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WAGNER TRANSFORMATIONS AND MAXIMAL CLIFFORD INVERSE SEMIGROUPS OF TRANSFORMATIONS

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Abstract

Using a technique developed by Victor Wagner, that combines full and partial transformations of a set, we classify all maximal Clifford inverse subsemigroups of the semigroup of all (full) transformations of a set. In addition, some computational results for properties of the Maximal Clifford inverse subsemigroups of the semigroup of all partial one-to-one transformations of a finite set are given.

1. Preliminaries

Recall that *a semigroup* is a set S with an associative binary operation. An element $e \in S$ is called *an idempotent* if $e^2 = e$. We denote the set of all idempotents of S with $E(S)$.

A semigroup S is called (von Neumann) *regular* if for every $s \in S$ there exists $t \in S$ such that $sts = s$. Clearly, taking $t' = tst$ we obtain $st's = s$ and $t'st' = t'$ (see [5], where this fact was observed for the first time).

Thus we can give an alternative definition: a semigroup S is regular if for every $s \in S$ there exists $t \in S$ such that $sts = s$ and $tst = t$. In this case we say that t is an *inverse* of s (and, clearly, then s is an inverse of t). Observe that

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an inverse in the sense of group theory is an inverse in the sense of semigroup theory. However, an element of a semigroup may have more than one inverse.

A semigroup S is *inverse* if each of its elements $s \in S$ has a uniquely determined inverse (which we denote as s^{-1}). Observe that $(ss^{-1})^2 = ss^{-1}$ and $(s^{-1}s)^2 = s^{-1}s$ for all $s \in S$, that is, ss^{-1} and $s^{-1}s$ are idempotents of S. These idempotents do not have to coincide, that is, $ss^{-1} = s^{-1}s$ need not be true for all $s \in S$.

Alternatively, an inverse semigroup can be defined as a regular semigroup with commuting idempotents (that is, $ef = fe$ for any two idempotents e and f). This is the original definition of inverse semigroups given by Wagner [6], while our first definition belongs to Liber [3].

Recall that *(lower) semilattice* is a partially ordered set S in which each pair of elements, a and b, has the greatest lower bound, $a \wedge b$.

For the set of all idempotents $E(S)$ of a semigroup S, *the natural partial order* on $E(S)$ is defined by $e \preceq f$ if and only if $ef = fe = e$, for any $e, f \in E(S)$.

Thus, any semilattice is a commutative semigroup of idempotents with respect to this meet operation and the partial order on it is the natural partial order of the semilattice. And, conversely, any commutative semigroup B of idempotents is a semilattice with $a \wedge b = ab$ where the meet is with respect of the natural partial order on B. Therefore, we can identify the classes of semilattices and commutative semigroups of idempotents.

The natural (or *canonical*) *partial order* on an inverse semigroup S is defined by $a \preceq b$ if and only if $a = aa^{-1}b$, for any $a, b \in S$. Wagner, who was the first to introduce $\preceq$, gave in [7] many equivalent definitions of the order relation.

Observe that although idempotents of an inverse semigroup commute, idempotents do not have to commute with non-idempotent elements. In other words, idempotents of an inverse semigroup S do not necessarily belong to the *center* (denoted $C(S)$) of the inverse semigroup. If idempotents commute with *each* element, the semigroup is called a *Clifford inverse semigroup*. Clifford was the first to consider this class of inverse semigroups in [1]. Clifford inverse semigroups have many alternative definitions. For example, an inverse semigroup S is Clifford precisely when $ss^{-1} = s^{-1}s$ for all $s \in S$. Also, an inverse semigroup is Clifford precisely when it is a union of its subgroups. In this case the semigroup is a disjoint union of its maximal subgroups.

A *partial transformation* of (or a *function in*) a set A is a mapping of any subset $B \subseteq A$ into A. In particular, the mapping of an empty subset of A into A is the *empty* partial transformation of A.

The set $B \subseteq A$ on which the partial transformation f is defined is called the *first projection of* f and denoted: $pr_1 f = B$.

The set $C \subseteq A$ that is the range of f is called the *second projection of* f, and denoted: $pr_2 f = C$. Thus, $pr_2 f = f(pr_1 f)$.

We denote with Δ_B the identity map on B, that is $\Delta_B(b) = b$ for every $b \in B$. Clearly $pr_1\Delta_B = pr_2\Delta_B = B$.

The set $\mathcal{F}_A$ of all partial transformations of A is a semigroup under the natural composition of functions. If $f, g \in \mathcal{F}_A$ then their composition $g \circ f \in \mathcal{F}_A$ is defined as follows: $g \circ f(a) = g(f(a))$ for every $a \in A$. Both sides of that equality are defined or not defined simultaneously, that is, $g \circ f(a)$ is defined exactly when both $f(a)$ and $g(f(a))$ are defined. In particular, the product of two nonempty functions may be the empty function.

A partial transformation f is called *one-to-one* if it satisfies $(\forall a, b \in A)\ [f(a) = f(b) \Rightarrow a = b]$, i.e. if $f(a) = f(b)$ implies $a = b$ for any $a, b \in A$ for which both $f(a)$ and $f(b)$ are defined. In other words, a one-to-one partial transformation of A is a one-to-one mapping of a subset of A into A. It is easy to see that the set $\mathcal{I}_A$ of all one-to-one partial transformations of a set A is a subsemigroup of $\mathcal{F}_A$. Moreover, $\mathcal{I}_A$ is an inverse semigroup called the *symmetric inverse semigroup* of (all) one-to-one partial transformations of A.

Observe that for any two partial transformations $f, g \in \mathcal{I}_A$,

$$f \preceq g \iff f \subseteq g.$$

We consider a partial transformation as a binary relation. In particular, $f = \{(a, f(a)) \mid a \in pr_1 f\}$. This explains our notation $f \subseteq g$.

Here $f \subseteq g$ means that f is a restriction of g to a smaller domain (or, equivalently, g is an extension of f to a larger domain). In other words, $f(a) = g(a)$ for every $a \in pr_1\ f$.

The set of all (full) transformations of A together with the natural composition of functions is also a semigroup called the *full transformation semigroup* of a set A or (the *symmetric semigroup* of A) and denoted by $\mathcal{T}_A$.

Clearly, $\mathcal{T}_A$ is a subsemigroup of $\mathcal{F}_A$ but it is not a subsemigroup of $\mathcal{I}_A$. Also it is not hard to see that $\mathcal{T}_A$ is a regular semigroup, but it is not an inverse semigroup, unless $|A| \leq 1$.

A one-to-one mapping of A onto itself is called a *permutation* of A. The set $\mathcal{G}_A$ of all permutations of A forms a group (the *symmetric group*) on A. This group is the group of units (that is, the elements invertible in the sense of group theory) of $\mathcal{I}_A$. The group $\mathcal{G}_A$ is contained in both, $\mathcal{I}_A$ and $\mathcal{T}_A$.

Clearly, the groups are precisely those inverse semigroups in which the natural partial order is equality.

