

**COUPLING THE LQ REGULARIZATION  
METHOD AND THE OUTER  
APPROXIMATION METHOD FOR SOLVING  
PSEUDOMONOTONE VARIATIONAL  
INEQUALITIES**

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**Abstract**

In our recent papers [2, 3] we have proposed a modified logarithmic-quadratic method for solving monotone generalized variational inequalities and quasimonotone multivalued variational inequalities on polyhedral. The method is based on the special logarithmic quadratic function which replaces the usual quadratic. In this paper we combine this result with the outer approximation method to obtain a new interior approximation algorithm for solving pseudomonotone variational inequalities satisfying a certain Lipschitz condition on a closed convex set. Next, to avoid the Lipschitz condition we combine this technique with linesearch technique to obtain a convergent algorithm for pseudomonotone variational inequalities.

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## 1 Introduction

Classical variational inequalities, denoted by  $(VIP)$ , are to find a vector  $x^* \in C$  such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C,$$

where  $C$  is a nonempty closed convex subset of  $\mathbb{R}^n$  and  $F$  is a continuous mapping from  $\mathbb{R}^n$  into itself. Problems  $(VIP)$  include nonlinear complementarity problems (shortly  $(NCP)$ , when  $C = \mathbb{R}_+^n$ ) and system of nonlinear equations (when  $C = \mathbb{R}^n$ ). Throughout this paper we assume that  $C$  is bounded, that  $F(x)$  is continuous, pseudomonotone on  $C$  and the solution set of  $(VIP)$ , denoted by  $S^*$ , is nonempty. Variational inequalities have many important applications in economics, operation researches and nonlinear analysis, and have been studied by many researchers (see [5, 11, 12, 13, 14, 15, 18, 24]).

First, let us recall the well known concepts that will be used in the sequel (see [19, 20]).

**Definition 1.1.** Let  $C \subseteq \mathbb{R}^n$  and  $F : C \rightarrow \mathbb{R}^n$ . The function  $F$  is said to be (a) pseudomonotone on  $C$  if for each  $x, y \in C$  the inequality

$$\langle F(y), x - y \rangle \geq 0$$

implies

$$\langle F(x), x - y \rangle \geq 0.$$

(b) Lipschitz on  $C$  with constant  $L$  (shortly  $L$ -Lipschitz) if for each  $x, y \in C$

$$\|F(x) - F(y)\| \leq L\|x - y\|.$$

Among powerful approaches to  $(NCP)$  is the logarithmic-quadratic proximal method (shortly  $(LQ)$ ) presented originally by Auslender et al. in Ref. [7, 8, 9] under that the operator is monotone on  $C := \mathbb{R}_+^n$ , which is starting with any point  $x^0 \in C$  and  $\lambda_k \geq \lambda > 0$ , iteratively updates  $x^{k+1}$  conforming the following problem:

$$0 \in \lambda_k F(x) + \nabla_x d_\phi(x, x^k),$$

where

$$d_\phi(x, y) = \sum_{i=1}^n y_i^2 \phi(y_i^{-1} x_i)$$

and

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu(t - \log t - 1) & \text{if } t > 0 \\ +\infty & \text{otherwise,} \end{cases} \quad (1.1)$$

with  $\nu > \mu > 0$ . Then the term  $d_\phi$  forces the iteratives  $\{x^{k+1}\}$  to stay in the interior of  $C$ .

In our recently papers [1, 2, 3, 4], we have proposed a new type of proximal interior method for solving (VIP) and equilibrium problems (shortly (EQ)) on polyhedral  $C := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ , where  $A$  is an  $p \times n$  matrix,  $b \in \mathbb{R}^p$  through replacing function  $d_\phi(x, y)$  by  $D(x, y)$  which is defined as  $D(x, y) = d(l(x), l(y))$ , where

$$\begin{aligned} l_i(x) &= b_i - \langle a_i, x \rangle \quad i = 1, 2, \dots, p, \\ l(x) &= (l_1(x), l_2(x), \dots, l_p(x))^T, \end{aligned}$$

and

$$d(x, y) = \begin{cases} \frac{1}{2} \|x - y\|^2 + \mu \sum_{i=1}^n y_i^2 \left( \frac{x_i}{y_i} \log \frac{x_i}{y_i} - \frac{x_i}{y_i} + 1 \right) & \text{if } x > 0, \\ +\infty & \text{otherwise,} \end{cases} \quad (1.2)$$

with  $\mu \in (0, 1)$ .

