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SEMISIMPLE FINITE DIMENSIONAL HOPF ALGEBRAS

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Abstract

The paper presents a survey of recent results on a classification of finite dimensional Hopf algebras with some restrictions on the category of its finite dimensional modules.

1. Introduction

Hopf algebras play a substantial role in the theory of noncommutative rings, noncommutative algebraic geometry and quantum groups.

One of the main problems in the theory of finite dimensional Hopf algebras is a classification of these algebras up to an isomorphism. In order to solve the problem it is necessary to find a list of main series of these algebras and to show that any algebra belongs to this list if its dimension is sufficiently large. Nowadays there is no satisfactory hypothesis concerning the main series. So it is necessary to find a classification of finite dimension semisimple Hopf algebra of under certain restrictions.

Suppose that H is a finite dimensional semisimple Hopf algebra over an algebraically closed field k. It is assumed that $\operatorname{char} k$ is coprime with the dimension of H. We shall assume that in each dimension $d > 1$ there exist at most one irreducible H-modules of dimension d. Under these assumptions there is found an explicit form of the counit and the antipode in H. In the paper we are summarizing recent results on this problem.

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The topic of the paper is motivated by the results of G.M. Seitz on a characterization of finite groups G having only one irreducible complex representation of degree $d > 1$. A group G with this property is either an extraspecial 2-group of order $2^{2m+1,}$ $d = 2^m$, or $|G| = d(d+1)$, where $d + 1 = p^f$, p a prime.

It is necessary to mention that semisimple Hopf algebras of dimension less than 60 were considered in [11]. Another approach to a classification of semisimple Hopf algebras is presented in [10].

2. Hopf algebras

In this section we shall remind basic notion in Hopf algebra theory from [9]. An associative algebra H over a filed k is a Hopf algebra if there exist an algebra homomorphisms of *comultiplication* $\Delta : H \to H^{\otimes 2}$, of a *counit* $\varepsilon : H \to k$ and algebra anti-homomorphism $S : H \to H$, an *antipode*, such that the following diagrams are commutative:

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H^{\otimes 2} \arrow[d, "1\otimes\Delta"] \\ H^{\otimes 2} \arrow[r, "\Delta\otimes 1"'] & H^{\otimes 3} \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "1"] & H^{\otimes 2} \arrow[d, "1\otimes\varepsilon"] \\ H^{\otimes 2} \arrow[r, "\varepsilon\otimes 1"'] & H \end{tikzcd}$$

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[drr, "\varepsilon"] & H^{\otimes 2} \arrow[r, "1\otimes S"] & H^{\otimes 2} \arrow[d, "\mu"] \\ H^{\otimes 2} \arrow[r, "S\otimes 1"'] & H^{\otimes 2} \arrow[r, "\mu"'] & k \end{tikzcd}$$

where $\mu : H^{\otimes 2} \to H$ is the multiplication map, [9]. Throughout the paper we shall use Sweedler's notation for comutiplication. If $h \in H$ then

$$\Delta(h) = \sum_h h_{(1)} \otimes h_{(2)} \in H \otimes H \tag{1}$$

For example any group algebra kG of a group G over a field k is a Hopf algebra where

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}$$

for any $g \in G$.

Another series of examples of Hopf algebras are universal (restricted) enveloping algebra U [3] of a Lie (restricted) algebra L. Comultiplication, counit and antipode in U is defined as follows

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x$$

for all $x \in L$ [3, Chapter 2,1].

Suppose that $k[G]$ is an algebra of regular functions on an algebraic group G. Then $k[G] \otimes k[G]$ is the algebra of regular functions on $G \times G$. So $k[G]$ is a Hopf algebra in which

$$\Delta(f)(x,y) = f(xy), \quad \varepsilon(f) = f(1), \quad S(f)(x) = f\left(x^{-1}\right).$$

If H has finite dimension then the dual space H^* is a Hopf algebra with *convolutive* multiplication $l_1 * l_2$, comultiplication Δ^*, counit ε^* and an antipode S^* which are defined as follows

$$\begin{gathered} l_1 * l_2 = \mu \cdot (l_1 \otimes l_2) \cdot \Delta, \quad \Delta(l)(x \otimes y) = l(xy), \\ (S^* l)(x) = l(S(x)), \quad \varepsilon^*(l) = l(1) \end{gathered}$$

for all $x, y \in H$.

