East-West J. of Mathematics: Vol. 12, No 1 (2010) pp. 59-68

ON RIGHT STRONGLY PRIME TERNARY SEMIRINGS

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Abstract

In this paper we introduce the notion of right strongly prime ternary semiring and study some interesting properties of right strongly prime ternary semiring.

1 Introduction

The literature of the theory of ternary operations is vast and scatter over diverse areas of mathematics. The notion of ternary semiring is introduced by T. K. Dutta and S. Kar [1] in the year 2003 as a natural generalization of ternary ring introduced by W. G. Lister [15] in 1971. Subsequently, many notions of semiring and ring have been generalized to ternary semirings. Some earlier works of ternary semiring may be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14]. The notion of strongly prime ring was introduced by Handelman and Lawrence [11] in the year 1975 and they characterized strongly prime rings. The class of strongly prime rings is an interesting class of rings because it has

Key words: Ternary Semiring, Prime Ideal, Strongly Prime Ternary Semiring, Strongly Prime Ideal.
2000 AMS Mathematics Subject Classification: 16Y30

some interesting properties which are similar to the properties of commutative domains. In 2007, T. K. Dutta and M. L. Das [10] introduced and studied (right) strongly prime semiring.

In this paper we introduce and study the notion of right strongly prime ternary semiring.

2 Preliminaries

Definition 2.1. A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions :

(i) $(abc)de = a(bcd)e = ab(cde)$;
(ii) $(a+b)cd = acd + bcd$,
(iii) $a(b+c)d = abd + acd$,
(iv) $ab(c+d) = abc + abd$ for all $a, b, c, d, e \in S$.

Definition 2.2. Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$ then '0' is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.

Throughout the paper S will always denote a ternary semiring with zero and $S^* = S \setminus \{0\}$.

We note that a ternary semiring may not contain an identity but there are certain ternary semirings which generate identities in the sense defined below :

Definition 2.3. A ternary semiring S admits an identity provided that there exist elements $\{(e_i, f_i) \in S \times S \ (i = 1, 2, ..., n)\}$ such that $\sum_{i=1}^{n} e_i f_i x = \sum_{i=1}^{n} e_i x f_i = \sum_{i=1}^{n} x e_i f_i = x$ for all $x \in S$. In this case, the ternary semiring S is said to be a ternary semiring with identity $\{(e_i, f_i) : i = 1, 2, ..., n\}$.

In particular, if there exists an element $e \in S$ such that $eex = exe = xee = x$ for all $x \in S$, then 'e' is called a unital element of the ternary semiring S.

It is easy to see that $xye = (exe)ye = ex(eye) = exy$ and $xye = x(eye)e = xe(yee) = xey$ for all $x, y \in S$. So we have the following result :

Proposition 2.4. *If e is a unital element of a ternary semiring S, then $exy = xey = xye$ for all $x, y \in S$.*

Example 2.5. Let $\mathbb{Z}_0^-$ be the set of all negative integers with zero. Then with the usual binary addition and ternary multiplication, $\mathbb{Z}_0^-$ forms a ternary semiring with zero element '0' and unital element '-1'.

Definition 2.6. An additive subsemigroup T of a ternary semiring S is called a ternary subsemiring if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.7. An additive subsemigroup I of a ternary semiring S is called a left (right, lateral) ideal of S if s_1s_2i (respectively is_1s_2, s_1is_2) $\in I$ for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, a right, a lateral ideal of S, then I is called an ideal of S.

Proposition 2.8. Let S be a ternary semiring and $a \in S$. Then the principal

(i) left ideal generated by 'a' is given by $< a >_l = SSa + na$

(ii) right ideal generated by 'a' is given by $< a >_r = aSS + na$

(iii) two-sided ideal generated by 'a' is given by $< a >_t = SSa + aSS + SSaSS + na$

(iv) lateral ideal generated by 'a' is given by $< a >_m = SaS + SSaSS + na$

(v) ideal generated by 'a' is given by $< a > = SSa + aSS + SaS + SSaSS + na$, where $n \in \mathbb{Z}_0^+$ (set of all positive integers with zero).

