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A NOTE ON COVER-AVOIDING PROPERTIES OF FINITE GROUPS

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Abstract

A subgroup H of a group G is said to be a CAP^*-subgroup of a group G if, for any non-Frattini chief factor K/L of G, we have $HK = HL$ or $H \cap K = H \cap L$. In this paper, some new characterizations for finite groups are obtained based on the assumption that some subgroups are CAP^*-subgroups of G.

1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G is said to have the cover-avoiding property in G if H covers or avoids every chief factor of G, in short, H is a CAP-subgroup of G. There has been much interest in the past in investigating the structure of finite groups when some subgroups have the cover-avoiding property, and many interesting results have been made, for example[1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 15, 16, 17].

In [14], Li and Liu introduced the CAP^*-subgroup.

Definition 1.1 *A subgroup H of a group G is said to be a CAP^*-subgroup of G if, for any non-Frattini chief factor K/L of G, we have $HK = HL$ or*

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$H \cap K = H \cap L$.

The authors had set up some meaningful results under the assumption of some subgroups are CAP^*-subgroup. In this paper, some new characterizations are obtained based on the assumption that some subgroups are CAP^*-subgroups of G.

Recall that a class of groups $\mathcal{F}$ is a formation if $\mathcal{F}$ contains all homomorphic images of group in $\mathcal{F}$, and if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$ [11, VI, Satz 7.1 and 7.2].

2. Basic definitions and preliminary results

Let K and L be normal subgroups of a group G with $K \leq L$. Then K/L is called a normal factor of G. A subgroup H of G is said to cover K/L if $HK = HL$. On the other hand, if $H \cap K = H \cap L$, then H is said to avoid K/L. If K/L is a chief factor of G and $K/L \leq \Phi(G/L)$ (respectively $K/L \nleq \Phi(G/L)$), then K/L is said to be a Frattini (respectively non-Frattini) chief factor of G.

Lemma 2.1 [14, Lemma 2.1] *Let N be a normal subgroup of a group G. If H is a CAP^*-subgroup of G, then:*

(1) HN/N is a CAP^*-subgroup of G/N.

(2) $H \cap N$ is a CAP^*-subgroup of G.

(3) If $N \leq \Phi(G)$ or $gcd(|H|, |N|) = 1$, then HN is a CAP^*-subgroup of G, where $gcd(-, -)$ denotes the greatest common divisor.

The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G.

Lemma 2.2 *Let G be a group and let M be a subgroup of G.*

(1) If M is normal in G, then $F^*(M) \leq F^*(G)$.

(2) $F^*(G) \neq 1$ if $G \neq 1$, in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$.

(3) $F^*(F^*(G)) = F^*(G) \geq F(G)$; If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.

(4) $C_G(F^*(G)) \leq F(G)$.

(5) Let $N = Z(E(G))\Phi(F(G))$. Then $F^*(G/N) = F^*(G)/N$, where $E(G)$ is the layer of G.

(6) $E(G)/Z(E(G))$ is the direct product of non-abelian simple groups.

Proof. By [12, X.13], (1)-(4) and (6) follow. By [6, Proposition 4.10], (5) is obtained. □

Lemma 2.3 [18, Chapter1, Theorem 7.15] *Let H be a normal subgroup of G. If every chief factor of G contained in H is cyclic, then $G/C_G(H)$ is supersolvable.*

Lemma 2.4 [8, Lemma 3.12] *Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G. If $|P| \leq p^2$ and G is A_4-free, then G is p-nilpotent.*

Lemma 2.5 [8, Lemma 3.16] *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type. Also let H a normal subgroup of a group G such that $G/H \in \mathcal{F}$. If G is A_4-free and all 2-maximal subgroups of every Sylow subgroup of H are CAP-subgroups of G, then G is in $\mathcal{F}$.*

3. Results

Theorem 3.1 *Let H be a normal subgroup of a group G such that G/H is supersolvable. Suppose that every maximal subgroup of any Sylow subgroups of $F^*(H)$ is a CAP^*-subgroup of G, then G is supersolvable.*

Proof. Suppose that the theorem is false and let G be a counterexample with smallest order. Then:

(1) $Z(E(H)) = 1$, in particular, $E(H)$ is the direct product of non-abelian simple groups.