In other words $l_1 * l_2$ is obtained as a composition of maps

$$H \xrightarrow{\quad\Delta\quad} H^{\otimes 2} \xrightarrow{\quad l_1 \otimes l_2\quad} k \otimes k \xrightarrow{\quad\mu\quad} k$$

An element $g \in H$ is a *group-like element* if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. The set $G(H)$ of all group-like elements is a multiplicative group in H. If H is a group algebra kG of a group G then $G(kG) = G$.

It is easy to check that an group-like elements of of the dual Hopf algebra H^* are just algebra homomorphisms $H \to k$. In what follows we shall denote by G the group $G(H^*)$ of group-like elements of H^*.

An associative algebra A with a unit element is a *left H-module algebra* if A is a left H-module and

$$h \cdot (ab) = \sum_h \left(h_{(1)} \cdot a\right) \left(h_{(2)} \cdot b\right), \quad h \cdot 1 = \varepsilon(h) 1$$

for $h \in H$ and $a, b \in A$. Here we use notation (1). Similarly one can define a *right H-module algebra.*

If A is a left H-module algebra then each group-like element acts ac an algebra automorphism of A. If H is a universal enveloping algebra of a Lie algebra L then A is a left H-module algebra if and only if there is a representation of L by derivations of the algebra A.

If an associative algebra A is a left (right) H-module algebra then we also say that there a l*eft (right) action of H on A*, or H *measures* A.

There are left and right actions of the dual Hopf algebra H^* on the algebra H which are denoted by $H^* \rightharpoonup H,\ H \leftharpoonup H^*$. These actions are defined as follows: if $f \in H^*,\ x \in H$ then in notations of (1) we have

$$f \rightharpoonup x = \sum_x x_{(1)} \langle f, x_{(2)} \rangle, \quad x \leftharpoonup f = \sum_x \langle f, x_{(1)} \rangle x_{(2)} \tag{2}$$

In particular if $g \in G(H^*)$ then $g \rightharpoonup x,\ x \leftharpoonup g$ are algebra automorphisms of H.

If H is a Hopf algebra then tensor product of any two left H-modules P, Q is again H-module. Namely, if $h \in H$ is from (1) and $p \in P$, $q \in Q$, then

$$h \cdot (p \otimes q) = \sum_h \big(h_{(1)} \cdot p\big) \otimes \big(h_{(2)} \cdot q\big) \in P \otimes Q.$$

It means that the category ${}_H\mathcal{M}$ of left H-modules is a tensor category with associative tensor product.

3. Direct decompositions and categories of modules

We shall consider the case when Hopf algebra H is a semisimple finite dimensional algebra over an algebraically closed field k. It is assumed that either $\operatorname{char} k = 0$ or $\operatorname{char} k > \dim H$. Consider the set of its irreducible modules. Modules of dimension 1 are in one-to-one correspondence with algebra homomorphism $H \to k$. Thus we have one-dimensional H-modules $E_g,\ g \in G$, assigned to elements g of the group G of group-like elements of G^*. More precise if $x \in E_g$ then $hx = \langle h, g \rangle x$ for any $h \in H$. The number of 1-dimensional non-isomorphic H-modules $E_g,\ g \in G$, is equal to the order of G.

Denote by $M_1, \dots, M_n$ be irreducible H-modules of dimensions greater than 1. We shall assume that $1 < d_1 = \dim M_1 < \cdots < d_n = \dim M_n$.

Since the basic field k is algebraically closed H as a k-algebra has a semisimple decomposition

$$\begin{gathered} H = (\oplus_{g \in G} k e_g) \oplus \operatorname{Mat}(d_1, k) \oplus \cdots \oplus \operatorname{Mat}(d_n, k), \\ 1 < d_1 < \cdots < d_n. \end{gathered} \tag{3}$$

where $\{e_g \mid g \in G\}$ is a system of central orthogonal idempotents in H. Throughout the rest of the paper we shall fix decomposition (3)

The number of 1-dimensional non-isomorphic H-modules E_g, $g \in G$, is equal to the order of G.

The problem of a classification of Hopf algebra H with direct decomposition (3) means that it is necessary to find an explicit form of a comultiplication Δ, of a counit ε and of an antipode S. In the next section we shall expose result on forms of Δ, ε, S. These results are based on a structure of a tensor category of H-modules.

