

INVERTIBLE MATRICES OVER SEMIFIELDS

R.I. Sararnrakskul*, S. Sombatboriboon[‡]
and P. Lertwichitsilp[†]

*Department of Mathematics, Faculty of Science,
Srinakharinwirot University, Bangkok 10110, Thailand
e-mail: *ruangvarin@gmail.com*

[‡]*Department of Mathematics, Faculty of Science,
Chulalongkorn University, Bangkok 10330, Thailand*

[†]*Department of Mathematics, Faculty of Science,
Kasetsart University, Bangkok 10900, Thailand
e-mail: 1patch2@hotmail.com*

Abstract

A *semifield* is a commutative semiring $(S, +, \cdot)$ with zero 0 and identity 1 such that $(S \setminus \{0\}, \cdot)$ is a group. Then every field is a semifield. It is known that a square matrix A over a field F is an invertible matrix over F if and only if $\det A \neq 0$. In this paper, invertible matrices over a semifield which is not a field are characterized. It is shown that if S is a semifield which is not a field, then a square matrix A over S is an invertible matrix over S if and only if every row and every column of A contains exactly one nonzero element.

1 Introduction

A *semiring* is a triple $(S, +, \cdot)$ such that $(S, +)$ and (S, \cdot) are semigroups and for all $x, y, z \in S$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$. A semiring $(S, +, \cdot)$ is called *additively* [*multiplicatively*] *commutative* if $x + y = y + x$ [$x \cdot y = y \cdot x$] for all $x, y \in S$. We call $(S, +, \cdot)$ *commutative* if it is both additively and multiplicatively commutative. An element $0 \in S$ is called a *zero*

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of a semiring $(S, +, \cdot)$ if $x+0 = 0+x = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$ and by an *identity* of $(S, +, \cdot)$ we mean an element $1 \in S$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in S$. Note that a zero and an identity of a semiring are unique.

If a semiring $(S, +, \cdot)$ has a zero 0 [an identity 1], we say that an element $x \in S$ is *additively* [*multiplicatively*] *invertible* over S if there exists an element $y \in S$ such that $x+y = y+x = 0$ [$x \cdot y = y \cdot x = 1$]. Note that such a y is unique and may be written as $-x$ [x^{-1}]. Observe that if x is additively invertible, then for all $a \in S$, $ax+a(-x) = a(x-x) = a0 = 0$, $a(-x)+ax = a(-x+x) = a0 = 0$, $xa + (-x)a = (x-x)a = 0a = 0$ and $(-x)a + xa = (-x+x)a = 0a = 0$. Thus $-ax = a(-x)$ and $-xa = (-x)a$. Since \cdot is distributive over $+$ in a semiring $(S, +, \cdot)$, the following fact holds.

Proposition 1.1. *Let S be a additively commutative semiring with zero 0 .*

If x_1, \dots, x_k are additively invertible over S , then $\sum_{i=1}^k a_i x_i$ and $\sum_{i=1}^k x_i a_i$ are additively invertible over S for all $a_1, \dots, a_k \in S$. Moreover, $-\sum_{i=1}^k a_i x_i =$

$$\sum_{i=1}^k a_i (-x_i) \text{ and } -\sum_{i=1}^k x_i a_i = \sum_{i=1}^k (-x_i) a_i.$$

Proof Let x_1, \dots, x_k be additively invertible in S and $a_1, \dots, a_k \in S$. Then

$$\begin{aligned} \sum_{i=1}^k a_i (-x_i), \sum_{i=1}^k (-x_i) a_i &\in S. \text{ Since } S \text{ is additively commutative,} \\ \sum_{i=1}^k a_i x_i + \sum_{i=1}^k a_i (-x_i) &= \sum_{i=1}^k (a_i x_i + a_i (-x_i)) = \sum_{i=1}^k a_i (x_i - x_i) = \sum_{i=1}^k a_i 0 = 0 \\ \sum_{i=1}^k x_i a_i + \sum_{i=1}^k (-x_i) a_i &= \sum_{i=1}^k (x_i a_i + (-x_i) a_i) = \sum_{i=1}^k (x_i - x_i) a_i = \sum_{i=1}^k 0 a_i = 0. \end{aligned}$$

Hence the lemma is proved. \square

A commutative semiring $(S, +, \cdot)$ with zero 0 and identity 1 is called a *semifield* if $(S \setminus \{0\}, \cdot)$ is a group. Then every field is a semifield. It is clearly seen that the following fact holds in any semifield.

