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ON 2-PRIMAL SKEW POLYNOMIAL RINGS

V. K. Bhat

School of Mathematics, SMVD University, P/o SMVD University, Katra, J and K, India- 182320
e-mail: vijaykumarbhat2000@yahoo.com

Abstract

In this article, we discuss minimal prime ideals of a Noetherian ring R. We recall $\sigma(*)$ property on a ring R, where σ is an automorphism of R (i.e. $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where P(R) is the prime radical of R). We ultimately show that if R is a Noetherian ring satisfying this property, then $R[x;\sigma]$ is a 2-primal ring.

Introduction

A ring R always means an associative ring with identity. The field of rational numbers and the set of natural numbers are denoted by $\mathbb{Q}$ and $\mathbb{N}$ respectively. The set of prime ideals of R is denoted by Spec(R). The sets of minimal prime ideals of R is denoted by Min.Spec(R). Prime radical and the set of nilpotent elements of R are denoted by P(R) and N(R) respectively. Let R be a ring and σ be an automorphism of R. Let I be an ideal of R such that $\sigma^m(I) = I$ for some $m \in \mathbb{N}$. We denote $\cap_{i=1}^{m}\sigma^i(I)$ by I^0. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J.

This article concerns the study of skew polynomial rings in terms of 2-primal rings. 2-primal rings have been studied in recent years and are being treated by authors for different structures. In [14] , Greg Marks discusses the 2-primal property of $R[x;\sigma;\delta]$, where R is a local ring, σ is an automorphism of R and δ is a σ-derivation of R. Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [10]. 2-primal near rings have been discussed by Argac and Groenewald in [2]. Recall that a ring R is 2-primal if and only if N(R) = P(R) if and only if the prime radical is a completely semiprime ideal.

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An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We also note that a reduced is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [7, 9, 11, 16].

Before proving the main result, we find a relation between the minimal prime ideals of R and those of the skew polynomial ring $R[x;\sigma]$, where R is a Noetherian ring and σ is an automorphism of R. This is proved in Theorem 2. Recall that $R[x;\sigma]$ is the usual polynomial ring with coefficients in R, in which multiplication is subject to the relation $ax = x\sigma(a)$ for all $a \in R$. We take any $f(x) \in R[x;\sigma]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote $R[x;\sigma]$ by S. Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. For example [1, 3, 4, 5, 8, 10, 12, 13]. Recall that in [12], a ring R is called σ-rigid if there exists an endomorphism of R with the property that $a\sigma(a) = 0$ implies a $= 0$ for $a \in R$. In [13], Kwak defines a $\sigma(*)$-ring R to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to the skew-polynomial ring $R[x;\sigma]$.

We consider the above property when σ is an automorphism of R and ultimately investigate the 2-primal property of $R[x;\sigma]$ when R is a Noetherian ring and prove the following:

(1) Let R be a Noetherian ring. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.

(2) Let R be a Noetherian $\sigma(*)$-ring. Then $R[x;\sigma]$ is 2-primal.

These results are proved in Theorems 5 and 7, respectively.

Skew polynomial rings

We begin with the following definition:

Definition 1 Let R be a ring, σ an automorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$.

Recall that an ideal I of a ring R is called σ-invariant if $\sigma(I) = I$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$.

We also note that if R is a Noetherian ring, then Min.Spec(R) is finite (Theorem (2.4) of [6]) and for any automorphism σ of R and for any $U \in Min.Spec(R)$, we have $\sigma^i(U) \in Min.Spec(R)$ for all $i \in \mathbb{N}$, therefore, it follows that there exists some $m \in N$ such that $\sigma^m(U) = U$ for all $U \in Min.Spec(R)$. As mentioned earlier we denote $\cap_{i=0}^{m}\sigma^i(U)$ by U^0.

We recall that an ideal J of a ring is called a σ-prime ideal of R if J is σ-invariant and for any σ-invariant ideals K and L with $KL \subseteq J$, we have $K \subseteq J$ or $L \subseteq J$. With this we have the following:

Theorem 2 *Let R be a Noetherian ring and σ an automorphism of R. Then:*

(1) If $P \in Min.Spec(S)$, then $P = (P \cap R)S$ and there exists $U \in Min.Spec(R)$ such that $P \cap R = U^0$.

