(t_1) \geq 1$ and $FRp(t_3) \geq 1$. $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider three cases.

Case 4.1: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1\}$. By (2.14), $var(b) = var((\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_i\}$, a contradiction.

Case 4.2: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_2\}$. If $GLp(u) = 0$, then $\hat{\sigma}_{x_1,b}[u] \in X_2$. We have $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. If $GLp(u) \geq 1$, $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4.3: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1, x_2\}$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.14. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b) = 0, GLp(b) = 1, GRp(b) > 1, FRp(b) > 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1, GRp''_f(b) > 1$, then $\sigma_{x_1,b}$ is not regular.*

Proof. This can be proved in the same manner as Lemma 2.13. □

Theorem 2.2. *Let* $b \in W_{(2,2)}(X_2)$ *be the such that* $op(b) > 1, var(b) = \{x_1, x_2\}$ *and* $FLp(b) = 0$*,* $GLp(b) = 1$ *and* $GRp(b) > 1$*. Then* $\sigma_{x_1,b}$ *is regular if and only if*

(i) $FRp(b) = 1$, $rightmost(b) = x_j$, $FRp'_f(b) = 1$, $rightmost'_f(b) = x_k$, $j \neq k$,

(ii) $FRp(b) = 1$, $leftmost(b) = x_i$, $FRp'_f(b) > 1$, $GRp''_f(b) = 1$, $rightmost''_f(b) = x_l$, $l \neq i$,

(iii) $FRp(b) > 1$, $leftmost(b) = x_i$, $FRp'_f(b) > 1$, $GRp''_f(b) = 1$, $rightmost''_f(b) = x_l$, $l \neq i$.

Proof. Assume that $\sigma_{x_1,b}$ is regular. Then there exists $\sigma_{u,v} \in Hyp(2,2)$ such that $\sigma_{x_1,b}\sigma_{u,v}\sigma_{x_1,b} = \sigma_{x_1,b}$. Using Lemma 2.1,

$$\begin{aligned} &GLp(\hat{\sigma}_{u,v}[b]) = 1, GLp'_g(\hat{\sigma}_{u,v}[b]) = 1,\\ &leftmost(\hat{\sigma}_{u,v}[b]) = x_1, leftmost'(\hat{\sigma}_{u,v}[b]) = x_2. \end{aligned} \tag{2.15}$$

If $FRp(b) = 0$, then $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$ or $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. Then we consider the following cases:

(1) $FRp(b) = 1, rightmost(b) = x_j, FRp'_f(b) = 1$, $rightmost'_f(b) = x_k$, $j \neq k$.

(2) $FRp(b) = 1, rightmost(b) = x_j, FRp'_f(b) = 1$, $rightmost'_f(b) = x_l$, $j = l$.

(3) $FRp(b) = 1, leftmost(b) = x_i, FRp'_f(b) > 1, GRp''_f(b) = 1$, $rightmost''_f(b) = x_l$, $l \neq i$.

(4) $FRp(b) = 1, leftmost(b) = x_i, FRp'_f(b) > 1, GRp''_f(b) = 1$, $rightmost''_f(b) = x_k$, $k = i$.

(5) $FRp(b) = 1, FRp'_f(b) > 1, GRp''_f(b) > 1$.

(6) $FRp(b) > 1, FRp'_f(b) = 1$.

(7) $FRp(b) > 1, leftmost(b) = x_i, FRp'_f(b) > 1, GRp''_f(b) = 1, rightmost''_f(b) = x_l, l \neq i$.

(8) $FRp(b) > 1, leftmost(b) = x_i, FRp'_f(b) > 1, GRp''_f(b) = 1, rightmost''_f(b) = x_k$, $i = k$.

(9) $FRp(b) > 1, FRp'_f(b) > 1, GRp''_f(b) > 1$.

If b satisfies one of the cases (2), (4), (5), (6), (8) and (9), then by Lemma 2.9, 2.10, 2.11, 2.12, 2.13 and 2.14 (respectively) we have $\sigma_{x_1,b}$ is not regular.