Definition 2.9. An ideal I of a ternary semiring S is called a k-ideal if $x+y \in I$; $x \in S$, $y \in I$ imply that $x \in I$.

Definition 2.10. A proper ideal P of a ternary semiring S is called a prime ideal of S if for any three ideals A, B, C of S; $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

Definition 2.11. A ternary semiring S is called a prime ternary semiring if the zero ideal $\{0\}$ is a prime ideal of S.

3 Right Strongly Prime Ternary Semirings

Definition 3.1. A ternary semiring S is called right strongly prime if for every $a \in S^*$, there exist finite subsets F_1, F_2, F_3 of S such that $aF_1F_2F_3y = \{0\} \Longrightarrow y = 0$ for all $y \in S$.

Proposition 3.2. *A ternary semiring S is right strongly prime if and only if for every $a \in S^*$, there exists a finite subset F of S such that $aFFFy = \{0\} \Longrightarrow y = 0$ for all $y \in S$.*

Proof Suppose S is a right strongly prime ternary semiring. Let $a \in S^*$. Then there exist finite subsets F_1, F_2, F_3 of S such that $aF_1F_2F_3y = \{0\} \Longrightarrow y = 0$ for all $y \in S$. Let $F = F_1 \cup F_2 \cup F_3$. Then $F_1, F_2, F_3 \subseteq F$ and F is finite. Suppose $aFFFy = \{0\}$ for all $y \in S$. Then $aF_1F_2F_3y \subseteq aFFFy = \{0\}$ for all $y \in S$. This implies that $y = 0$ for all $y \in S$.

Converse part is obvious.

Example 3.3. Let $S = \{ri : r \in \mathbb{R},\ i^2 = -1\}$, where $\mathbb{R}$ is the set of all real numbers. Then together with usual binary addition and ternary multiplication, S forms a ternary semiring. Let $ri(\neq 0) \in S$ and $F = \{ri\}$. Then $(ri)FFFy = 0$ implies that $y = 0$ for all $y \in S$. Hence S is a right strongly prime ternary semiring.

Theorem 3.4. *Every right strongly prime ternary semiring is a prime ternary semiring.*

Proof Suppose that S is a right strongly prime ternary semiring. Let A, B, C be three ideals of S such that $ABC = \{0\}$. Suppose that $A \neq \{0\}$ and $B \neq \{0\}$. Since $A \neq \{0\}$, there exists $a(\neq 0) \in A$. Since S is a right strongly prime ternary semiring, by Proposition 3.2, there exists a finite subset F of S such that $aFFFy = \{0\}$ implies that $y = 0$ for all $y \in S$. Now $aFFF(BSC) = (aFF)(FBS)C \subseteq (ASS)(SBS)C \subseteq ABC = \{0\}$. This implies that $BSC = \{0\}$. Again, since $B \neq \{0\}$, there exists $b(\neq 0) \in B$ and for this $b \neq 0$, there exists a finite subset F' of S such that $bF'F'F'c \subseteq BSSSC \subseteq BSC = \{0\}$ for $c \in C$. This implies that $c = 0$. Since c is an arbitrary element of C, we find that $C = \{0\}$. This shows that $\{0\}$ is a prime ideal of S and hence S is a prime ternary semiring.

Theorem 3.5. *Let S be a ternary semiring with unital element 'e'. Then the following are equivalent :*

(i) *S is right strongly prime;*

(ii) *If I is a non-zero ideal of S, there exist finite subsets F' of I and F of S such that*

$F'Fy = \{0\}$ *implies that* $y = 0$ *for all* $y \in S$*;*

(iii) *If $x \in S^*$, there exists $s \in S$ and finite subsets F', F of S such that $xsF'Fy = \{0\}$*

implies that $y = 0$ for all $y \in S$.

Proof (i) $\Longrightarrow$ (ii).