2. Maximal Clifford inverse subsemigroups of $\mathcal{T}_A$

2.1 Wagner's transformation $\tilde{\varphi}$

Now let A be an arbitrary set and $\mathcal{T}_A$ the full transformation semigroup on the set A.

Suppose that Φ is an inverse subsemigroup of $\mathcal{T}_A$ and $\mathcal{E}_\Phi$ is the semilattice of idempotents of Φ. For every $\varphi \in \Phi$ define $\tilde{\varphi} = \varphi\Delta_{pr_2\varphi^{-1}}$. Let $\tilde{\Phi} = \{\tilde{\varphi} : \varphi \in \Phi\}$.

Then $\tilde{\Phi}$ is an inverse semigroup of partial one-to-one transformations of A and the correspondence $\varphi \mapsto \tilde{\varphi}$ is an isomorphism of Φ onto $\tilde{\Phi}$. This result belongs to Wagner [6].

Let Ψ be an inverse subsemigroup of $\mathcal{I}_A$. Is it possible to find an inverse subsemigroup Φ of $\mathcal{T}_A$ such that $\Psi = \tilde{\Phi}$? If yes, is Φ uniquely determined? Generally speaking, we have negative answers to both questions. For example, $\mathcal{I}_A \neq \tilde{\Phi}$ for all $\Phi \subset \mathcal{T}_A$. Also, it is not difficult to find two inverse semigroups $\Phi_1, \Phi_2 \subset \mathcal{T}_A$ such that $\Phi_1 \neq \Phi_2$ but $\tilde{\Phi}_1 = \tilde{\Phi}_2$ as shown next.

Example 2.1.1. Let $|A| = 3$ and formally let $A = \{1, 2, 3\}$. It is easy to check that $\Phi_1 = \{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\}$ and $\Phi_2 = \{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\}$ are inverse semigroups and $\Phi_1 \neq \Phi_2$ but $\tilde{\Phi}_1 = \tilde{\Phi}_2 = \{\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$.

Thus, given $\tilde{\Phi}$, we cannot "recover" Φ. However, if we know both $\tilde{\Phi}$ and $\mathcal{E}_\Phi$, then it is possible to reconstruct Φ. Indeed, $\varphi = \tilde{\varphi}\varphi^{-1}\varphi$ for every $\varphi \in \Phi$. Now, $\varphi^{-1}\varphi$ is the only idempotent in $\mathcal{E}_\Phi$ such that $pr_2\varphi^{-1}\varphi = pr_2\varphi^{-1}$ and also $pr_2\varphi = pr_2\tilde{\varphi}$. Thus, given $\tilde{\varphi} \in \tilde{\Phi}$, we take the only idempotent $\chi \in \mathcal{E}_\Phi$ such that $pr_2\chi = pr_2\tilde{\varphi}^{-1}\tilde{\varphi}$ and see that $\varphi = \tilde{\varphi}\chi$.

It follows that every inverse semigroup Φ of transformations of a set A can be constructed from the isomorphic inverse semigroup $\tilde{\Phi}$ of one-to-one partial transformations of A and the subsemilattice $\mathcal{E}_\Phi$ of $\mathcal{T}_A$.

This technique is also discribed in detail in [2] and used in [4].

We will use it in the next section to classify the Maximal Clifford inverse subsemigroups of $\mathcal{T}_A$ with maximal semilattices of idempotents.

2.2 Maximal Clifford inverse subsemigroups of $\mathcal{T}_A$

Recall that inverse semigroup is called Clifford inverse if its idempotents belong to the center of the semigroup. Since $\varphi^{-1}\varphi = \varphi\varphi^{-1}$ for any φ in a Clifford inverse subsemigroup Φ of $\mathcal{T}_A$ it is not hard to see that $\tilde{\varphi}^{-1}\tilde{\varphi} = \tilde{\varphi}\tilde{\varphi}^{-1}$ and hence $pr_1\tilde{\varphi} = pr_2\tilde{\varphi} = pr_2\varphi$.

The following proposition gives necessary and sufficient conditions for an inverse subsemigroup of $\mathcal{T}_A$ to be a Clifford inverse semigroup.

Proposition 2.2.1. *An inverse subsemigroup Φ of $\mathcal{T}_A$ is Clifford if and only if, for every $\varphi \in \Phi$ and $\psi \in \mathcal{E}_\Phi$ such that $pr_2\psi \subseteq pr_2\varphi$, $\tilde{\varphi}\psi = \psi\varphi$.*

Proof. If Φ is Clifford and $pr_2\psi \subseteq pr_2\varphi$, then $\tilde{\varphi}\tilde{\varphi}^{-1} = \tilde{\varphi}^{-1}\tilde{\varphi}$ and hence $pr_1\tilde{\varphi} = pr_2\tilde{\varphi}$ so that $pr_2\psi \subseteq pr_2\varphi = pr_2\tilde{\varphi} = pr_1\tilde{\varphi}$, and hence $\tilde{\varphi}\psi = \varphi\psi = \psi\varphi$

because idempotents of Φ are central, that is, they commute with all elements of Φ.

Conversely, suppose that $\tilde{\varphi}\psi = \psi\varphi$ for all $\varphi \in \Phi$ and $\psi \in \mathcal{E}_\Phi$ such that $pr_2\psi \subseteq pr_2\varphi$. Clearly, $pr_1\psi\varphi = A$, and hence $pr_1\tilde{\varphi}\psi = A$. Thus $pr_2\psi \subseteq pr_1\tilde{\varphi} = pr_2\varphi^{-1}$. Let $\psi = \varphi^{-1}\varphi$. Then $pr_2\psi = pr_2\varphi^{-1} = pr_1\tilde{\varphi}$, so that $\tilde{\varphi}\psi = \varphi\psi = \psi\varphi$. Thus $\varphi = \varphi\varphi^{-1}\varphi = \varphi^{-1}\varphi\varphi$. It follows that $\varphi\varphi^{-1} = \varphi^{-1}\varphi\varphi\varphi^{-1}$, that is, $\varphi\varphi^{-1} \preceq \varphi^{-1}\varphi$ for all $\varphi \in \Phi$. Here $\preceq$ denotes the natural (canonical) order relation of Φ, as defined in the Preliminary Section. Replacing φ^{-1} by φ we obtain that $\varphi^{-1}\varphi \preceq \varphi\varphi^{-1}$, that is, $\varphi^{-1}\varphi = \varphi\varphi^{-1}$ for all $\varphi \in \Phi$, and hence Φ is Clifford. $\square$

Let Φ be a Clifford inverse subsemigroup of $\mathcal{T}_A$ with the semilattice $\mathcal{E}_\Phi$ of idempotents. If $\varphi \in \Phi$ then $\varphi^{-1}\varphi = \varphi\varphi^{-1} = f$ is an idempotent in $\mathcal{E}_\Phi$. Observe that $\tilde{\varphi}^{-1}\tilde{\varphi} = \tilde{\varphi}\tilde{\varphi}^{-1}$ and hence $pr_1\tilde{\varphi} = pr_2\tilde{\varphi} = pr_2\varphi$. Now let $f \in \mathcal{E}_\Phi$ with $pr_2 f = B$ and let $\alpha \in \mathcal{G}_B$ be a permutation of B.