Applying to (VIP), in this paper we consider function  $D(x, y)$  for every  $x, y \in T_k \quad \forall k = 0, 1, \dots$ , where  $T_k$  is a polyhedral sequence defined by

$$T_k = \{x \in \mathbb{R}^n \mid b_i - a_i x \geq 0 \quad i = 1, 2, \dots, n + k\},$$

$a_i$  ( $i = 1, 2, \dots, n + k$ ) are the rows of matrix  $A_k$ ,  $l_i(x) = b_i - \langle a_i, x \rangle \quad i = 1, 2, \dots, n + k$  and  $l(x) = (l_1(x), l_2(x), \dots, l_{n+k}(x))^T$ . For given  $x^{k,j} \in T_k$  ( $j = 0, 1, \dots$ ), we denote by  $\nabla_x D(x, x^{k,j})$  the gradient of  $D(\cdot, x^{k,j})$  at  $x$ . Then we have

$$\nabla_x D(x, x^{k,j}) = -A_k^T (l(x) - l(x^{k,j})) + \mu X_{k,j} \log \frac{l(x)}{l(x^{k,j})},$$

where

$$\begin{aligned} X_{k,j} &= \text{diag}(l_1(x^{k,j}), \dots, l_{n+k}(x^{k,j})), \\ \log \frac{l(x)}{l(x^{k,j})} &= \left( \log \frac{l_1(x)}{l_1(x^{k,j})}, \dots, \log \frac{l_{n+k}(x)}{l_{n+k}(x^{k,j})} \right). \end{aligned}$$

## 2 An interior proximal algorithm

To solve the generalized variational inequalities:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0 \quad \forall x \in C,$$

where  $C \subset \mathbb{R}^n$  is a polyhedral defined  $C := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ ,  $\varphi : C \rightarrow \mathbb{R}$  is convex,  $F : C \rightarrow \mathbb{R}^n$  and the matrix  $A := (a_{ij})_{p \times n}$  such that  $\text{rank} A = n$ , the LQ regularization algorithm is described in our paper (see [2]) as the following.

**Algorithm 2.1. Step 0.** Choose  $x^0 \in C, k := 0$ , a positive sequence  $\{c_k\}$  such that  $c_k \rightarrow c > 0$  as  $k \rightarrow +\infty$ .

**Step 1.** Solve the strongly convex program:

$$\min\{\langle F(x^k), y - x^k \rangle + \varphi(y) + \frac{1}{c_k}D(y, x^k) \mid y \in C\} \quad (2.1)$$

to obtain the unique solution  $y^k$ .

If  $y^k = x^k$ , then terminate:  $x^k$  is a solution to problems (VIP).

Otherwise go to Step 2.

**Step 2.** Find  $x^{k+1}$  which is the unique solution to the strongly convex program:

$$\min\{\langle F(y^k), y - y^k \rangle + \varphi(y) + \frac{1}{c_k}D(y, x^k) \mid y \in C\}.$$

**Step 3.** Set  $k := k + 1$ , and return to Step 1.

The following lemma establishes convergence of the algorithm.

**Lemma 2.2.** ([2], Theorem 2.7) Suppose that the function  $F$  is pseudomonotone and  $L$ -Lipschitz on  $C$ . Choose  $\epsilon, \mu$  and sequence  $\{c_k\}$  such that

$$0 < \epsilon, 0 < \mu < \min\left\{\frac{1 - \epsilon - c_k \|\bar{A}^{-1}\|^2}{3}, \frac{1 - \epsilon - c_k \bar{L}^2}{5}\right\} \quad \forall k = 1, 2, \dots,$$

where  $\bar{A} := (a_{ij})_{n \times n}$  is a submatrix of  $A$  such that  $\text{rank } \bar{A} = n$  and

$$\|\bar{A}^{-1}\| = \sup_{\|x\|=1} \|\bar{A}^{-1}x\|.$$

Then

(i) If Algorithm 2.1 terminates at Step 1, then  $x^k$  is a solution to (VIP).