Suppose that S is a right strongly prime ternary semiring and I be a non-zero ideal of S. Since I is a non-zero ideal of S, there exists $x(\neq 0) \in I$. Again since S is right strongly prime, there exists a finite subset F of S such that $xFFFy = 0$ implies that $y = 0$ for all $y \in S$. Let $F' = xFF$. Then $F' = xFF \subseteq IFF \subseteq I$ i.e. F' is a finite subset of I. Thus there exist finite subsets F' of I and F of S such that $F'Fy = \{0\}$ implies that $y = 0$ for all $y \in S$.

(ii) $\Longrightarrow$ (iii).

Suppose that condition (ii) holds. Let $a(\neq 0) \in S$. Then $< a >$ is a non-zero ideal of S. Now by (ii), there exists finite subsets F' of $< a >$ and F of S such that $F'Fy = \{0\}$ implies that $y = 0$ for all $y \in S$. If possible, let $aSS = \{0\}$. Then $< a > SS = \{0\}$. Since $F'Fa \subseteq < a > SS = \{0\}$, we have $F'Fa = \{0\}$. This implies that $a = 0$, a contradiction. Therefore, $aSS \neq \{0\}$.

Thus there exists $r, x \in S$ such that $arx \neq 0$. Then $I =< arx >$ is a non-zero ideal of S. By (ii), there exists a finite subset G' of I and a finite subset G of S such that $G'Gy = \{0\}$ implies that $y = 0$ for all $y \in S$. Since G' is a finite subset of I, we find that

$$G' = \left\{ narx + \sum_{i=1}^{m} arx\alpha_i\beta_i + \sum_{j=1}^{l} \gamma_j\delta_j arx + \sum_{k=1}^{s} \mu_k arx\nu_k + \sum_{p=1}^{t} c_p d_p arxu_p v_p \right\};$$

where $n, m, l, s, t \in \mathbb{Z}_0^+$; $\alpha_i, \beta_i, \gamma_j, \delta_j, \mu_k, \nu_k, c_p, d_p, u_p, v_p \in S$.

$$= \left\{ narx + \sum_{i=1}^{m} arx\alpha_i\beta_i + \sum_{j=1}^{l} \gamma_j\delta_j arx + \sum_{k=1}^{s} e(\mu_k arx\nu_k)e + \sum_{p=1}^{t} c_p d_p arxu_p v_p \right\}.$$

Let $F'' = \{x, x\alpha_i\beta_i, x\nu_k e, xu_p v_p : i = 1, 2, ..., m; k = 1, 2, ..., s; p = 1, 2, ..., t; m, s, t \in \mathbb{Z}_0^+\}$, and let $arF''Gy = \{0\}$. Then $G'Gy = \{0\}$. By (ii), we have $y = 0$.

$(iii) \Longrightarrow (i)$.

Suppose (iii) holds. Let $a \in S^*$. Then by (iii), there exists $s \in S$ and finite subsets F', F of S such that $xsF'Fy = \{0\}$ implies that $y = 0$ for all $y \in S$. Now taking $F_1 = \{s\}$, $F_2 = F'$ and $F_3 = F$ we find that there exist finite subset F_1, F_2, F_3 of S such that $aF_1F_2F_3y = \{0\}$ implies that $y = 0$. Hence S is right strongly prime.

Example 3.6. *Let S be the set of all 2×2 matrices over $\mathbb{Z}_0^-$, set of all non-positive integers. Then S is a ternary semiring with usual matrix addition and matrix multiplication. Let I be a non-zero ideal of S. Then I has a non-zero element, say $(a_{ij})_{2\times 2}$. Then $(a_{ij})_{2\times 2}$ has at least one non-zero element, say a_{rs}. Since I is an ideal of S, $E_{11}E_{1r}(a_{ij})_{2\times 2}E_{s1}E_{11} \in I$, where E_{rs} is the 2×2 matrix whose $(r, s)^{th}$ element is 1 and all others elements are zero. This shows that I has an element, say f_1 whose $(1, 1)^{th}$ element is non-zero and all others elements are zero. Similarly, we can get an element, say f_2 in I whose $(2, 2)^{th}$ element is non-zero and all others elements are zero. Let $f_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z}^- \right\}$, $f_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} : b \in \mathbb{Z}^- \right\}$. Let $F = \{f_1, f_2\}$ and $G = \{g_1, g_2\}$, where $g_1 = \left\{ \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} : c \in \mathbb{Z}^- \right\}$, $g_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} : d \in \mathbb{Z}^- \right\}$. Suppose that $FGz = 0$, where $z = \begin{pmatrix} u & v \\ x & y \end{pmatrix} \in S$. Then $f_1g_1z = f_2g_2z = f_1g_2z = f_2g_1z = 0$. This implies that $acu = acv = bdx = bdy = 0$. Since $a, b, c, d \in \mathbb{Z}^-$, we must have $u = v = x = y = 0$. Consequently, $z = 0$ and hence S is a right strongly prime ternary semiring.*