Definition 2.2.1. We say that α, as described above, is a *local automorphism* of $e \in \mathcal{E}_\Phi$ if $pr_2 e \subseteq B$ and $\alpha e \alpha^{-1} = e_{|B}$. In other words, α is a local automorphism of e precisely when $pr_2 e \subseteq B$ and $[\forall a, b \in B]\ e(a) = b \Leftrightarrow e\alpha(a) = \alpha(b)$.

Observe from the definition that any local automorphism is a partial one-to-one map, that is, any local automorphism is an element of $\mathcal{I}_A$. Next we prove that for an element φ of a Clifford inverse subsemigroup of $\mathcal{T}_A$, the corresponding Wagner's transformation $\tilde{\varphi}$ is a local automorphism for all idempotents of that subsemigroup which are no grater (in canonical sense) then the idempotent $\varphi^{-1}\varphi$.

Proposition 2.2.2. *Let Φ be a Clifford inverse subsemigroup of $\mathcal{T}_A$ with the semilattice $\mathcal{E}_\Phi$ of idempotents and let φ be an element of Φ such that $\varphi^{-1}\varphi = f \in \mathcal{E}_\Phi$. Then $\tilde{\varphi}e\tilde{\varphi}^{-1} = e_{pr_2\varphi}$ for every $e \in \mathcal{E}_\Phi$ such that $e \preceq f$. In other words, $\tilde{\varphi}$ is a local automorphism of e for every $e \preceq f$.*

Proof. Let Φ be a Clifford inverse subsemigroup of $\mathcal{T}_A$ with the semilattice $\mathcal{E}_\Phi$ of idempotents and let φ be an element of Φ such that $\varphi^{-1}\varphi = f \in \mathcal{E}_\Phi$. Suppose that $\tilde{\varphi}f\tilde{\varphi}^{-1}(a)$ exists for some $a \in A$. Then $a = \tilde{\varphi}(b)$ for some $b \in B$. Therefore, $b \in pr_2 f$, and hence $f(b) = b$. It follows that $\tilde{\varphi}f\tilde{\varphi}^{-1}(a) = \tilde{\varphi}f(b) = \tilde{\varphi}(b) = a$. Conversely, if $a \in pr_2\varphi$ then $\tilde{\varphi}^{-1}(a)$ exists because $pr_2\tilde{\varphi} = pr_2\varphi$. Let $\tilde{\varphi}^{-1}(a) = b$. Then $f(b) = b$ because $b \in pr_1\tilde{\varphi} = pr_2 f$. Thus $\tilde{\varphi}f\tilde{\varphi}^{-1}(a) = \tilde{\varphi}f(b) = \tilde{\varphi}(b) = a$. Therefore, $\tilde{\varphi}f\tilde{\varphi}^{-1} = \Delta_{pr_2\varphi}$.

By Proposition 2.2.1, $\tilde{\varphi}e = e\varphi$ for every $e \in \mathcal{E}_\Phi$ such that $pr_2 e \subseteq pr_2 f$. Recall that $\varphi = \tilde{\varphi}f$ and hence $\tilde{\varphi}e = e\tilde{\varphi}f$. Multiplying this by $\tilde{\varphi}^{-1}$ on the right we obtain $\tilde{\varphi}e\tilde{\varphi}^{-1} = e\tilde{\varphi}f\tilde{\varphi}^{-1} = e\Delta_{pr_2\varphi} = e_{|B}$, where $B = pr_1\tilde{\varphi} = pr_2\varphi$. $\square$

Now we are at the point where we can classify the maximal Clifford inverse subsemigroups of $\mathcal{T}_A$ with maximal semilattices of idempotents.

Theorem 2.2.1. *Let $\mathcal{E}$ be a subsemilattice of $\mathcal{T}_A$ and $\tilde{\Phi}_{\mathcal{E}}$ the set of all local automorphisms of $\mathcal{E}$. For each $\tilde{\alpha} \in \tilde{\Phi}_{\mathcal{E}}$ there is a unique element $f_{\tilde{\alpha}} \in \mathcal{E}$ such that $pr_2 f_{\tilde{\alpha}} = pr_1 \tilde{\alpha}$. Denote $\hat{\alpha} = \tilde{\alpha} f_{\tilde{\alpha}}$. Then the set $\hat{\Phi}_{\mathcal{E}}$ is the greatest among all Clifford inverse subsemigroups Ψ of $\mathcal{T}_A$, for which $\mathcal{E}$ is their semilattice of idempotents (that is, $\mathcal{E}_\Psi = \mathcal{E}$).*

Proof. First observe that if α is a local automorphism of $\mathcal{E}$ so is α^{-1}, since $\alpha e \alpha^{-1} = e_{|B} = \alpha^{-1} e \alpha = \alpha^{-1} e (\alpha^{-1})^{-1}$.

So for every $\hat{\alpha} = \tilde{\alpha} f \in \hat{\Phi}_{\mathcal{E}}$, both $\tilde{\alpha}$ and $\tilde{\alpha}^{-1}$ belong to $\tilde{\Phi}_{\mathcal{E}}$ and also $pr_1\tilde{\alpha} = pr_2\tilde{\alpha} = pr_1\tilde{\alpha}^{-1} = pr_2\tilde{\alpha}^{-1} = pr_2 f = B$, since $\tilde{\alpha}$ and $\tilde{\alpha}^{-1}$ are permutations of $B = pr_2 f$. Clearly, $\hat{\beta} = \tilde{\alpha}^{-1} f$ is an inverse of $\hat{\alpha}$, since

$$\hat{\alpha}\hat{\beta}\hat{\alpha} = \tilde{\alpha} f \tilde{\alpha}^{-1} f \tilde{\alpha} f = \tilde{\alpha} f [\tilde{\alpha}^{-1} f \tilde{\alpha}] f = \tilde{\alpha} f f|_B f = \tilde{\alpha} f = \hat{\alpha}$$

and is the only element with this property (since $\tilde{\alpha}^{-1}$ is the only inverse of $\tilde{\alpha}$.) Hence, for any $\hat{\alpha} \in \hat{\Phi}_{\mathcal{E}}$ we have $\hat{\alpha}^{-1} \in \hat{\Phi}_{\mathcal{E}}$.

To prove that $\hat{\Phi}_{\mathcal{E}}$ is closed under multiplication of transformations assume that $\hat{\alpha}, \hat{\beta} \in \hat{\Phi}_{\mathcal{E}}$. Then $\hat{\alpha} = \tilde{\alpha} f_{\tilde{\alpha}}$ and $\hat{\beta} = \tilde{\beta} f_{\tilde{\beta}}$ for uniquely determined $f_{\tilde{\alpha}}, f_{\tilde{\beta}} \in \mathcal{E}$ with $pr_2 f_{\tilde{\alpha}} = pr_1 \tilde{\alpha} = B \subseteq A$ and $pr_2 f_{\tilde{\beta}} = pr_1 \tilde{\beta} = C \subseteq A$.