Proposition 1.2. *If S is a semifield, then for all $x, y \in S$, $xy = 0$ implies $x = 0$ or $y = 0$.*

Let \mathbb{R} be the set of real numbers, \mathbb{Q} the set of rational numbers, $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ and $\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$. Then $(\mathbb{R}_0^+, +, \cdot)$ and $(\mathbb{Q}_0^+, +, \cdot)$ are semifields which are not fields.

For an $n \times n$ matrix A over a semiring S and $i, j \in \{1, \dots, n\}$, let A_{ij} be the entry of A in the i^{th} row and j^{th} column. Let A^t denote the transpose of A , that is, $A_{ij}^t = A_{ji}$ for all $i, j \in \{1, \dots, n\}$. Then $(A^t)^t = A$ and $(A + B)^t = A^t + B^t$ for all $n \times n$ matrices A, B over S . We have that for all $n \times n$ matrices A, B over a commutative semiring S , $(AB)^t = B^t A^t$.

Let $S = (S, +, \cdot)$ be a commutative semiring with zero 0 and identity 1 . An $n \times n$ matrix A over S is called *invertible* over S if there is an $n \times n$ matrix B over S such that $AB = BA = I_n$ where I_n is the identity $n \times n$ matrix over S . Note that such a B is unique.

It is well-known that a square matrix A over a field F is invertible if and only if $\det A \neq 0$. A generalization of this result can be found in [1, page 160] as follows: A square matrix A over a commutative ring R with identity 1 is invertible over R if and only if $\det A$ is a multiplicatively invertible in R , that is, there exists an element $r \in R$ such that $(\det A)r = r(\det A) = 1$. Characterizations of invertible matrices over some kinds of semirings can be found in [2] and [4].

The above examples of semifields which are not fields have the property that 0 is the only additively invertible element, that is, for $x, y \in S$, $x + y = 0$ implies $x = y = 0$. In fact, this property is generally true.

Proposition 1.3. ([5]) *If S is a semifield which is not a field, then 0 is the only additively invertible element of S .*

The purpose of this paper is to show that a square matrix A over a semifield S which is not a field is invertible over S if and only if every row and every column of A contains exactly one nonzero element.

2 Main Result

First, we give some necessary conditions for a square matrix over a commutative semiring S with zero and identity to be invertible over S .

Proposition 2.1. *Let S be a commutative semiring with zero 0 and identity 1 and A an $n \times n$ matrix over S . If A is invertible over S , then for all $i, j, k \in \{1, \dots, n\}$, $j \neq k$, $A_{ij}A_{ik}$ and $A_{ji}A_{ki}$ are additively invertible.*

Proof Let B be an $n \times n$ matrix over S such that $AB = BA = I_n$. Then for all distinct $p, q \in \{1, \dots, n\}$, $(AB)_{pq} = 0 = (BA)_{pq}$, so

$$\sum_{l=1}^n A_{pl}B_{lq} = \sum_{l=1}^n B_{pl}A_{lq} = 0.$$

This shows that

for all $l, p, q \in \{1, \dots, n\}$ with $p \neq q$, $A_{pl}B_{lq}$ and $B_{pl}A_{lq}$ are additively invertible in S . (1)