Theorem 3.1 [V. A. Artamonov] *If $g \in G$ and $i = 1, \dots, n$ then there are H-module isomorphisms*

$$\begin{gathered} E_g \otimes M_i \simeq M_i \otimes E_g \simeq M_i, \\ M_i \otimes M_j \simeq \delta_{ij} \left(\oplus_{g \in G} E_g \right) \oplus \left(\oplus_{t=1}^n m_{ij}^t M_t \right), \\ m_{ij}^t = \dim_k \operatorname{Hom}_H \left(M_i \otimes M_j, M_t \right) \geqslant 0. \end{gathered}$$

In particular $d_i d_j = \delta_{ij}|G| + \sum_t m_{ij}^t d_t$ and $|G| \leqslant d_1^2$.

The multiplicities m_{ij}^t satisfy the equations

$$m_{ij}^s = m_{js}^i, \quad \delta_{ij}\delta_{ls}|G| + \sum_{t=1}^n m_{ij}^t m_{ts}^l = \delta_{js}\delta_{li}|G| + \sum_{t=1}^n m_{js}^t m_{it}^l$$

for all $i, j, s, l = 1, \dots, n$. In particular $m_{ij}^s = m_{js}^i = m_{si}^j$ and

$$\delta_{ij}\delta_{ls}|G| + \sum_{t=1}^n m_{ti}^j m_{ts}^l = \delta_{js}\delta_{li}|G| + \sum_{t=1}^n m_{st}^j m_{it}^l \tag{4}$$

If $i, j, p = 1, \dots, n$, then $m_{ij}^p \leqslant d_{\min(i,j,p)}$. Each H-module M_i is equipped with a non-degenerate (skew-)symmetric bilinear function $[x, y]$ such that

$$[hx, y] = [x, S(h)y]$$

for all $h \in H$ and for all $x, y \in M$.

4. Classification theorem

We shall use some significant elements in tensor square of a full matrix algebra. Denote by $\mathcal{R}_q$ the element

$$\mathcal{R}_q = \frac{1}{d_q} \sum_{i,j=1}^{d_q} E_{ij} \otimes E_{ji} \in \operatorname{Mat}(d_q, k)^{\otimes 2}. \tag{5}$$

Here $E_{ij} \in \operatorname{Mat}(d_q, k)$ stands for matrix unit. Up to a scalar multiple element $\mathcal{R}$ such that such that $(A \otimes B)\mathcal{R}_q = \mathcal{R}_q(B \otimes A)$ for all $A, B \in \operatorname{Mat}(d_q, k)$.

Each matrix constituent $\operatorname{Mat}(d_q, k)$ in (3) is stable under the antipode S. Moreover $S^2 = 1$. By Noether-Skolem theorem there exist a (skew-)symmetric matrix $U_q \in \mathrm{GL}(d_q, k)$ such that $S(x) = U_q\, {}^t x U_q^{-1}$ for all $x \in \mathrm{Mat}(d_q, k)$. Here ${}^t x$ is the transpose of a matrix x.

It can be also shown that $S(e_g) = e_{g^{-1}}$ for any central idempotent e_g from (3). So the form of the antipode is defined.

The next theorem present the form of a comultiplication Δ and a counit ε in H.

Theorem 4.1 [V. A. Artamonov] *Let $g \in G$ and $x \in \mathrm{Mat}(d_r, k)$ where $r = 1, \dots, n$. Put $\Delta_q = (1 \otimes S)\mathcal{R}_q$. Then*

$$\varepsilon(e_g) = \delta_{1,g}, \quad \varepsilon(x) = 0$$

and

$$\Delta(e_g) = \sum_{f \in G} e_f \otimes e_{f^{-1}g} + \sum_{t=1,\dots,n} (1 \otimes (g \rightharpoonup))\, \Delta_t,$$

$$\Delta(x) = \sum_{g \in G} [(g \rightharpoonup x) \otimes e_g + e_g \otimes (x \leftharpoonup g)] + \sum_{i,j=1}^{n} \Delta^r_{ij}(x),$$

where $\Delta^r_{ij}(x) \in \mathrm{Mat}(d_i, k) \otimes \mathrm{Mat}(d_j, k)$.