Since $f_{\tilde{\beta}}$ and $f_{\tilde{\alpha}}$ are elements of a subsemilattice, their product is also an idempotent in $\mathcal{E}$. Thus $e = f_{\tilde{\beta}} f_{\tilde{\alpha}} \in \mathcal{E}$ and also, $pr_2 e = D = pr_2 f_{\tilde{\alpha}} \cap pr_2 f_{\tilde{\beta}}$, that is, $D = B \cap C$.

By $e \preceq f_{\tilde{\alpha}}$ and $\tilde{\alpha}$ local automorphism of $\mathcal{E}$ it follows that $\tilde{\alpha}$ is a local automorphism of e, and hence $\tilde{\alpha} e \tilde{\alpha}^{-1} = e_{|B}$. Similarly, $pr_2 e \subseteq pr_2 f_{\tilde{\beta}}$ and so $\tilde{\beta} e \tilde{\beta}^{-1} = e_{|C}$.

The product of the partial maps $\tilde{\alpha}$ and $\tilde{\beta}$ is defined on the inverse image under $\tilde{\beta}$ of the intersection of the second projection of the first with the first projection of the second, and so $pr_1 \tilde{\gamma} \subseteq D$ where $\tilde{\gamma} = \tilde{\alpha}\tilde{\beta}$. So $\tilde{\gamma}$ is a permutation on $D_1 = pr_2 \tilde{\gamma} \subseteq D$ and $\tilde{\gamma}$ is a local automorphism of every $e' \preceq e$ where $e' \in \mathcal{E}$ and $pr_2 e' \subseteq D_1$. To clarify this last statement observe that $\tilde{\alpha}$ is a local automorphism of e' and $\tilde{\beta}$ is a local automorphism of e' and if $e'(a) = b$ for every $a, b \in D$,

$$e' \tilde{\alpha} \tilde{\beta}(a) = \tilde{\alpha}(e' \tilde{\beta}(a)) = \tilde{\alpha}(\tilde{\beta}(b)) = \tilde{\alpha}\tilde{\beta}(b).$$

So $\tilde{\gamma} \in \tilde{\Phi}_{\mathcal{E}}$. Finally, $\hat{\alpha}\hat{\beta} = \tilde{\alpha} f_{\tilde{\alpha}} \tilde{\beta} f_{\tilde{\beta}} = \tilde{\gamma} e \in \hat{\Phi}_{\mathcal{E}}$.

On the other hand, for every $\hat{\alpha} \in \hat{\Phi}_{\mathcal{E}}$ we have

$$\hat{\alpha}\hat{\alpha}^{-1} = \tilde{\alpha} f_{\tilde{\alpha}} \tilde{\alpha}^{-1} f_{\tilde{\alpha}} = f_{\tilde{\alpha}} = \hat{\alpha}^{-1}\hat{\alpha}.$$

Therefore, $\hat{\Phi}_{\mathcal{E}}$ is a Clifford inverse subsemigroup of $\mathcal{T}_A$.

Further, suppose that $\hat{\Phi}_{\mathcal{E}} \subset \Psi$, where Ψ is a Clifford inverse semigroup with semilattice of idempotents $\mathcal{E}$. Then there exists $\varphi \in \Psi$ such that $\varphi \notin \hat{\Phi}_{\mathcal{E}}$ and

$\varphi^{-1}\varphi \in \mathcal{E}$. By Proposition 2.2.2 $\tilde{\varphi}$ is a local automorphism of $\varphi^{-1}\varphi$ and $\tilde{\varphi} \in \tilde{\Phi}_{\mathcal{E}}$ since $\tilde{\Phi}_{\mathcal{E}}$ is the set of all local automorphisms of $\mathcal{E}$. Thus $\varphi = \varphi(\varphi^{-1}\varphi) = \tilde{\varphi}(\varphi^{-1}\varphi)$ is an element of $\hat{\Phi}_{\mathcal{E}}$, which contradicts the assumption that $\varphi \notin \hat{\Phi}_{\mathcal{E}}$. Therefore, $\hat{\Phi}_{\mathcal{E}}$ is a maximal Clifford inverse subsemigroup of $\mathcal{T}_A$.

□

Straightforward corollary from the above theorem follows.

Corollary. *If $\mathcal{E}$ is a maximal subsemilattice of $\mathcal{T}_A$ then $\hat{\Phi}_{\mathcal{E}}$ is a maximal Clifford inverse subsemigroup of $\mathcal{T}_A$. If S is a maximal Clifford inverse subsemigroup of $\mathcal{T}_A$ with semilattice of idempotents $\mathcal{E}$ then $S = \hat{\Phi}_{\mathcal{E}}$.*

3. Maximal Clifford inverse subsemigroups of $\mathcal{I}_A$

The goal in this section is to classify maximal Clifford inverse subsemigroups of $\mathcal{I}_A$, that is, the semigroup of all partial one-to-one transformations of a set A onto itself, without any restrictions over the cardinality of A. In addition, we study the relationships between Clifford inverse subsemigroups of $\mathcal{I}_A$ and the equivalence relations on A. A classification of the maximal Clifford inverse subsemigroups of $\mathcal{I}_A$ in the finite case is obtained in [8]. Here we also generalize some of the computational results in [8] for a finite set A and add a few additional results.

Let ε be an equivalence relation on a set A. Consider the set $\mathcal{G}_\varepsilon$ of all permutations of A that map each equivalence class of ε onto itself. In other words, $\varphi \in \mathcal{G}_\varepsilon$ if and only if $\varphi(K) = K$ for every ε-class K. Obviously, $\mathcal{G}_\varepsilon$ is a permutation group.

A subset $B \subseteq A$ is called *ε-saturated* if it is a union of equivalence classes of ε. Thus B is ε-saturated exactly when it satisfies the following condition: $a \in B$ and $(a, b) \in \varepsilon$ imply $b \in B$ for all $a, b \in A$.

Let $\mathcal{I}_\varepsilon$ denote the set of all restrictions of permutations from $\mathcal{G}_\varepsilon$ to arbitrary ε-saturated subsets of A. The elements of $\mathcal{I}_\varepsilon$ are precisely permutations φ of arbitrary ε-saturated subsets B of A such that φ preserves all ε-classes contained in B. It is easy to see that $\mathcal{I}_\varepsilon$ is a Clifford inverse semigroup of one-to-one partial transformations of A. Indeed, all of its elements are such partial transformations. If $\varphi \in \mathcal{I}_\varepsilon$ then $\varphi^{-1} \in \mathcal{I}_\varepsilon$, where φ^{-1} denotes the one-to-one partial transformation inverse to φ. Clearly, $\varphi^{-1} \circ \varphi = \varphi \circ \varphi^{-1} = \Delta_B$, where B is the domain, denoted $pr_1\varphi$, (and hence also the range, denoted $pr_2\varphi$) of φ. Finally, if $\varphi, \psi \in \mathcal{I}_\varepsilon$, where φ is a permutation of B and ψ a permutation of C for some ε-saturated subsets B and C of A, then $\psi \circ \varphi$ is a permutation of the ε-saturated subset $B \cap C$. Clearly, this product of two partial permutations is an element of $\mathcal{I}_\varepsilon$.