Definition 3.7. Let A be a non-empty subset of a ternary semiring S. Then the right annihilator of A with respect to $B(\subseteq S)$ in S, denoted by $r_a(A, B)$, is defined by $r_a(A, B) = \{x \in S : ABx = \{0\}\}$.

Proposition 3.8. *The right annihilator of a subset A with respect to a subset B of a ternary semiring S is a right ideal of S.*

Proof We note that $0 \in r_a(A, B)$, since $AB0 = \{0\}$. So $r_a(A, B)$ is non-empty. Let $s, t \in r_a(A, B)$. Then $ABs = ABt = \{0\}$. Now $AB(s + t) = ABs + ABt = \{0\} + \{0\} = \{0\}$ implies that $s + t \in r_a(A, B)$. Again, $AB(sxy) = (ABs)xy = \{0\}xy = \{0\}$ for all $x, y \in S$ implies that $sxy \in r_a(A, B)$. Hence $r_a(A, B)$ is a right ideal of S.

Proposition 3.9. *The right annihilator of a subset A with respect to a right ideal B of a ternary semiring S with unital element 'e' is an ideal (i.e. left ideal, lateral ideal and right ideal) of S.*

Proof From Proposition 3.8, it follows that $r_a(A, B)$ is a right ideal of S. Now it remain to show that $r_a(A, B)$ is a left and a lateral ideal of S. Let $s \in r_a(A, B)$. Then $ABs = \{0\}$. Now since B is a right ideal of S, we find that $AB(xys) = A(Bxy)s \subseteq A(BSS)s \subseteq ABs = \{0\}$ for all $x, y \in S$ implies that $xys \in r_a(A, B)$. This implies that $r_a(A, B)$ is a left ideal of S. Again, since B is a right ideal of S, we find that $AB(xsy) = AB(exe)(ese)y = A(Bex)(eesey) \subseteq A(BSS)(eesey) \subseteq AB(eesey) = A(Bee)(sey) \subseteq A(BSS)(sey) \subseteq AB(sey) = (ABs)ey = \{0\}ey = \{0\}$ for all $x, y \in S$ implies that $xsy \in r_a(A, B)$. This implies that $r_a(A, B)$ is a lateral ideal of S. Hence $r_a(A, B)$ is an ideal of S.

Definition 3.10. A ternary semiring S is said to satisfy descending chain condition (DCC) on right ideals of S if for each sequence of right ideals $I_1, I_2, I_3, ...$ of S with $I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$ there exists a positive integer n such that $I_n = I_{n+1} = ...$

We have shown that every right strongly prime ternary semiring is a prime ternary semiring. But a prime ternary semiring may not be a right strongly prime ternary semiring.

In particular we have the following result :

Theorem 3.11. *If S is a prime ternary semiring with descending chain condition (DCC) on right annihilator ideals of S then S is a right strongly prime ternary semiring.*

Proof Let I be a non-zero ideal of S and let $\mathcal{C}_{r_a}$ denotes the class of all right annihilators of the form $r_a(F', F)$, where F' and F are finite subsets of I and S respectively. Since S satisfies descending chain condition (DCC) on right annihilator ideals of S, $\mathcal{C}_{r_a}$ contains a minimal element $J = r_a(F'_0, F_0)$, say. We claim that $J = \{0\}$. If possible, let $J \neq \{0\}$. Since S is a prime ternary semiring, $ISJ \neq \{0\}$. Then there exist $x \in I$, $s \in S$ and $y \in J$ such that $xsy \neq 0$. Let $F'' = F'_0 \cup \{x\}$ and $F''' = F_0 \cup \{s\}$. Since $F'_0 F_0 s \subseteq F'' F''' s$, $r_a(F'', F''') \subseteq J$. Again, $y \in J$ and $xsy \neq 0$ imply that $r_a(F'', F''') \subset J$ which contradicts the minimality of J. Hence $J = \{0\}$. Therefore, by using Theorem 3.5(ii), we find that S is right strongly prime.