Theorem 4.2 [V. A. Artamonov] *Let H be a semisimple Hopf algebra with semisimple decomposition* (3)*.*

Suppose that there exists a matrix constituent $\mathrm{Mat}(d_i, k)$ *which is a Hopf ideal in H. Then $n = 1$.*

Now we shall get more information of actions (2) of elements $g \in G$. As it was mentioned above $g \rightharpoonup$, $\leftharpoonup g$ are algebra automorphisms of H. It follows immediately that each matrix component $\mathrm{Mat}(d_i, k)$ is stable under these actions. There exists a projective representation $\Phi_i : G \to \mathrm{PGL}(d_i, k) = \mathrm{PGL}(M_i)$ such that $g \rightharpoonup x = \Phi_i(g) x \Phi_i(g)^{-1}$ for all $x \in \mathrm{Mat}(d_i, k)$. Moreover elements $\Phi_i(g)$, $S(\Phi_i(v))$ commute in $\mathrm{PGL}(M_i)$. We can always assume that $\Phi_i(1)$ is the identity matrix and $\Phi_i\left(g^{-1}\right) = \Phi_i(g)^{-1}$ for all $g \in G$. It is easy to check that the trace $\operatorname{tr} \Phi_i(g) = \delta_{g1} d_i$ for any $g \in G$.

According to Shur's theory [4, Chapter 7, §53] there exists a group G^* with a central normal subgroup $N \simeq H^2(G, k^*)$ such that $G^*/N \simeq G$ and projective representations of G are induced by a linear representation of G^*. If the group G is Abelian then G^* is nilpotent and therefor its irreducible representation are monomial [4, Chapter 7, §52]. Using this theory we can prove

Theorem 4.4 [V. A. Artamonov] *Suppose that the group G is nilpotent. Taking an isomorphic copy of each matrix component in* (3) *we can assume that matrices $\Phi_i(g)$, $S(\Phi_i(g))$ are monomial for any i and any $g \in G$.*

Recall that a matrix is *monomial* if in each row it contain only one non-zero element. Each monomial matrix is a product of a diagonal matrix and a permutation matrix, which is obtained from identity matrix by a permutation of rows.

Theorem 4.3 [V. A. Artamonov] *Suppose that Φ_i induces an irreducible projective representation of the group G on M_i. Then $i = 1$, the order of G is equal to d_1^2 and $\Delta_{11}^t = 0$ for all $t = 1, \dots, n$. In particular $J = \oplus_{j\geqslant 2} \operatorname{Mat}(d_j, k)$ is a Hopf ideal in H and H/J is the Hopf algebra from Theorem 5.1 — 5.3.*

5. The case $n = 1$.

Hopf algebras with $n = 1$ in the decomposition (3) were considered by several authors. For simplicity we put $d_1 = d$. If the order of G has maximal possible value d^2 then the group G is Abelian. In the paper [12] Hopf algebra H is classified in terms of bicharacters of the group G using monoidal category ${}_H\mathcal{M}$.

If $d = 2$ then by [12] there exist up to equivalence four classes of semisimple Hopf algebras H of dimension 8, namely group algebras of Abelian groups of order 8, group algebras of dihedral group D_4 and of quaternions Q_8, and G. Kac Hopf algebra H generated by elements x, y, z with defining relations

$$\begin{aligned} x^2 = y^2 = 1,\ xy = yx,\ zx = yz, zy = xz, \\ z^2 = \frac{1}{2}(1 + x + y - xy), \\ \varepsilon(z) = 1,\ S(z) = z^{-1}, \\ \Delta(z) = \frac{1}{2}\left((1 + y) \otimes 1 + (1 - y) \otimes x\right)(z \otimes z), \end{aligned}$$

and x, y are group-like elements.

In the paper [13] there is given an explicit form of H if the order of G is d^2 and either d is odd or the group G is an elementary Abelian 2-group.

Interesting results were obtained in [8]. Let H be a semisimple Hopf algebra of dimension $2p^2$, where p is an odd integer. Then either H has a semisimple decomposition (3) with $n = 1,\ d = p$ and $|G| = p^2$ or H is its dual and H has a semisimple decomposition with $2p$ one-dimensional components and $\frac{p(p-1)}{2}$ components isomorphic ro $\operatorname{Mat}(2, k)$.