Next, let $i, j, k \in \{1, \dots, n\}$ be such that $j \neq k$. Then

$$\begin{aligned}
A_{ij}A_{ik} &= (A_{ij}A_{ik})(AB)_{ii} = A_{ij}A_{ik}\left(\sum_{l=1}^n A_{il}B_{li}\right) \\
&= A_{ij}A_{ik}A_{ik}B_{ki} + \sum_{\substack{l=1 \\ l \neq k}}^n A_{ij}A_{ik}A_{il}B_{li} \\
&= A_{ik}^2(B_{ki}A_{ij}) + \sum_{\substack{l=1 \\ l \neq k}}^n A_{ij}A_{il}(B_{li}A_{ik})
\end{aligned} \tag{2}$$

$$\begin{aligned}
A_{ji}A_{ki} &= (BA)_{ii}A_{ji}A_{ki} = \left(\sum_{l=1}^n B_{il}A_{li}\right)A_{ji}A_{ki} \\
&= \sum_{\substack{l=1 \\ l \neq j}}^n B_{il}A_{li}A_{ji}A_{ki} + B_{ij}A_{ji}A_{ji}A_{ki} \\
&= \sum_{\substack{l=1 \\ l \neq j}}^n A_{ki}A_{li}(A_{ji}B_{il}) + A_{ji}^2(A_{ki}B_{ij}).
\end{aligned} \tag{3}$$

From (1), (2), (3) and Proposition 1.1, we deduce that $A_{ij}A_{ik}$ and $A_{ji}A_{ki}$ are both additively invertible in S . \square

Example 1. Define \oplus on $[0, 1]$ by

$$x \oplus y = \max\{x, y\} \text{ for all } x, y \in [0, 1].$$

Then $([0, 1], \oplus, \cdot)$ is clearly a commutative semiring with zero 0 and identity 1. Moreover, 0 is the only additively invertible element of $([0, 1], \oplus, \cdot)$. Let A be an $n \times n$ matrix whose entries are in $[0, 1]$. Assume that A is invertible over $([0, 1], \oplus, \cdot)$. Then $AB = BA = I_n$ for some $n \times n$ matrix B over $[0, 1]$. Thus A and B contain neither a zero row nor a zero column. Since 0 is the only additively invertible in $([0, 1], \oplus, \cdot)$, by Proposition 2.1, every row and every column of A and B contain exactly one nonzero element. Since for $x, y \in [0, 1]$, $xy = 1$ implies $x = y = 1$, we deduce that a nonzero element of A and B in each row and each column must be 1.

If A is an $n \times n$ matrix over $[0, 1]$ of this form, then A is invertible over

$([0, 1], \oplus, \cdot)$. In fact, this is true for such an A in any commutative semiring with zero 0 and identity 1 that $AA^t = A^tA = I_n$. Since for $i, j \in \{1, \dots, n\}$,

$$(AA^t)_{ij} = \sum_{l=1}^n A_{il}A_{lj}^t = \sum_{l=1}^n A_{il}A_{jl} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

$$(A^tA)_{ij} = \sum_{l=1}^n A_{il}^tA_{lj} = \sum_{l=1}^n A_{li}A_{lj} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

it follows that $AA^t = A^tA = I_n$.

Theorem 2.2. *Let S be a semifield which is not a field and A an $n \times n$ matrix over S . Then A is invertible over S if and only if every row and every column of A contains exactly one nonzero element.*

Proof It is evident if $n = 1$. Assume that $n > 1$ and A is invertible over S . Let B be an $n \times n$ matrix over S such that $AB = BA = I_n$. Note that every row and every column must contain at least one nonzero element. To show that every row of A has exactly one nonzero element, suppose on the contrary that there are $p, q, q' \in \{1, \dots, n\}$ such that $q \neq q'$, $A_{pq} \neq 0$ and $A_{pq'} \neq 0$. Let $j \in \{1, \dots, n\}$ be such that $j \neq p$. Then

$$0 = (I_n)_{pj} = (AB)_{pj} = \sum_{l=1}^n A_{pl}B_{lj}.$$

By Proposition 1.3, $A_{pl}B_{lj} = 0$ for all $l \in \{1, \dots, n\}$. In particular, $A_{pq}B_{qj} = 0$. Since $A_{pq} \neq 0$, by Proposition 1.2, $B_{qj} = 0$. This shows that

$$B_{qj} = 0 \text{ for all } j \in \{1, \dots, n\} \text{ with } j \neq p. \quad (1)$$