Conversely, we have to show that $\sigma_{x_1,b}\sigma_{u,v}\sigma_{x_1,b} = \sigma_{x_1,b}$ for some $u, v \in W_{(2,2)}(X_2)$. If b satisfies (i), let $u = g(x_k, x_j)$ and $v = x_2$; if b satisfies (ii), let $u = x_1$ and $v = g(x_i, x_l)$; if b satisfies (iii), let $u = x_1$ and $v = g(x_i, x_l)$. □

Lemma 2.15. *If* $b \in W_{(2,2)}(X_2)$ *such that* $FLp(b) = 0, GLp(b) > 1, GRp(b) = 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) = 1, rightmost'_f(b) = x_j$*, then* $\sigma_{x_1,b}$ *is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) > 1, GRp(b) = 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) = 1, rightmost'_f(b) = x_j$, we let

$$b = g(g(t_1, t_2), f(t_3, x_j))) \text{ where } t_1, t_2, t_3 \in W_{(2,2)}(X_2).$$

Then $b =$

$$\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]], x_j)). \quad (2.16)$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (2.16), $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]] = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$. Clearly, $b \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. Then $GLp(u) \geq 1$. We have

$$b = \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]], x_j).$$

We consider two cases.

Case 3.1: $x_1 \notin var(\hat{\sigma}_{x_1,b}[u])$. Clearly, $var(b) = \{x_j\}$, a contradiction.

Case 3.2: $x_1 \in var(\hat{\sigma}_{x_1,b}[u])$. Since $FRp(t_3) = 0$, $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]] = x_j$. Then $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_j\}$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider three cases.

Case 4.1: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1\}$. By (2.16), $var(b) = \{x_i\}$, a contradiction.

Case 4.2: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_2\}$. If $GLp(u) = 0$, then $\hat{\sigma}_{x_1,b}[u] \in X_2$. If $\hat{\sigma}_{x_1,b}[u] = x_1$, then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$ because $GRp(t_3) \geq 1$, a contradiction. If $\hat{\sigma}_{x_1,b}[u] = x_2$, then $var(b) = \{x_j\}$, a contradiction. If $GLp(u) \geq 1$, $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4.3: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1, x_2\}$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.16. *If* $FLp(b) = 0, GLp(b) > 1, GRp(b) = 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1$*, then* $\sigma_{x_1,b}$ *is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) > 1, GRp(b) = 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1$, we let

$$b = g(t_1, f(f(t_2, t_3), x_j)) \text{ or } b = g(t_1, f(g(t_2, t_3), x_j))$$

where $t_1, t_2, t_3 \in W_{(2,2)}(X_2)$ and $FLp(t_1) = 0$.

Case A. $b = g(t_1, f(f(t_2, t_3), x_j))$. Then $b =$

$$\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]]), x_j))(2.17)$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (2.17), $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]] = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$. Clearly, $b \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. Then $GLp(u) \geq 1$. We have

$$b = \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]]), x_j).$$

We consider two cases.

Case 3.1: $x_1 \notin var(\hat{\sigma}_{x_1,b}[u])$. Clearly, $var(b) = \{x_j\}$, a contradiction.

Case 3.2: $x_1 \in var(\hat{\sigma}_{x_1,b}[u])$. We have $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider three cases.

Case 4.1: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1\}$. By (2.17), $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$ because $GLp(t_1) \geq 1$, a contradiction.

Case 4.2: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_2\}$. If $GLp(u) = 0$, $\hat{\sigma}_{x_1,b}[u] \in X_2$. If $\hat{\sigma}_{x_1,b}[u] = x_1$, then we consider $g \in t_2$ or $g \notin t_2$. If $g \in t_2$, then $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. If $g \notin t_2$, then $var(b) = var(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) = \{x_l\}$, a contradiction. If $\hat{\sigma}_{x_1,b}[u] = x_2$, then $var(b) = \{x_j\}$, a contradiction. If $GLp(u) \geq 1$, $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4.3: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1, x_2\}$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case B. $b = g(t_1, f(g(t_2, t_3), x_j))$. Then $b =$

$$\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]]), x_j)) \quad (2.18)$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (2.18), $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]] = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$. Clearly, $b \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. Then $GLp(u) \geq 1$. We have

$$b = \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]]), x_j).$$

We consider two cases.

Case 3.1: $x_1 \notin var(\hat{\sigma}_{x_1,b}[u])$. Clearly, $var(b) = \{x_j\}$, a contradiction.