Definition 3.12. Let I be a proper ideal of a ternary semiring S. Then the congruence on S, denoted by ρ_I and defined by $s\rho_I s'$ if and only if $s+a_1 = s'+a_2$ for some $a_1, a_2 \in I$, is called the Bourne congruence on S defined by the ideal I.

We denote the Bourne congruence (ρ_I) class of an element r of S by r/ρ_I or simply by r/I and denote the set of all such congruence classes of S by S/ρ_I or simply by S/I.

Definition 3.13. For any proper ideal I of a ternary semiring S if the Bourne congruence ρ_I, defined by I, is proper i.e. $0/I \neq S/I$, then we define the addition and ternary multiplication on S/I by $a/I + b/I = (a+b)/I$ and $(a/I)(b/I)(c/I) = (abc)/I$ for all $a, b, c \in S$.

With these two operations S/I forms a ternary semiring and is called the Bourne factor ternary semiring or simply the factor ternary semiring.

Definition 3.14. An ideal I of a ternary semiring S is called a right strongly prime ideal if the factor ternary semiring S/I is right strongly prime.

Theorem 3.15. *Let Q be a k-ideal of a ternary semiring S. Then Q is a right strongly prime ideal of S if and only if for every ideal I of S not contained in Q, there exist finite subsets F' of I and F of S such that $F'Fy \subseteq Q$ implies that $y \in Q$ for all $y \in S$.*

Proof Let Q be a right strongly prime ideal of the ternary semiring S. Then the factor ternary semiring S/Q is right strongly prime. Let I be an ideal of S not contained in Q. Then $(I+Q)/Q$ is a non-zero ideal of the right strongly prime factor ternary semiring S/Q. Thus there exist finite subsets $F'' = \{(i_1+q_1)/Q, (i_2+q_2)/Q, ..., (i_n+q_n)/Q\}$ of $(I+Q)/Q$ and F/Q of S/Q such that $F''(F/Q)(y/Q) = 0/Q$ implies that $y/Q = 0/Q$ for all $y/Q \in S/Q$. Let $F' = \{i_1, i_2, ..., i_n\}$. Then F' is a finite subset of I. Let $i \in F'$. Then $i/Q = (i+q)/Q$, since $i\rho_Q(i+q)$ as $i+q = (i+q)+0$, where $q \in Q$. Let $F'Fy \subseteq Q$. Then $(F'/Q)(F/Q)(y/Q) = 0/Q$ i.e. $F''(F/Q)(y/Q) = (F'/Q)(F/Q)(y/Q) = 0/Q$ which implies that $y/Q = 0/Q$ for all $y/Q \in S/Q$. Since Q is a k-ideal of S, $y \in Q$ for all $y \in S$.

Conversely, let I/Q be a non-zero ideal of S/Q. Then I is an ideal of S not contained in Q. Then by hypothesis there exist finite subsets F' and F of I and S respectively such that $F'Fy \subseteq Q$ implies that $y \in Q$ for all $y \in S$. Since F' is a finite subset of I, F'/Q is a finite subset of I/Q. Let $(F'/Q)(F/Q)(y/Q) = 0/Q$. Then $F'Fy \subseteq Q$ and hence $y \in Q$ i.e. $y/Q = 0/Q$. Thus S/Q is a right strongly prime ternary semiring. Hence Q is a right strongly prime ideal of S.