Also, we have

$$1 = (I_n)_{qq} = (BA)_{qq} = \sum_{l=1}^n B_{ql}A_{lq} \quad (2)$$

and

$$0 = (I_n)_{qq'} = (BA)_{qq'} = \sum_{l=1}^n B_{ql}A_{lq'}. \quad (3)$$

Then (1) and (2) yield $B_{qp}A_{pq} = 1$. Also, from Proposition 1.3 and (3), we have $B_{qp}A_{pq'} = 0$. Hence

$$A_{pq'} = 1A_{pq'} = (B_{qp}A_{pq})A_{pq'} = A_{pq}(B_{qp}A_{pq'}) = A_{pq}0 = 0$$

which is a contradiction. Hence every row contains exactly one nonzero element.

Since $A^t B^t = (BA)^t = (AB)^t = B^t A^t = (I_n)^t = I_n$, from the above proof, we have that every row of A^t contains exactly one nonzero element. Hence every column of A contains exactly one nonzero element.

Conversely, assume that every row and every column contains exactly one nonzero element of S . Then

$$\begin{aligned} \text{for each } i \in \{1, \dots, n\}, \text{ there is a unique } k_i \in \{1, \dots, n\} \\ \text{such that } A_{ik_i} \neq 0 \end{aligned} \quad (4)$$

and

$$\text{for all distinct } i, j \text{ in } \{1, \dots, n\}, k_i \neq k_j. \quad (5)$$

Define an $n \times n$ matrix B over S by

$$B_{ij} = \begin{cases} A_{ji}^{-1} & \text{if } A_{ji} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Let $i, j \in \{1, \dots, n\}$ be given. Then it follows from (4) and (6) that $(AB)_{ij} = \sum_{l=1}^n A_{il} B_{lj} = A_{ik_i} B_{k_i j}$, or

$$(AB)_{ij} = \begin{cases} A_{ik_i} A_{jk_i}^{-1} & \text{if } A_{jk_i} \neq 0, \\ 0 & \text{if } A_{jk_i} = 0, \end{cases} = \begin{cases} A_{ik_i} A_{ik_i}^{-1} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} = (I_n)_{ij}.$$

From (4) and (5), we have $\{k_1, \dots, k_n\} = \{1, \dots, n\}$. It follows that $i = k_s$ and $j = k_t$ for some $s, t \in \{1, \dots, n\}$, thus it follows from (4) and (6) that

$$(BA)_{ij} = (BA)_{k_s k_t} = \sum_{l=1}^n B_{k_s l} A_{lk_t} = B_{k_s t} A_{tk_t} \text{ or}$$

$$(BA)_{ij} = \begin{cases} A_{tk_s}^{-1} A_{tk_t} & \text{if } A_{tk_s} \neq 0, \\ 0 & \text{if } A_{tk_s} = 0, \end{cases} = \begin{cases} A_{tk_t}^{-1} A_{tk_t} & \text{if } k_s = k_t, \\ 0 & \text{if } k_s \neq k_t, \end{cases} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} =$$

$(I_n)_{ij}$. This shows that $AB = BA = I_n$. Hence A is invertible over S , proving our Theorem. \square

We note here that Reutenauer and Straubing [3] have shown that if A and B are $n \times n$ matrices over any commutative semiring with zero and identity, then $AB = I_n$ implies $BA = I_n$. However, its given proof is quite complicated.

Example 2. Let $n > 1$ and

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Since $\det A = 1$, A is invertible over the field $\mathbb{R}[\mathbb{Q}]$. However, by Theorem 2.2, A is not invertible over the semifield $(\mathbb{R}_0^+, +, \cdot)$ $[(\mathbb{Q}_0^+, +, \cdot)]$. If

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & n \\ n-1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n-2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

then B is invertible over the semifield $(\mathbb{R}_0^+, +, \cdot)$ $[(\mathbb{Q}_0^+, +, \cdot)]$, so B is invertible over the field $(\mathbb{R}, +, \cdot)$ $[(\mathbb{Q}, +, \cdot)]$.

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