Case 3.2: $x_1 \in var(\hat{\sigma}_{x_1,b}[u])$. Since $FRp(t_3) \geq 1$, $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider three cases.

Case 4.1: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1\}$. By (2.18), $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$ because $GLp(t_1) \geq 1$, a contradiction.

Case 4.2: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_2\}$. If $GLp(u) = 0$, $\hat{\sigma}_{x_1,b}[u] \in X_2$. If $\hat{\sigma}_{x_1,b}[u] = x_1$, then $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. If $\hat{\sigma}_{x_1,b}[u] = x_2$, then $var(b) = \{x_j\}$, a contradiction. If $GLp(u) \geq 1$, $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4.3: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1, x_2\}$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.17. *If $b \in W_{(2,2)}(X_2)$ such that $FLp(b)=0, GLp(b)>1, GRp(b)>1$. then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b)=0, GLp(b)>1, GRp(b)>1$, we let

$$b = g(g(t_1,t_2), g(t_3,t_4)) \text{ where } t_1,t_2,t_3,t_4 \in W_{(2,2)}(X_2).$$

Then

$$\begin{aligned}\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \quad &= \quad \hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]]), \\ &\quad\quad \hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_4]])). \qquad (2.19)\end{aligned}$$

There are two cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] \in X_2$. We consider three cases.

Case 1.1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (2.19), since $FRp(t_1)=0$, $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]] \in X_2$. Then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction.

Case 1.2: $\hat{\sigma}_{x_1,b}[v] = x_2$. Since $FRp(t_4)=0$, $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_4]] \in X_2$. Then $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider three cases.

Case 2.1: $x_2 \notin var(\hat{\sigma}_{x_1,b}[v])$. By (2.19), since $GLp(v) \geq 1$, $\hat{\sigma}_{x_1,b}[v] = b(r_1,r_2)$ for some $r_1,r_2 \in W_{(2,2)}X_2$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 2.2: $x_2 \notin var(\hat{\sigma}_{x_1,b}[v])$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.18. *If $FLp(b)=0, GLp(b)>1, GRp(b)>1, FRp(b)=1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b)=1, rightmost'_f(b) = x_j$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b)=0, GLp(b)>1, GRp(b)>1, FRp(b)=1, leftmost(b)=x_i, rightmost(b)=x_j, FRp'_f(b)=1, rightmost'_f(b) = x_j$, we let

$$b = g(t_1, f(g(t_2,t_3),t_4))$$

where $t_1,t_2,t_3,t_4 \in W_{(2,2)}(X_2)$, $GLp(t_1) \geq 1$ and $FRp(t_4)=0$. Then

$$\begin{aligned}b \quad &= \quad \hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]], \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]], \\ &\quad\quad \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]]), \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_4]])). \qquad (2.20)\end{aligned}$$

There are four cases to consider.

Case 1: $\hat{\sigma}_{x_1,b}[v] = x_1$. By (2.20), $b = \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_1]] = x_i$, a contradiction.

Case 2: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \in X_2$. Clearly, $b \in X_2$, a contradiction.

Case 3: $\hat{\sigma}_{x_1,b}[v] = x_2$ and $\hat{\sigma}_{x_1,b}[u] \notin X_2$. Then $GLp(u) \geq 1$. We have

$$b \quad = \quad \hat{\sigma}_{x_1,b}[u](\hat{\sigma}_{x_1,b}[v](\hat{\sigma}_{x_1,b}\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_2]], \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]]), \hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_4]]).$$

We consider two cases.

Case 3.1: $x_1 \notin var(\hat{\sigma}_{x_1,b}[u])$. Since $FRp(t_4) = 0$, $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_4]] = x_j$. Then $var(b) = \{x_j\}$, a contradiction.

Case 3.2: $x_1 \in var(\hat{\sigma}_{x_1,b}[u])$. Since $FRp'_f(b) = 1$, $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[t_3]] = x_j$ because $FRp(t_3) = 0$. Then $var(b) = \{x_j\}$, a contradiction.

Case 4: $\hat{\sigma}_{x_1,b}[v] \notin X_2$. Then $GLp(v) \geq 1$. We consider three cases.