(2) If $U \in Min.Spec(R)$, then $U^0 S \in Min.Spec(S)$.

Proof. (1) Let $P \in Min.Spec(S)$. Then $x \notin P$, as it is not a zero-divisor, therefore $P \cap R$ is a σ-prime ideal of R and $(P \cap R)S$ is a prime ideal of S by Lemma (10.6.4)(ii, iii) and Proposition (10.6.12) of [15]. Hence $P = (P \cap R)S$. Now $(P \cap R)S$ is prime, so it the intersection $\cap_{i=1}^{n} U_i$ of the primes that are minimal over it and these form a single orbit under σ. Therefore $P \cap R = U_i^0$ for each i. Let B be a minimal prime ideal of R with $B \subseteq U_i$. Then B^0 is σ-prime and $B^0 \subseteq U_i^0 = P \cap R$. Therefore $B^0 S$ is a prime ideal contained in $P = (P \cap R)S$. So $B^0 S = (P \cap R)S$ and, hence $B^0 = P \cap R$.

(2) Let $U \in Min.Spec(R)$. Then U^0 is σ-prime and $U^0 S$ is a prime ideal of S by Proposition (10.6.12) of [15]. Now it must contain a minimal prime ideal P of S (Proposition (2.3) of [6]). Now by paragraph (1) above $P = (P \cap R)S$ and $P \cap R = B^0$ for some $B \in Min.Spec(R)$. Therefore $B^0 S \subseteq U^0 S$ and $B^0 \subseteq U^0$. So $\sigma^i(B) \subseteq U$ for some i and therefore $\sigma^i(B) = U$ by the minimality of U. Hence $B^0 = U^0$ and $U^0 S = P$ is minimal. □

Proposition 3 *Let R be a ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring implies R is 2-primal.*

Proof. Let $a \in R$ be such that $a^2 \in P(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R).$$

Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. □

Proposition 4 *Let R be a $\sigma(*)$-ring and $U \in Min.Spec(R)$ be such that $\sigma(U) = U$. Then $US = U[x;\sigma]$ is a completely prime ideal of $S = R[x;\sigma]$.*

Proof. R is 2-primal by Proposition 3 and further more U is completely prime by Proposition (1.11) of Shin [16]. Now we note that σ can be extended to an automorphism $\overline{\sigma}$ of R/U. Now it is well known that $S/US \simeq (R/U)[x;\overline{\sigma}]$ and hence US is a completely prime ideal of S. □

We now give a necessary and sufficient condition for a Noetherian ring to be a $\sigma(*)$-ring in the following Theorem:

Theorem 5 *Let R be a Noetherian ring. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.*

Proof. Let R be a Noetherian ring such that for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R. Let $a \in R$ be such that $a\sigma(a) \in P(R) = \cap_{i=1}^{n} U_i$, where U_i are the minimal primes of R. Now for each i, $a \in U_i$ or $\sigma(a) \in U_i$ and U_i is completely prime. Now $\sigma(a) \in U_i = \sigma(U_i)$ implies that $a \in U_i$. Therefore $a \in P(R)$. Hence R is a $\sigma(*)$-ring.

Conversely, suppose that R is a $\sigma(*)$-ring and let $U = U_1$ be a minimal prime ideal of R. Now by Proposition 3, P(R) is completely semiprime. Let $U_2, U_3, ..., U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Now suppose that $U = U_1$ is not completely prime. Then there exist $a, b \in R \setminus U$ with $ab \in U$. Let c be any element of $b(U_2 \cap U_3 \cap ... \cap U_n)a$. Then $c^2 \in \cap_{i=1}^{n} U_i = P(R)$. So $c \in P(R)$ and, thus $b(U_2 \cap U_3 \cap ... \cap U_n)a \subseteq U$. Therefore $bR(U_2 \cap U_3 \cap ... \cap U_n)Ra \subseteq U$ and, as U is prime, $a \in U$, $U_i \subseteq U$ for some $i \neq 1$ or $b \in U$. None of these can occur, so U is completely prime. □

We now prove the following Theorem, which is crucial in proving Theorem (**??**).