Otherwise, $Z(E(H)) \neq 1$. Let $N = \Phi(F(H))Z(E(H))$. It is clear that $G/N\big/H/N \cong G/H$ is supersolvable. Let M/N be a maximal subgroup of a Sylow subgroup PN/N of $F^*(H)/N$, where P is a Sylow subgroup of $F^*(H)$. We can see that $M \cap P$ is a maximal subgroup of P. By hypothesis, $M \cap P$ is a CAP^*-subgroup of G. Applying Lemma 2.1, M/N is a CAP^*-subgroup of $F^*(H)/N$. Furthermore, $F^*(H)/N = F^*(H/N)$ by Lemma 2.2. It follows that G/N satisfies the hypothesis of our theorem for normal subgroup H/N. Thus, by the minimality of G, G/N is supersolvable and therefore G is solvable. This implies that $F^*(H) = F(H)$. We can finish the argument by following:

(a) All minimal normal subgroups of G contained in $F^*(H)$ are cyclic of prime order and non-Frattini.

Let N be a minimal normal subgroup of G contained in $F(H)$. Then N is a p-group for some prime p. If $N \leq \Phi(G)$, then $F(H/N) = F(H)/N$ by [Huppert, III, satz 4.2]. We can see that G/N satisfies the hypothesis of our theorem. By the minimal choice of G, G/N is supersolvable and therefore G is supersolvable, a contradiction. Hence we may assume that $N/1$ is a non-Frattini chief factor of G. There exists a maximal subgroup P_1 of a Sylow p-subgroup P of $F(H)$ such that $P_1 \cap N = 1$, this implies that $|N| = p$, as desired.

(b) A contradiction.

Let P be a Sylow p-subgroup of $F(H)$ and let K/L be a chief factor of G contained in P. We can choose a maximal subgroup P_1 of P such that $L \leq P_1$ and $K \nleq P_1$. If P_1 covers K/L, then $P_1K = P_1$ and so $K \leq P_1$, a contradiction. It follows from P_1 avoids K/L that $P_1 \cap K = L$. By comparing the order, we can see that $|K/L| = p$. Hence every chief factor of G under $F(H)$ is cyclic of prime order. On the one hand, by Lemma 2.3, $G/C_G(F(H))$ is supersolvable and therefore $G/H \cap C_G(F(H)) = G/C_H(F(H))$ is supersolvable. On the other hand, $C_H(F(H)) \leq F(H)$, it is clear that $G/F(H)$ is supersolvable. Therefore G is supersolvable, another contradiction. Hence $E(H) = 1$ and $E(H)$ is the direct product of non-abelian simple groups by Lemma 2.2.

(2) $F^*(H) = F(H)$.

Suppose that $E(H) \neq 1$. Let N be a minimal normal subgroup of G contained in $E(H)$, then N is a product of some non-abelian simple groups. It is clear that $N \nleq \Phi(G)$. If every maximal subgroup of Sylow subgroup P of $F^*(H)$ covers $N/1$, then $N \leq \Phi(P)$ and so $N \leq \Phi(G)$, a contradiction. Thus, there exists a maximal subgroup P_1 of P such that $P_1 \cap N = 1$ for every Sylow subgroup P of $F^*(H)$. This implies that N is the subgroup with square-free order and therefore N is solvable, a contradiction.

By (1) and (2), we can finish our proof. □

Corallary 3.2 *Let H be a solvable normal subgroup of a group G such that G/H is supersolvable. Suppose that every maximal subgroup of any Sylow subgroups of $F^*(H)$ is a CAP^*-subgroup of G, then G is supersolvable.*

Remark 3.3 The condition "H is solvable" in Corollary 3.2 can not be removed. For example, let $G = H = GL(2,4)$. Then $F(H) \cong Z_3$, where Z_3 is a cyclic group of order 3. It is clear that G satisfies the hypothesis of the Corollary 3.2 for normal subgroup H, but G is not supersolvable.

If M is a maximal subgroup of G and H is a maximal subgroup of M, then we call H a 2-maximal subgroup of G. We say the group G is A_4-free if there is no subgroup in G for which A is an isomorphic image. We prove the following results.

Theorem 3.4 *Let H be a normal subgroup of a group G and let p be the smallest prime dividing the order of H. If every 2-maximal subgroup of every Sylow p-subgroup of H is a CAP^*-subgroup of G and G is A_4-free, then H is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample with smallest order. Then:

(1) $O_{p'}(H) = 1$.

Otherwise, $O_{p'}(H) \neq 1$. We can see that $G/O_{p'}(H)$ satisfies the theorem for normal subgroup $H/O_{p'}(H)$. By the choice of G, $H/O_{p'}(H)$ is p-nilpotent and therefore H is p-nilpotent, as desired.