(ii) If the algorithm does not terminate, then the sequence  $\{x^k\}$  converges to a solution to problems (VIP).

Note that auxiliary problems in Algorithm 2.1 can be solved efficiently by using available softwares.

When  $K$  is not polyhedral, we suggest approximating  $C$  by polyhedral convex sets. Polyhedral outer approximation methods of a convex set are based upon the fact that any nonempty closed convex set can be approximated by polyhedral convex sets. This technique has been widely used in convex programming and variational inequalities (see [16, 23]). In this section, we embed upper results in a polyhedral outer approximation procedure in order to solve problems (VIP). To this end, we suppose, as usual, that the closed convex set  $C$  is given as

$$C := \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j \in J\},$$

where  $J$  is a finite index set and the functions  $g_j$  ( $j \in J$ ) are convex and subdifferentiable on  $\mathbb{R}^n$ . By taking  $g(x) := \max_{j \in J} g_j(x)$ , we can write  $C = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ . Suppose now that Slater's condition is satisfied, i.e., that there exists  $v^0$  such that  $g(v^0) < 0$ .

For getting the convergence of the polyhedral approximation algorithm described below, we need the following result.

**Lemma 2.3.** ([23] Theorem 6.1, p.180) *Let  $C \subseteq \mathbb{R}^n$ ,  $\{x^j\} \subset \mathbb{R}^n \setminus C$  be a bounded sequence, and  $v^0 \in \text{int}C$ ,  $y^j \in [v^0, x^j] \setminus \text{int}C$ ,  $p^j \in \partial g(y^j)$  and  $0 \leq \alpha_j \leq g(y^j)$  such that  $\alpha_j - g(y^j) \rightarrow 0$  as  $j \rightarrow +\infty$ . If, for every  $j$ , the affine functions  $l_j(x) := \langle p^j, x - y^j \rangle + \alpha_j$  satisfy*

$$l_j(x^j) > 0, \quad l_j(x^{j+1}) \leq 0, \quad l_j(x) \leq 0 \quad \forall x \in C,$$

then every accumulation point of the sequence  $\{x^k\}$  belongs to  $C$ .

Now we are in a position to describe the polyhedral approximation algorithm.

**Algorithm 2.4.** *Initialization.* Choose a box  $T_0 := \{x \in \mathbb{R}^n \mid A_0 x \leq b_0\}$  containing bounded set  $C$ , where  $A_0$  is identity matrix, vector  $b_0 \in \mathbb{R}^n$ .

**Iteration (Outer Iteration)**  $k = 0, 1, \dots$

*Step 0.* Pick  $u^{k,0} = x^k \in T_k$  and  $j = 0$ .

*Step 1. (Inner iteration)*

*Solve the strongly convex quadratic program*

$$y^{k,j} = \operatorname{argmin}\{\langle F(u^{k,j}), y - u^{k,j} \rangle + \frac{1}{c_k} D(y, u^{k,j}) \mid y \in T_k\}.$$

*If  $y^{k,j} = u^{k,j}$ , then go to Step 2.*

*Otherwise, solve the strongly convex quadratic program*

$$y^{k,j+1} = \operatorname{argmin}\{\langle F(y^{k,j}), y - y^{k,j} \rangle + \frac{1}{c_k} D(y, u^{k,j}) \mid y \in T_k\},$$

*$j := j + 1$  and return to Step 1.*

*Step 2.*  $x^{k+1} := u^{k,j}$ .

*If  $x^{k+1} \in C$ , then stop.*

*If  $x^{k+1} \notin C$ , then construct a hyperplane  $l_{k+1}$  such that*

$$l_{k+1}(x) \leq 0 \quad \forall x \in C, \quad l_{k+1}(x^{k+1}) > 0.$$

*Set*

$$T_{k+1} := \{x \in T_k \mid l_{k+1}(x) \leq 0\}.$$

*Increase  $k$  by 1 and go to Iteration  $k$ .*

Convergence of Algorithm 2.4 is ensured by the following theorem.