Corollary 3.16. *A k-ideal I of a ternary semiring S is a right strongly prime ideal if for $a \notin I$, there exist finite subsets F' of $< a >$ and F of S such that $F'Fb \subseteq I$ implies that $b \in I$.*

Proof : Since $a \notin I$, $< a >$ is not properly contained in I. Then by above Theorem 3.15, there exist finite subsets F' and F of $< a >$ and S respectively such that $F'Fb \subseteq I$ implies that $b \in I$.

Definition 3.17. *[3]* A non-empty subset A of a ternary semiring S is called an m-system if for each $a, b, c \in A$ there exist elements x_1, x_2, x_3, x_4 of S such that $ax_1bx_2c \in A$ or $ax_1x_2bx_3x_4c \in A$ or $ax_1x_2bx_3cx_4 \in A$ or $x_1ax_2bx_3x_4c \in A$.

Theorem 3.18. *[3] A proper ideal P of a ternary semiring S is prime if and only if its complement $S \setminus P$ is an m-system.*

A similar type of result we obtain for right strongly prime ternary semiring. For this we introduce the following notion :

Definition 3.19. *A non-empty subset G of a ternary semiring S is called an sp-system if for any $g \in G$ there is a finite subset $F_1 \subseteq < g >$ and a finite subset F_2 of S such that $F_1F_2z \cap G \neq \emptyset$ for all $z \in G$.*

Theorem 3.20. *A proper ideal I of a semiring S is a right strongly prime if and only if $S \setminus I$ is an sp-system.*

Proof Let I be a right strongly prime ideal of S. Let $g \in S \setminus I$. Then $g \notin I$. So there exist finite subsets F' of $< g >$ and F of S such that $F'Fb \subseteq I$ implies that $b \in I$, by using Corollary 3.15. This implies that $F'Fz \cap (S \setminus I) \neq \emptyset$ for all $z \in (S \setminus I)$. Hence $S \setminus I$ is an sp-system.

Conversely, suppose that $(S \setminus I)$ is an sp-system. Let $a \notin I$. Then $a \in S \setminus I$. So there exist a finite subset F' of $< a >$ and F of S such that $F'Fz \cap (S \setminus I) \neq \emptyset$ for all $z \in S \setminus I$. Let $F'Fb \subseteq I$. Then $F'Fb \cap (S \setminus I) = \emptyset$. If possible, let $b \notin I$. Then $b \in S \setminus I$ which implies that $F'Fb \cap (S \setminus) \neq \emptyset$, a contradiction. Consequently, $b \in I$ and hence I is a right strongly prime ideal of S.

Now we consider the matrix ternary semiring $M_n(S)$, where S is a ternary semiring.

Let S be a ternary semiring with unital element 'e' and $M_n(S)$ be the set of all square matrices of order n ($n \in \mathbb{N}$) with entries from S.

Suppose $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n} \in M_n(S)$.

We define binary addition and ternary multiplication in $M_n(S)$ as follows :

$$(a_{ij})_{n\times n} + (b_{ij})_{n\times n} = (a_{ij} + b_{ij})_{n\times n} \text{ and } (a_{ij})_{n\times n}(b_{ij})_{n\times n}(c_{ij})_{n\times n} = (d_{ij})_{n\times n},$$

$$\text{where } d_{ij} = \sum_{k,\ l=1}^{n} a_{ik}b_{kl}c_{lj};\ \ 1 \leq i, j \leq n.$$

It can be easily verified that together with above defined addition and multiplication $M_n(S)$ is a ternary semiring with unital element. We call $M_n(S)$ the matrix ternary semiring.

Let $x \in S$. Then the notation xE_{ij} will be used to denote the $n \times n$ matrix in which the (i,j)-th entry is x and all other entries are zero. Then we can write $A = (a_{ij})_{n\times n} = \sum_{i,j=1}^{n} a_{ij}E_{ij}$ and it can be easily verified that

$$(xE_{pq})(yE_{rs})(zE_{uv}) = \begin{cases} (xyz)E_{pv}, & \text{if } \quad q = r \text{ and } s = u \\ 0, & \text{if } \quad q \neq r \quad \text{or} \quad s \neq u \end{cases} \quad \text{for all } x, y, z \in S.$$