Theorem 5.1 [V. A. Artamonov, 2009] *Let H be from (3) with $n = 1$ and $G = G(H^*)$. The order of G is divisible by d and is a divisor of d^2.*

The order of G is equal to d^2 if and only if $\Delta_{11}^1 = 0$ in Theorem 4.1. Under these restrictions any two Hopf algebras of the same dimensions $2d^2$ are deformations of each other.

Recall that a Hopf algebra H' with a comultiplication Δ', counit ε' and an antipode S' is a *deformation* of H if $H' = H$ as an algebra and there exists an invertible element

$$J = \sum J_i \otimes J_i' \in H \otimes H, \quad J_i, J_i' \in H,$$

such that

$$\left[\sum_i (\Delta(J_i) \otimes J_i'\right] (J \otimes 1) = \left[\sum_i J_i \otimes \Delta(J_i')\right] (1 \otimes J) \in H^{\otimes 3},$$
$$\sum_i \varepsilon(J_i) J_i' = \sum_i J_i \varepsilon(J_i') = 1 \in H.$$

Moreover $\varepsilon' = \varepsilon$ and

$$\Delta'(h) = J^{-1}\Delta(h)J \in H \otimes H, \quad h \in H,$$
$$S'(h) = v^{-1}S(h)v \in H,\ h \in H, \quad v = \sum_i S(J_i)J_i' \in H.$$

Theorem 5.2 [Artamonov V.A., I.A. Chubarov, R. Mukhatov, 2007-2009] *Let H be from* (3), *$n = 1$ and $G = G(H^*)$. The projective representation Φ_i from § is faithful and irreducible representation . In particular $G = A_1 \times A_2$ for some Abelian groups $A_1 \simeq A_2$ of order d. Moreover let N be the full preimage of A_1 in the Shur group G^*. Then there exists a one dimensional kN-module M' in M_1 and an isomorphism of kG^*-modules $M_1 \simeq kG^* \otimes_{kN} M'$. In other terms Φ is induced by 1-dimensional representation of N.*

Consider a converse situation.

Theorem 5.3 [Artamonov V.A., I.A. Chubarov, R. Mukhatov, 2007-2009] *Suppose that an algebra H has semisimple decomposition* (3) *with $n = 1$ and $G = A \times A$ is an Abelian group, $|A| = d$. Take an irreducible projective faithful representation Φ of degree n as in Theorem 5.2. Define left $g \rightharpoonup x$ as $\Phi_i(g)x\Phi_i(g)^{-1}$. Put $S(e_g) = e_{g^{-1}}$ for any $g \in G$.*

Then there exists a (skew-)symmetric matrix $U \in \mathrm{GL}(d, k)$ with the following properties. The algebra H with

(i) *a counit $\varepsilon(e_g) = \delta_{g1},\ \varepsilon(\mathrm{Mat}(d, k)) = 0$;*

(ii) *with an antipode $S(e_g) = e_{g^{-1}},\ S(x) = U^t x U^{-1}$;*

(iii) *with right action $x \leftharpoonup g = S(A_g)xS(A_g)^{-1}$;*

(iv) *with comuliplication*

$$\Delta(e_g) = \sum_{f \in G} e_f \otimes e_{f^{-1}g} + \sum_{\alpha,\beta=1,\ldots,n} S(E_{\alpha\beta}) \otimes (g \rightharpoonup E_{\beta\alpha}),$$
$$\Delta(x) = \sum_{g \in G} [(g \rightharpoonup x) \otimes e_g + e_g \otimes (x \leftharpoonup g)]$$

is a Hopf algebra. Here $x \in \mathrm{Mat}(d,k),\ g \in G$.

Note that semisimple Hopf algebras of dimension $2p^2$ for an odd prime p in the same case as in Theorem 5.3 were also classified in [8] in different terms. All semisimple Hopf algebras of dimension $2p^2$ for an odd prime p were classified in [N0]. All Hopf algebra of dimension $2p^2$ for an odd prime p were classified in [7]. In this Theorem 5.3 we expand these results of an arbitrary n using the language of projective representations of the group G.