Case 4.1: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1\}$. By (2.20), $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$ because $GLp(t_1) \geq 1$, a contradiction.

Case 4.2: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_2\}$. If $GLp(u) = 0$, $\hat{\sigma}_{x_1,b}[u] \in X_2$. We have $op(b) = op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. If $GLp(u) \geq 1$, $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction.

Case 4.3: $var(\hat{\sigma}_{x_1,b}[v]) = \{x_1, x_2\}$. Then $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, a contradiction. □

Lemma 2.19. *If $FLp(b) = 0, GLp(b) > 1, GRp(b) > 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1$, then $\sigma_{x_1,b}$ is not regular.*

Proof. Assume that $\sigma_{x_1,b}$ is regular. Since $FLp(b) = 0, GLp(b) > 1, GRp(b) > 1, FRp(b) = 1, leftmost(b) = x_i, rightmost(b) = x_j, FRp'_f(b) > 1$, we let

$$b = g(t_1, f(f(t_2, t_3), t_4)) \text{ or } b = g(t_1, f(g(t_2, t_3), t_4))$$

where $t_1, t_2, t_3, t_4 \in W_{(2,2)}(X_2)$, $GLp(t_1) \geq 1$ and $FRp(t_4) = 0$. Then it can be similarly proved as Lemma 2.18. □

If $FLp(b) = 0$, $GLp(b) > 1$, $GRp(b) = 1$ and $FRp(b) = 0$, then it can be proved that $\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]] \in X_2$ or $op(\hat{\sigma}_{x_1,b}[\hat{\sigma}_{u,v}[b]]) > op(b)$, this implies that $\sigma_{x_1,b}$ is not regular.

Theorem 2.3. *Let $b \in W_{(2,2)}(X_2)$ be the such that $op(b) > 1, var(b) = \{x_1, x_2\}$ and $FLp(b) = 0$, $GLp(b) > 1$ and $FRp(b) = 1$. Then $\sigma_{x_1,b}$ is regular if and only if* (i) *or* (ii)*:*

(i) $GRp(b) = 1$, $FRp'_f(b) = 1$, $rightmost'_f(b) \neq rightmost(b)$.

(ii) $GRp(b) > 1, FRp'_f(b) = 1, rightmost'_f(b) \neq rightmost(b)$.

Proof. Assume $\sigma_{x_1,b}$ is regular. Then there exists $\sigma_{u,v} \in Hyp(2,2)$ such that $\sigma_{x_1,b}\sigma_{u,v}\sigma_{x_1,b} = \sigma_{x_1,b}$. Using Lemma 2.1, $GLp(\hat{\sigma}_{u,v}[b]) = 1$, $GLp'_g(\hat{\sigma}_{u,v}[b]) = 1$, $leftmost(\hat{\sigma}_{u,v}[b]) = x_1$ and $leftmost'(\hat{\sigma}_{u,v}[b]) = x_2$. Consider

(1) $GRp(b) = 1, rightmost(b) = x_j$, $FRp'_f(b) = 1, rightmost'_f(b) = x_k$, $k \neq j$.

(2) $GRp(b) = 1, rightmost(b) = x_j$, $FRp'_f(b) = 1, rightmost'_f(b) = x_l$, $l = j$.

(3) $GRp(b) = 1, FRp'_f(b) > 1$.

(4) $GRp(b) > 1, FRp(b) = 0$,

(5) $GRp(b) > 1, rightmost(b) = x_j$, $FRp'_f(b) = 1, rightmost'_f(b) = x_l$, $l \neq j$.

(6) $GRp(b) > 1, rightmost(b) = x_j$, $FRp'_f(b) = 1, rightmost'_f(b) = x_k$, $k = j$.

(7) $GRp(b) > 1, FRp'_f(b) > 1$.

If b satisfies one of the cases (2), (3), (4), (6) and (7), then by Lemma 2.15, 2.16, 2.17, 2.18 and 2.19 (respectively) we get $\sigma_{x_1,b}$ is not regular.

Conversely, if b satisfies (i), let $u = g(x_k, x_j)$ and $v = x_2$ and if b satisfies (ii), we let $u = g(x_l, x_j)$ and $v = x_2$. □

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