The following theorem gives a classification of the maximal Clifford inverse subsemigroups of $\mathcal{I}_A$ in terms of the structures $\mathcal{I}_\varepsilon$ described above and it is the central result in this section.

Theorem 3.1 *$\mathcal{I}_\varepsilon$ is a maximal Clifford inverse subsemigroup of $\mathcal{I}_A$. Conversely, every maximal Clifford inverse subsemigroup of $\mathcal{I}_A$ coincides with $\mathcal{I}_\varepsilon$ for a suitable equivalence relation ε on A.*

Proof. Suppose that $\mathcal{I}_\varepsilon \subset \Psi$, where Ψ is a Clifford inverse subsemigroup of $\mathcal{I}_A$. Let $\psi \in \Psi\backslash\mathcal{I}_\varepsilon$. Then $\psi^{-1} \circ \psi \in \Psi$. Here $\psi^{-1} \circ \psi = \Delta_B$, where $B = pr_1\psi$. Since Δ_B is an idempotent in Ψ it follows that Δ_B commutes with each element φ of $\mathcal{I}_\varepsilon$. Clearly, Δ_B is not the empty transformation (and thus $B \neq \emptyset$) since otherwise ψ would be the empty transformation and so ψ would belong to $\mathcal{I}_\varepsilon$ which would contradict the original assumption. Thus, there are (at least one) ε-classes, which have non-empty intersection with B. If B is not ε-saturated, then there exists a non-singleton ε-class, say K, such that $a, b \in K$ and $a \notin B, b \in B$. Consider the transformation $\varphi = (a, b)$, where (a, b) is a transposition of K interchanging a and b. Clearly $\varphi \in \mathcal{I}_\varepsilon$, and hence $\varphi \circ \Delta_B = \Delta_B \circ \varphi$ on one hand, but on the other the left handside of the identity, $\varphi \circ \Delta_B(a)$ is undefined at a while the right handside $\Delta_B \circ \varphi(a) = b$. The last contradiction shows that B is ε-saturated.

Thus B is a union of ε-classes. Choose one of these classes, say, L. Then $\Delta_L \in \mathcal{I}_\varepsilon$, and hence $\psi \circ \Delta_L = \Delta_L \circ \psi$. The last identity is possible if and only if $\psi(L) = L$, therefore, ψ is a permutation of L. Thus the domain of ψ is a union of ε-classes and $\psi(L) = L$ for each of these ε-classes. It follows that $\psi \in \mathcal{I}_\varepsilon$, and hence $\Psi = \mathcal{I}_\varepsilon$. That proves the first claim of Theorem 3.1.

Now suppose that Ψ is a maximal Clifford inverse subsemigroup of $\mathcal{I}_A$. Choose a binary relation π_Ψ on A defined as follows:

$$(a, b) \in \pi_\Psi \Leftrightarrow (\forall\varphi \in \Psi)\{a \in pr_1\varphi \Leftrightarrow b \in pr_1\varphi\}.$$

Clearly, π_Ψ is an equivalence relation on A.

Let us write π instead of π_Ψ when the semigroup Ψ is specified. We are going to prove that $\Psi = \mathcal{I}_\pi$.

Let $\psi \in \Psi$. From the definition of π it follows directly that the domain of ψ is π-saturated. Further, let $a \in pr_1\psi$ and $\psi(a) = c$. Then $\psi^{-1}(c) = a$ and from $\psi \circ \psi^{-1} = \psi^{-1} \circ \psi$ it follows that $a \in pr_1\psi^{-1}$ and $c \in pr_1\psi$.

Finally, for any $\varphi \in \Psi$ such that $a \in pr_1\varphi$, recall that $\varphi^{-1} \in \Psi$ and $\varphi^{-1} \circ \varphi$ is an idempotent in Ψ that commutes with any element of Ψ, thus

$$c = \psi \circ \varphi^{-1} \circ \varphi(a) = \varphi^{-1} \circ \varphi \circ \psi(a) = \varphi^{-1} \circ \varphi(c),$$

that is $\varphi(c)$ is defined and $c \in pr_1\varphi$. Hence $a\pi c$ and ψ preserves the π classes K.

Thus $\psi(K) = K$, and hence $\psi \in \mathcal{I}_\pi$ so that $\Psi \subseteq \mathcal{I}_\pi$. By the maximality of Ψ, $\Psi = \mathcal{I}_\pi$. $\square$

Clearly the relationship π_Ψ is an equivalence relation and $\Psi \subseteq \mathcal{I}_{\pi_\Psi}$ for *every* Clifford inverse subsemigroup Ψ of $\mathcal{I}_A$. It turns out that π is not necessarily the only equivalence relation of A with this property.

Define the following binary relation τ_Ψ on A: $\tau_\Psi = \bigcup \Psi$, that is, τ_Ψ is the transitivity relation of Ψ. In other words,

$$\tau_\Psi = \{(a, b) \mid \psi(a) = b \text{ for some } \psi \in \Psi\}.$$

Indeed, τ is obviously a transitive binary relation. It is symmetric because $\psi(a) = b \Rightarrow \psi^{-1}(b) = a$. It is reflexive in the case of maximal Clifford inverse subsemigroup Ψ of $\mathcal{I}_A$, because $\Delta_A \in \Psi$ for, otherwise, $\Psi \cup \{\Delta_A\}$ is a Clifford inverse subsemigroup of $\mathcal{I}_A$ that properly contains Ψ, which contradicts the maximality of Ψ. Thus τ is an equivalence relation on A.

In general τ is not necessarily reflexive for an arbitrary Clifford inverse subsemigroup, since some element of A may not be mapped into itself for any element of Ψ. So we redefine τ as:

$$\tau_\Psi = \{(a, b) \mid \psi(a) = b \text{ for some } \psi \in \Psi\} \cup \Delta_A,$$

where $\Delta_A = \{(x, x) \mid x \in A\}$, the diagonal on A, (the smallest equivalence relation on A).

Now τ_Ψ is an equivalence relation and $\Psi \subseteq \mathcal{I}_{\tau_\Psi}$ for *every* Clifford inverse subsemigroup Ψ of $\mathcal{I}_A$.