Theorem 3.21. *Let S be a ternary semiring with unital element e. Then S is a right strongly prime ternary semiring if and only if $M_n(S)$ is a right strongly prime ternary semiring.*

Proof Let S is a right strongly prime ternary semiring. We shall show that $M_n(S)$ is a right strongly prime ternary semiring. Let $(a_{ij})_{n\times n} \neq (0)_{n\times n}$, $(b_{ij})_{n\times n} \neq (0)_{n\times n} \in M_n(S)$. Then there exists $a_{pq} \neq 0 \in S$ and $b_{rs} \neq 0 \in S$ for some integers p, q, r, s such that $1 \leq p, q, r, s \leq n$. Since S is right strongly prime, there exist finite subsets $F = \{f_1, f_2, ..., f_k\}$, $G = \{g_1, g_2, ..., g_l\}$ and $H = \{h_1, h_2, ..., h_m\}$ of S such that $a_{pq}f_tg_uh_vb_{rs} \neq 0$ for some $f_t \in F, g_u \in G, h_v \in H$. Now $(ps)th$ entry of the matrix $(a_{ij})(f_tE_{q1})(g_uE_{11})(h_vE_{1r})(b_{ij})$ is $a_{pq}f_tg_uh_vb_{rs}$ which is non-zero. Therefore $F' = \{f_tE_{i1} : 1 \leq i \leq n\}$, $G' = \{g_uE_{11}\}$ and $H' = \{h_vE_{1j} : 1 \leq j \leq n\}$ are finite subsets of $M_n(S)$ such that for any $(a_{ij})_{n\times n} \neq (0)_{n\times n}$, $(b_{ij})_{n\times n} \neq (0)_{n\times n} \in M_n(S)$, we have

$$(a_{ij})_{n\times n}(f_tE_{i1})(g_uE_{11})(h_vE_{1j})(b_{ij})_{n\times n} \neq (0)_{n\times n}$$

and hence $M_n(S)$ is a right strongly prime ternary semiring.

Conversely, suppose that $M_n(S)$ is a right strongly prime ternary semiring and $a(\neq 0) \in S$, $b(\neq 0) \in S$. Then aE_{11} and bE_{11} are non-zero elements in $M_n(S)$. Since $M_n(S)$ is right strongly prime, there exist finite subsets $A = \{(a_{ij})_{n\times n}\}$, $B = \{(b_{ij})_{n\times n}\}$ and $C = \{(c_{ij})_{n\times n}\}$ of $M_n(S)$ such that $(aE_{11}(a_{ij})_{n\times n}(b_{ij})_{n\times n}(c_{ij})_{n\times n}(bE_{11}) \neq (0)_{n\times n}$.

Now $(aE_{11})(a_{ij})_{n\times n}(b_{ij})_{n\times n}(c_{ij})_{n\times n}(bE_{11}) = (d_{ij})_{n\times n}$, where

$$d_{ij} = \begin{cases} \sum_{s,\ l=1}^{n} aa_{1s}b_{sl}c_{l1}b, & \text{if } \ i = j = 1; \\ 0, & \text{elsewhere }. \end{cases}$$

Since $(d_{ij})_{n\times n} \neq (0)_{n\times n}$, we have $\sum_{s,\ l=1}^{n} aa_{1s}b_{sl}c_{l1}b \neq 0$. Hence $aa_{1s}b_{sl}c_{l1}b \neq 0$ for some s and l; $1 \leq s, l \leq n$. Now choose $A' = \{a_{1j} : 1 \leq j \leq n\} \subseteq S$, $B' = \{b_{ij} : 1 \leq i, j \leq n\} \subseteq S$ and $C' = \{c_{i1} : 1 \leq i \leq n\} \subseteq S$. Then $a\alpha\beta\gamma b \neq 0$ for some $\alpha \in A'$, $\beta \in B'$ and $\gamma \in C'$ and hence S is a right strongly prime ternary semiring.

Acknowledgement : The authors are grateful to Professor T. K. Dutta for his valuable suggestions and continuous help through out the preparation of this paper. The third author is thankful to CSIR, India, for financial support.

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