Theorem 5.4 [1] *Let H be from Theorem 5.3. An element*

$$w = \sum_{g\in G} \chi_{g,w} e_g + Z_w \in H,$$

with $\chi_{g,w} \in k,\ Z_w \in \mathrm{Mat}(d,k)$ is a group-like element from H if and only if the following conditions are satisfied:

1) $\chi_{gh,w} = \chi_{g,w}\chi_{h,w}$ *for all $g,h \in G$ which means that $\chi_{*,w}$ is a character of G;*

2) $g \rightharpoonup Z_w = \chi_{g,w} Z_w = Z_w \leftharpoonup g$ *for every $g \in G$.*

3) $Z_w U\, {}^t Z_w = U$.

Theorem 5.5 [Puninsky E., 2009] *Under the assumption of Theorem 5.3 the order of $G(H)$ is equal to $2d$, provided d is an odd prime. The group $G(H)$ is Abelian.*

Let H has decomposition (3) with arbitrary n. Each space $\mathrm{Mat}(d_i,k)$ is equipped with a non-degenerate bilinear form

$$\langle A, B\rangle = \mathrm{tr}\,(A \cdot S(B)) = \mathrm{tr}(A \cdot U_i\, {}^t B U_i^{-1}) \tag{6}$$

where S is the antipode. We shall identify the dual space of full matrix algebra with itself via the form (6).

Proposition 5.6 [2] *Let $g \in G$ and $X, Y \in \mathrm{Mat}(d_i,k)$. The form* (6) *is symmetric and*

$$\langle X, Y \leftharpoonup g\rangle = \langle g \rightharpoonup X, Y\rangle$$

It means that the operators $g \rightharpoonup,\ \leftharpoonup g$ are adjoint with respect to the symmetric bilinear form (6).

Corollary 5.7 [2] *Let H be from Theorem 5.2. If $w, w' \in G(H)$ in Theorem 5.4 and $K = \{w \in G(H) \mid \chi_{w,g} = 1\, \forall g \in G\}$. If $w \notin w'K$ then $\langle Z_w, Z'_w\rangle = 0$. Moreover $\langle Z_w, Z_w\rangle = d_1$.*

Denote by $a * b$ the convolution multiplication in H^*. Note that ε is the unit element in H^* which is equal to $1 \in G$.

Proposition 5.8 [2] *Suppose that H is from Theorem 5.2. If $g, h \in G$ and $X, Y \in \mathrm{Mat}(d_1, k)^*$ then*

$$g * h = gh, \quad g * X = g \rightharpoonup X, \quad X * g = X \leftharpoonup g,$$
$$X * Y = \frac{1}{d_1} \sum_{g \in G} \langle Y \leftharpoonup g^{-1}, X \rangle g.$$

Corollary 5.9 [2] *Let H be from Theorem 5.2. Then H^* is a $\mathbb{Z}_2$-graded algebra with the grading $H^* = H_0^* \oplus H_1^*$, where $H_0^* = kG$ and $H_1^* = \mathrm{Mat}(d_1, k)$.*

Theorem 5.10 [2] *Let H be from Theorem 5.2 and $d > 2$. Then H^* is not isomorphic to any Hopf algebra belonging to the class of Hopf algebras from Theorem 5.2.*

Previous results are based on

Theorem 5.11 [5] *Let G be a finite Abelian group of and let k be an algebraically closed field such that $\mathrm{char}\, k$ does not divide the order of G. The group G admits a faithful irreducible projective representations of dimension d over k if and only G is a direct product of two isomorphic groups of order d. Dimensions of any irreducible projective representations of the group G are equal either to d or to 1.*

Theorem 5.12 [6] *A finite abelian group G of order d^2 has decomposition $G \simeq A \times A$ if and only if it admits a non-degenerate bilinear symmetric form. Any irreducible projective representation of G of degree d is obtained from another one by an automorphism of G.*

Theorem 5.13 [2] *Let G be a finite group whose order is coprime with $\mathrm{char}\, k$. A projective representation $\Omega : G \to \mathrm{PGL}(d, k)$ such that*

$$\Omega(g^{-1}) = \Omega(g)^{-1}, \ \Omega(E) = E,$$

is irreducible if and only if

$$\mathcal{R}_n = \frac{1}{|G|} \sum_{g \in G} \Omega(g^{-1}) \otimes \Omega(g)$$

where $\mathcal{R}_n$ is from (5).

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