Proposition 3.1. *Let ε be an equivalence relation on a set A. A Clifford inverse semigroup Ψ of partial permutations of A is contained in the maximal Clifford inverse semigroup $\mathcal{I}_\varepsilon$ if an only if $\tau_\Psi \subseteq \varepsilon \subseteq \pi_\Psi$. Thus, the equivalence relations ε on A such that $\Psi \subseteq \mathcal{I}_\varepsilon$ form the interval $[\tau_\Psi, \pi_\Psi]$ in the lattice of all equivalence relations on A.*

Proof. Let $\Psi \subseteq \mathcal{I}_\varepsilon$. Suppose that $(a, b) \in \tau_\Psi$, that is, $a = b$ or $\varphi(a) = b$ for some $\varphi \in \Psi$. In the former case $(a, b) \in \varepsilon$. In the latter case b belongs to the same ε-class as a because φ permutes this class. Thus $(a, b) \in \varepsilon$, and so $\tau_\Psi \subseteq \varepsilon$.

Now suppose that $(a, b) \in \varepsilon$ for some $a, b \in A$ and $a \in pr_1\varphi$ for an idempotent φ of Ψ. Since all elements of Ψ are partial one-to-one maps, then $\varphi(a) = a$ and, since $pr_1\varphi$ is ε-saturated, $b \in pr_1\varphi$. Similarly, $b \in pr_1\varphi \Rightarrow a \in pr_1\varphi$. Therefore, $(a, b) \in \pi_\Psi$ and so $\varepsilon \subseteq \pi_\Psi$.

Let η be an equivalence relation on A such that $\tau_\Psi \subseteq \eta$ and let K be an η-class. If $a \in A, \varphi \in \Psi$, and $\varphi(a)$ is defined, then $(a, \varphi(a)) \in \tau_\Psi \subseteq \eta$, and hence $\varphi(a) \in K$. Therefore, φ maps K into itself, that is, $\varphi_{|K}$ is a partial permutation of K. Now, if $\eta \subseteq \pi_\Psi$ and $\varphi(a)$ is defined, then $\varphi^{-1} \circ \varphi(a)$ is defined so that $a \in pr_1\varphi^{-1} \circ \varphi$. If $(a, b) \in \eta$, then $(a, b) \in \pi_\Psi$, and hence $b \in pr_1\varphi^{-1} \circ \varphi$. Therefore, $\varphi(b)$ is defined. We obtain $a \in pr_1\varphi \Leftrightarrow b \in pr_1\varphi$ for all $\varphi \in \Psi$ so that Ψ is η-saturated. $\square$

Remark Observe that we proved a little more: Let Ψ be a Clifford inverse semigroup of partial one-to-one transformations of a set A and ε an equivalence relation on A. Then

a. $\varphi(K) \subseteq K$ for every equivalence class K of ε and every $\varphi \in \Psi$ if and only if $\tau_\Psi \subseteq \varepsilon$;

b. Ψ is ε-saturated if and only if $\varepsilon \subseteq \pi_\Psi$.

$\square$

We may write τ and π instead of τ_Ψ and π_Ψ when the semigroup Ψ is specified.

Further, since $\tau \subseteq \pi$ for every Clifford inverse subsemigroup of partial permutations of $\mathcal{I}_A$, clearly every π-class is a union of τ-classes.

Consider a finite set A with $|A| > 0$ and let $K_1, K_2, \ldots, K_p$ be the equivalence classes of π and $L_1, L_2, \ldots, L_q$ those of τ, and let l_{k_i} be the number of L classes that are contained in the K_i class. Then the following result holds for every Clifford inverse semigroup Ψ of partial permutations of A :

Corollary 3.1. *The number of the maximal Clifford inverse subsemigroups of $\mathcal{I}_A$ that contain Ψ is:*

$$\prod_{i=1}^{p}[\sum_{j=1}^{l_{k_i}} S(l_{k_i}, j)],$$

where $S(n,k) = \dfrac{1}{k!}\displaystyle\sum_{i=0}^{k-1}(-1)^i \binom{k}{i}(k-i)^n$ *is the Stirling number of the second kind.*

Proof. Without loss of generality assume that the first l_{k_1} classes of τ form the K_1 class of π, i.e.

$$K_1 = \cup_{i=1}^{l_{k_1}} L_i.$$

Every equivalence relation ε for which $\tau \subseteq \varepsilon \subseteq \pi$ has a partition that contains the L_i classes and is contained in the K_1 class. The number of all such partitions is the sum of the Stirling numbers $S(l_{k_1}, i)$ from 1 to l_{k_1}. Combining all such partitions for every π-class we obtain the above formula. $\square$

Example 3.1. Clearly, each of the following Clifford inverse subsemigroups of $\mathcal{I}_A$:

$$\Psi_1 = \{\emptyset\}, \qquad \Psi_2 = \{\Delta_A\}, \qquad \Psi_3 = \Psi_1 \cup \Psi_2 = \{\emptyset,\ \Delta_A\},$$

is contained in every maximal Clifford inverse subsemigroup of $\mathcal{I}_A$.

For each of Ψ_i, $i = 1, 2, 3$ we have $\tau_{\Psi_i} = \Delta_A$ - the smallest equivalence relation on A and $\pi_{\Psi_i} = \omega$ the universal equivalence relation on A. Thus, $p = 1$

and $l_{k_i} = |A|$, and substituting into the formula from Corollary 3.1 we obtain that the total number of the maximal Clifford inverse subsemigroups of $\mathcal{I}_A$ is $\sum_{j=1}^{|A|} S(|A|, j)$.

To finish the discussion about the relationship of τ, π and Clifford inverse subsemigroups of $\mathcal{I}_A$, consider the following proposition.

Proposition 3.2. *Let Φ be a subsemigroup of $\mathcal{I}_A$. Then:*

1) If Φ is a maximal Clifford inverse subsemigroup of $\mathcal{I}_A$ then $\pi_\Phi = \tau_\Phi$;

2) If $|A| < \infty$ and $\pi_\Phi = \tau_\Phi$ then Φ is a Clifford inverse subsemigroup of $\mathcal{I}_A$ and $\Phi \subseteq \mathcal{I}_\pi$;

3) Φ is a maximal Clifford inverse subsemigroup of $\mathcal{I}_A$ if and only if $\Phi \cong \prod_{i\in I} \mathcal{G}^*_{K_i}$, where $\mathcal{G}^*_{K_i}$ is the symmetric group with adjoint empty transformation on the π-equivalence class K_i, that is, the group with zero of all bijective maps of the set K_i onto itself with the empty map as a zero element.

Proof. 1) We proved already that $\Phi = \mathcal{I}_\tau$. What is left to prove is that $\Phi \subseteq \mathcal{I}_\pi$.

Let $\varphi \in \Phi$. By the definition of π, $pr_1\varphi$ is $\pi-$saturated.

Let K be a $\pi-$class such that $K \subseteq pr_1\varphi$.

Assume that $\varphi(K) \neq K$. Then there exists $a \in K$ such that $\varphi(a) = b \notin K$. Since $(a, b) \notin \pi$ there exists $\psi \in \Phi$ such that $\psi(a)$ is undefined and $\psi(b)$ defined (or the other way around). Observe that in the first case we reach a contradiction with $\varphi \circ (\psi^{-1} \circ \psi)(a)$ that is undefined, but $(\psi^{-1} \circ \psi) \circ \varphi(a)$ defined, and in the later case with $\varphi^{-1} \circ (\psi^{-1} \circ \psi)(b)$ that is undefined, but $(\psi^{-1} \circ \psi) \circ \varphi^{-1}(b)$ defined.