Theorem 6 *Let R be a Noetherian $\sigma(*)$-ring, σ an automorphism of R. Then $R[x;\sigma]$ is 2-primal if and only if $P(R)[x;\sigma] = P(R[x;\sigma])$.*

Proof. Let $R[x;\sigma]$ be 2-primal. Now by Proposition 4 $P(R[x;\sigma]) \subseteq P(R)[x,\sigma]$. Let $f(x) = \sum_{j=0}^{n} x^j a_j \in P(R)[x;\sigma]$. Now R is a 2-primal subring of $R[x;\sigma]$ by Proposition 3, which implies that a_j is nilpotent and thus $a_j \in N(R[x;\sigma]) = P(R[x;\sigma])$, and so we have $x^j a_j \in P(R[x;\sigma])$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(R[x;\sigma])$. Hence $P(R)[x;\sigma] = P(R[x;\sigma])$.

Conversely suppose $P(R)[x;\sigma] = P(R[x;\sigma])$. We will show that $R[x;\sigma]$ is 2-primal. Let $g(x) = \sum_{i=0}^{n} x^i b_i \in R[x;\sigma]$, $b_n \neq 0$, be such that $(g(x))^2 \in P(R[x;\sigma]) = P(R)[x;\sigma]$. We will show that $g(x) \in P(R[x;\sigma])$. Now leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in Min.Spec(R)$. Now $\sigma(P) = P$ and P is completely prime by Proposition (1.11) of [16]. Therefore we have $b_n \in P$, for all $P \in Min.Spec(R)$; i.e. $b_n \in P(R)$. Now since $\sigma(P) = P$ for all $P \in Min.Spec(R)$, we get $(\sum_{i=0}^{n-1} x^i b_i)^2 \in P(R[x;\sigma]) = P(R)[x;\sigma]$ and as above we get $b_{n-1} \in P(R)$. With the same process in a finite number of steps we get $b_i \in P(R)$ for all i, $0 \leq i \leq n$. Thus we have $(g(x)) \in P(R)[x;\sigma]$; i.e. $(g(x)) \in P(R[x;\sigma])$. Therefore $P(R[x;\sigma])$ is completely semiprime. Hence $R[x;\sigma]$ is 2-primal. □

Theorem 7 *Let R be a Noetherian $\sigma(*)$-ring. Then $R[x;\sigma]$ is 2-primal.*

Proof. We use Theorem 2 to get that $P(R)[x;\sigma] = P(R[x;\sigma])$, and now the result is obvious by using Theorem 6. □

The following example shows that if R is a Noetherian ring, then $R[x;\sigma]$ need not be 2-primal.

Eample 8 Let $R = \mathbb{Q} \bigoplus \mathbb{Q}$ with $\sigma(a,b) = (b,a)$. Then the only σ-invariant ideals of R are 0 and R and, so R is σ-prime, $R[x;\sigma]$ is prime and $P(R[x;\sigma]) = 0$. But $(x(1,0))^2 = 0$. Therefore $R[x;\sigma]$ is not 2-primal.

The following example shows that if R is a Noetherian ring , then $R[x]$ need not be 2-primal.

Example 9 Let $R = M_2(\mathbb{Q})$, the set of 2×2 matrices over $\mathbb{Q}$. Then $R[x]$ is a prime ring with non-zero nilpotent elements and, so can not be 2-primal.

From these examples we conclude that if R is a Noetherian ring (even commutative and even a $\mathbb{Q}$-algebra), then $R[x;\sigma;\delta]$ need not be two primal, where σ is an automorphism of R and δ is a σ-derivation of R.

The above discussion leads to the further investigation:

Question If R is a 2-primal ring, is $R[x;\sigma;\delta]$ 2-primal (even if R is commutative, σ is the identity map or δ is the zero map, or the special case when R is Noetherian)?

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