(2) Let N be a minimal normal subgroup of G, then $N \nleq \Phi(G)$.

It is clear that G/N satisfies the hypothesis of the theorem for normal subgroup HN/N. By the minimality of G, HN/N is p-nilpotent. If $N \nleq H$, then $H \cap N = 1$ and so $H \cong HN/N$ is p-nilpotent, as desired. Hence we can see that $N \leq H$ and so H/N is p-nilpotent. Since the p-nilpotent group classes is saturate, $N \nleq \Phi(G)$. By (1), $N/1$ is a p-chief factor.

(3) Final contradiction.

Let $S \in Syl_p(N)$. If $|S| \leq p^2$, then N is p-nilpotent by Lemma 2.4, in contradiction to the fact that N is a minimal normal subgroup of G. Hence we may assume that $|S| \geq p^3$. If all 2-maximal subgroups cover $N/1$, then $N \leq \Phi(M)$ and so $N \leq \Phi(G)$, where M is a maximal subgroup of P, it is impossible. Thus, there exists a 2-maximal subgroup P_1 such that $P_1 \cap N = 1$. This implies that $|P_1N|_p = |P_1||S| > |P|$, a final contradiction. □

Remark 3.5 In Theorem 3.4, we can not remove the assumption that G is A_4-free in general. For example, $G = A_4$. It is clear that every 2-maximal subgroup of the Sylow 2-subgroup of A_4 is a CAP^*-subgroup of G. But G is not 2-nilpotent.

Corollary 3.6 *Let p be the smallest prime dividing the order of a group G. If G is A_4-free and every 2-maximal subgroup of every Sylow p-subgroup of G is a CAP^*-subgroup of G, then G is p-nilpotent.*

Corollary 3.7 *Let H be a normal subgroup of a group G. If G is A_4-free and every 2-maximal subgroup of every Sylow subgroup of H is a CAP^*-subgroup of G, then H is a Sylow tower group of supersolvable type.*

Proof. We use induction on $|H|$. Let p be the smallest prime dividing the order of H. By Theorem 3.4, H is p-nilpotent and so H possesses a normal Hall p'-subgroup K. It is clear that G satisfies the hypothesis of the corollary for the normal subgroup K, by induction, K has the Sylow tower property. Consequently, H has the Sylow tower property. □

Theorem 3.8 *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and let H be a normal subgroup of a group G such that $G/H \in \mathcal{F}$. If G is A_4-free and every 2-maximal subgroup of every Sylow subgroup of H is a CAP^*-subgroup of G, then $G \in \mathcal{F}$.*

Proof. Suppose that the theorem is not true and let G be a minimal counterexample. Then by corollary 3.7, we can see that H has a Sylow tower of supersolvable type. Let p be the largest prime in $\pi(H)$ and $P \in Syl_p(H)$. Then P is a normal subgroup of G and every 2-maximal subgroup of P is a CAP^*-subgroup of G. It is easy to see that all 2-maximal subgroups of every Sylow subgroup of H/P are CAP^*-subgroups of G/P and G/P is A_4-free. Thus, by the minimality of G, we have $G/P \in \mathcal{F}$.

Let N be a minimal normal subgroup of G contained in P, it is clear that G/N satisfies the hypothesis for normal subgroup H/N and $G/N \in \mathcal{F}$. If $N \leq \Phi(G)$, then $G \in \mathcal{F}$, a contradiction. It follows that $N \nleq \Phi(G)$. If every 2-maximal subgroup of P cover $N/1$, then $N \leq \Phi(G)$, a contradiction. Then there exits a 2-maximal subgroup P_1 such that $P_1 \cap N = 1$, this implies that $|N| \leq p^2$. By Lemma 2.5, $G \in \mathcal{F}$, a final contradiction. □

Remark 3.9 In Theorem 3.8, the group G is not necessary supersolvable. For example, let H be a direct product of two copies of a cyclic group of order 3. There exist elements a, b in H such that $a^3 = b^3 = [a, b] = 1$ and let $H = \langle a, b\rangle$. The group H has an automorphism of order 4 such that $a^c = b^{-1}$ and $b^c = a$. If we write $K = \langle c\rangle$, let G be the semidirect product $G = H \rtimes K$. Clearly, G satisfies the hypothesis of the Theorem 3.8 for normal subgroup H, but G is not supersolvable.

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