Therefore, $\varphi(K) = K$ and so $\Phi \subseteq \mathcal{I}_\pi$.

2) Let $|A| < \infty$, $\tau = \pi$ and let $\varphi \in \Phi$. Clearly π is an equivalence relation on A for any subsemigroup of $\mathcal{I}_A$ and since $\tau = \pi$, then τ is an equivalence relation on A.

Recall that the elements of $\mathcal{I}_A$ are partial one-to-one maps and every element has a unique inverse. Thus, $\varphi^{-1} \in \mathcal{I}_A$ and $pr_1\varphi = pr_2\varphi^{-1}$, $pr_2\varphi = pr_1\varphi^{-1}$. And also

$$\varphi \circ \varphi^{-1} = \Delta_{pr_1\varphi^{-1}} \text{ and } \varphi^{-1} \circ \varphi = \Delta_{pr_1\varphi}.$$

Further, let $a \in pr_1\varphi$ and $\varphi(a) = b \in pr_2\varphi$. Then $(a, b) \in \tau$ and so $(a, b) \in \pi$, therefore $b \in pr_1\varphi$ that is $pr_2\varphi \subseteq pr_1\varphi$. Since A is finite set, it follows that $pr_2\varphi = pr_1\varphi$. Finally, from τ an equivalence relation we have $(a, b) \in \tau$, which implies $(b, a) \in \tau$, that is, there exists $\psi \in \Phi$, such that $\psi(b) = a$. From

$(b, a) \in \pi$ it follows that $pr_1\psi = pr_1\varphi$. From Φ semigroup (in particular closed under composition) it follows that φ^{-1} belongs to Φ and $pr_1\varphi^{-1} = pr_1\varphi$. Thus $\varphi \circ \varphi^{-1} = \varphi^{-1} \circ \varphi$ for every $\varphi \in \Phi$, that is Φ is Clifford inverse semigroup.

On the other hand, the domain of φ is π-saturated and preserves the equivalence classes by construction, so $\Phi \subseteq \mathcal{I}_\pi$.

3) Let Φ be a maximal Clifford inverse subsemigroup of $\mathcal{I}_A$. By Theorem , $\Phi = \mathcal{I}_\pi$.

For every $\varphi \in \Phi$ let φ_{K_i} be the restriction of φ to the π-class K_i. Again by Theorem , such restriction is either an empty map or a permutation on K_i. Thus, $\Phi|_{K_i} \subseteq \mathcal{G}^*_{K_i}$. On the other hand $\Phi = \mathcal{I}_\pi$ contains all restrictions of a permutations of $\mathcal{G}_\pi$ to an arbitrary π-saturated subset of A. Thus, $\mathcal{G}^*_{K_i} \subseteq \Phi|_{K_i}$.

Clearly, $\varphi = \bigcup_{i \in I} \varphi_{K_i}$, that is, φ is a direct sum of its restrictions to the different equivalence classes K_i of π_Φ and two different maps of Φ will differ on at least one such restriction. So there exists a one-to-one correspondence $\varphi \leftrightarrow (\varphi_{K_i})_{i \in I}$. This correspondence is also surjective, since Φ is maximal Clifford inverse. Thus $\Phi \cong \prod_{i \in I} \mathcal{G}^*_{K_i}$.

Conversely, if $\Phi \cong \prod_{i \in I} \mathcal{G}^*_{K_i}$, then consider the map $\varphi \leftrightarrow (\varphi|_{K_1} \oplus \varphi|_{K_2} \oplus \cdots)$ that maps each element of Φ to the direct product of its restrictions over the equivalence classes of π_Φ. Clearly, this map is an isomorphism between Φ and $\prod_{i \in I} \mathcal{G}^*_{K_i}$, and so the elements of Φ are precisely permutations φ of arbitrary π-saturated subsets B of A such that φ preserves all π-classes contained in B, that is, Φ is maximal Clifford inverse subsemigroup of $\mathcal{I}_A$.

□

From here to the end of the work we consider a finite set A.

First we start with a computation of the elements of a maximal Clifford inverse subsemigroup of $\mathcal{I}_A$.

Corollary 3.1. *Let A be a finite set with $n > 0$ elements and ε an equivalence relation on A with m equivalence classes $K_1, K_2, \ldots, K_m$ of cardinalities $k_1, k_2, \ldots, k_m$, respectively (in particular, $\sum_{i=1}^m k_i = n$). The maximal Clifford inverse semigroup $\mathcal{I}_\varepsilon$ is isomorphic to $\prod_{i=1}^m \mathcal{G}^*_{K_i}$, and has $|\mathcal{I}_\varepsilon| = \prod_{i=1}^m (k_i! + 1)$ elements.*

Proof. The first statement follows from Proposition . Since $|A| < \infty$, obviously for every $\varphi \in \mathcal{I}_\varepsilon$ we have $\varphi = \bigcup_{i=1}^m \varphi_i$, and the one-to-one correspondence between $\mathcal{I}_\varepsilon$ and $\prod_{i=1}^m \mathcal{G}^*_{K_i}$ is defined as $\varphi \leftrightarrow (\varphi_1, \varphi_2, \ldots, \varphi_m)$. By Theorem , for every i either $pr_1\varphi_i = K_i$ or $pr_1\varphi_i = \emptyset$. In the former case, φ_i is a permutation of K_i, and hence $\varphi_i \in \mathcal{G}_{K_i}$. In the latter case, $\varphi_i = \emptyset$. If $\varphi_i \in \mathcal{G}_{K_i}$, then there are $n_i!$ ways to choose φ_i. If $K_i \nsubseteq pr_1\varphi$, then φ_i is the zero element of $\mathcal{I}_{K_i}$ and there is only one way to choose φ_i. Altogether we have $n_i + 1$ choices for each φ_i, and hence there exist $\prod_{i=1}^m (k_i! + 1)$ choices for arbitrary elements of $\mathcal{I}_\varepsilon$. □

Example 3.2. Let $\omega = A \times A$ be the universal equivalence relation on a set A of cardinality n. Thus, every two elements of A are ω-equivalent. Then $\mathcal{I}_\omega = \mathcal{G}_n \cup \{\emptyset\}$, that is, we obtain the symmetric group on A with an empty partial transformation of A adjoined. Clearly, $|\mathcal{I}_\omega| = n! + 1$.

Now let $\Delta_A = \{(a, b) \mid a = b\}$ be the equality relation on A. Then $\mathcal{I}_{\Delta_A} = \{\Delta_B : B \subseteq A\}$, and hence $\mathcal{I}_{\Delta_A}$ is isomorphic to the semilattice of all subsets of A under the operation of set-theoretical intersection. This follows from the fact that $\Delta_B \circ \Delta_C = \Delta_{B\cap C}$ for all $B, C \subseteq A$. Clearly, $|\mathcal{I}_{\Delta_A}| = 2^n$.

Maximal Clifford inverse semigroups on the same set of points can have very different numbers of elements.

The following corollary answers the question: For what equivalence relation on a finate set A does the corresponding maximal Clifford inverse subsemigroup of $\mathcal{I}_A$ have a maximal (minimal) order and what is that order?

Corollary 3.2. *Let A be a finite set with $n > 0$ elements, ε an equivalence relation on A, and $\mathcal{I}_\varepsilon$ a corresponding maximal Clifford inverse semigroup. If $n = 2m$ is even then the minimal order of $\mathcal{I}_\varepsilon$ is 3^m. In this case each equivalence class of ε consists of exactly two elements. If $n = 2m + 1$ is odd then the minimal order of $\mathcal{I}_\varepsilon$ is $2 \cdot 3^m$ and each equivalence class of ε consists of exactly two elements except one class that is a singleton.*

The maximal order of $\mathcal{I}_\varepsilon$ for $n \neq 2, 3$ is $n! + 1$ and it is achieved for $\varepsilon = \omega$. If n is 2 or 3, the maximal order of $\mathcal{I}_\varepsilon$ is 4 or 8, respectively.

Proof. If $n = 1$ then $\omega = \Delta_A$ is the only equivalence relation on A and $|\mathcal{I}_A| = 1! + 1 = 2$.

For $n = 2$ there are two different equivalence relations ω and Δ_A with $|\mathcal{I}_\omega| = 2! + 1 = 3$ and $|\mathcal{I}_{\Delta_A}| = (1! + 1)^2 = 4$.

For $n = 3$ there are three types of equivalence relations on A: ω, Δ_A, and an equivalence relation ε with two classes that contain one and two elements of A, respectively. It follows that $|\mathcal{I}_\omega| = 3! + 1 = 7$, $|\mathcal{I}_{\Delta_A}| = (1! + 1)^3 = 8$ and $|\mathcal{I}_\varepsilon| = (1! + 1)(2! + 1) = 6$.

For completeness' sake, observe that if $n = 0$ (that is, $A = \emptyset$), then $\mathcal{I}_\emptyset$ consists of a single empty transformation, and hence $\mathcal{I}_\emptyset$ is a singleton.

Therefore, we may assume that $n \geq 4$.

For $n > 4$, let $\mathcal{I}_\varepsilon$ be a maximal Clifford inverse semigroup of minimal order m. Suppose that ε has an equivalence class B with $k > 2$ elements in it. From the formula for the order of maximal Clifford inverse semigroup with a finite set A which we obtained in Corollary it follows that $k! + 1 | m$. Split B into two equivalence classes, one with two elements and the other with $k - 2$ elements. As a result, ε is replaced by another equivalence relation η. Observe that $k^2 - k \geq 6$ and $(k - 2)! \geq 1$ for $k \geq 3$. Thus

$$k!+1 > k!-3(k-2)!+3 = (k^2-k-3)(k-2)!+3 \geq 3(k-2)!+3 = (2!+1)((k-2)!+1),$$

and hence $|\mathcal{I}_\varepsilon| = m > \dfrac{m}{k!+1}(2!+1)((k-2)!+1) = |\mathcal{I}_\eta|$ contradicting the minimality of the order of $\mathcal{I}_\varepsilon$. It follows that each ε-class has no more than two elements.

If ε has two classes, $\{a\}$ and $\{b\}$, each of them a singleton, applying again the formula from Corollary , it follows that m is divisible by $(1!+1)^2 = 4$. Replace ε by a new equivalence relation η with an equivalence class $\{a, b\}$ and all other classes the same as in ε. Then $|\mathcal{I}_\varepsilon| = m > \dfrac{3}{4}m = \dfrac{m}{(1!+1)^2}(2!+1) = |\mathcal{I}_\eta|$, again contradicting the minimality of the order of $\mathcal{I}_\varepsilon$. Thus all ε-classes consist of two elements, except possibly one class consisting of a single element. This completes the proof of the first part of our Corollary (for the minimal order of a maximal Clifford semigroup).

Suppose now that $0 < k < n$. Then

$$\frac{n!}{k!(n-k)!} = \binom{n}{k} \geq n > 3 \geq 1 + \frac{1}{k!} + \frac{1}{(n-k)!} = \frac{k!(n-k)! + k! + (n-k)!}{k!(n-k)!}.$$

Therefore,

$$|\mathcal{I}_\omega| = n! + 1 > k!(n-k)! + k! + (n-k)! + 1 = (k!+1)((n-k)!+1) = |\mathcal{I}_\varepsilon| \quad (1)$$

for any equivalence relation ε with exactly two equivalence classes consisting of k and $n-k$ elements.

Assume, further, that

$$|\mathcal{I}_\omega| = n! + 1 > \prod_{i=1}^{m}(k_i! + 1) = |\mathcal{I}_\varepsilon| \quad (2)$$

is true for any equivalence relation ε with $m > 1$ equivalence classes, and consider an equivalence relation ε with $m+1$ equivalence classes, $(m > 1)$.

Let K_i be an equivalence class of ε with cardinality k_i, $1 \leq i \leq m+1$. Then the remaining m classes form an equivalence relation $\bar{\varepsilon}$ on $A \setminus K_i$ and by the inductive assumption (2) it follows that

$$|\mathcal{I}_{\omega_{A\setminus K_i}}| = (n-k_i)! + 1 > \prod_{j=1, j\neq i}^{m+1}(k_j! + 1) = |\mathcal{I}_{\bar{\varepsilon}}|. \quad (3)$$

Multiplying both sides of the inequality (3) by $(k_i! + 1)$ we obtain

$$(n_i! + 1)((n-k_i)! + 1) > (n_i! + 1)\prod_{j=1, j\neq i}^{m+1}(k_j! + 1) = \prod_{j=1}^{m+1}(k_j! + 1). \quad (4)$$

By (1) the lefthandside of (4) is smaller than $n! + 1$ and so $|\mathcal{I}_\omega| = n! + 1 > \prod_{i=1}^{m+1}(k_i! + 1) = |\mathcal{I}_\varepsilon|$ which completes the inductive steps.

Thus, $|\mathcal{I}_\omega| = n! + 1 > \prod_{i=1}^{m}(k_i! + 1) = |\mathcal{I}_\varepsilon|$ is satisfied for any equivalence relation ε with $m > 1$ equivalence classes. It follows that $|\mathcal{I}_\omega|$ is the maximal Clifford semigroup with the greatest number of elements. □

Remark Here are the orders of the smallest and the largest maximal Clifford semigroups $\min|\mathcal{I}_\varepsilon|$ and $\max|\mathcal{I}_\varepsilon|$, respectively, for $n \leq 10$:

n	1	2	3	4	5	6	7	8	9	10
$\min\|\mathcal{I}_\varepsilon\|$	2	3	6	9	18	27	54	81	162	243
$\max\|\mathcal{I}_\varepsilon\|$	2	4	8	25	121	721	5041	40,321	362,881	3,628,801

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