**Theorem 2.5.** *Suppose that the function  $F$  is pseudomonone and  $L$ -Lipschitz on  $C$ . We choose  $\epsilon, \mu$  and the sequence  $c_k > 0$  such that*

$$0 < \epsilon, 0 < \mu < \min\left\{\frac{1 - \epsilon - c_k \|\bar{A}_k^{-1}\|^2}{3}, \frac{1 - \epsilon - c_k \bar{L}_k^2}{5}\right\} \quad \forall k = 0, 1, \dots,$$

where polyhedral  $T_k := \{x \in \mathbb{R}^n \mid A_k x \leq b_k\}$ ,  $\|\bar{A}_k^{-1}\| = \sup_{\|x\|=1} \|\bar{A}_k^{-1}x\|$  and

$\bar{L}_k = L\|\bar{A}_k^{-1}\|$ . Then,

- (i) If  $x^k \in C$ , then  $x^k$  is a solution to problems (VIP).
- (ii) If the algorithm does not terminate, then the sequence  $\{x^k\}$  converges to a solution to problems (VIP).

*Proof.* (i) We suppose  $x^k \in C$ , Lemma 2.3 shows that  $x^k$  is a solution to the following variational inequality problems, denoted by (VIP<sub>k</sub>):

Find a vector  $\bar{x} \in T_k$  such that

$$\langle F(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \forall x \in T_k.$$

From  $C \subseteq T_k$  and  $x^k \in C$  is a solution to (VIP<sub>k</sub>), it follows that  $x^k$  is a solution to problems (VIP).

- (ii) Since  $x^k \in T_k$  for all  $k = 0, 1, \dots$  and the sequence  $\{T_k\}$  satisfies all of assumptions of Lemma 2.3, outer proximal method shows that the sequence  $\{x^k\}$  must converge to  $x^* \in C$  and

$$T_0 \supseteq T_1 \supseteq \dots \supseteq T_k \supseteq \dots \supseteq C,$$

where  $x^*$  belongs to  $T_k$  for all  $k = 0, 1, \dots$ . Then  $x^*$  is a solution to problems (VIP).

The proof is complete. □

### 3 The interior-outer proximal linesearch method

Convergence of Algorithm 2.4 requires that the function  $F$  satisfies the Lipschitz condition on  $C$ . This condition depends on positive constant  $L$  and in cases, it is unknown or difficult to approximate. So in this section, in order to avoid this assumption, we combine the interior proximal method, the outer proximal method and the linesearch technique. The interior-linesearch technique has been used widely in descent method for solving variational inequalities (VIP) on  $C := \mathbb{R}_+^n$  (see [13, 18]).

In case  $C$  is a polyhedral, we construct iteratively a sequence converging to a solution to (VIP) without assuming Lipschitz continuity of  $F$ . The sequence

$\{y^k\}$  is described in Algorithm 2.1, the iterate  $x^k$  is defined as the following Amijo-type linesearch technique:

Find  $\lambda_k \in (0, 1)$  as the smallest number such that

$$\langle F((1 - \lambda_k)x^k + \lambda_k y^k), y^k - x^k \rangle + \frac{1}{2c_k} D(y^k, x^k) \leq 0.$$

Set

$$\begin{aligned} z^k &:= (1 - \lambda_k)x^k + \lambda_k y^k, \\ \delta_k &:= \gamma_k \frac{\lambda_k \langle F(z^k), z^k - y^k \rangle}{(1 - \lambda_k) \|F(z^k)\|^2}, \\ x^{k+1} &:= P_C(x^k - \delta_k F(z^k)). \end{aligned}$$

The convergent of the sequence  $\{x^k\}$  is defined as the following.

**Lemma 3.1.** ([2], Theorem 3.5) *Suppose that the sequences  $\gamma_k \in (0, 2)$  such that  $\liminf \gamma_k(2 - \gamma_k) > 0$ ,  $c_k \rightarrow \bar{c}$  as  $k \rightarrow \infty$ , and function  $F$  is pseudomonotone on  $C$ . Then,*

- (i) *If  $x^k = y^k$ , then  $x^k$  is a solution to problems (VIP).*
- (ii) *if  $x^k \neq y^k$  for all  $k = 0, 1, \dots$ , then the iterate  $x^k$  converges to  $x^*$  which is a solution to problems (VIP).*

Now we are in a position to consider  $C$  which is a convex subset of  $\mathbb{R}^n$ . We embed Amijo-type linesearch technique in a polyhedral outer approximation procedure in order to solve problems (VIP). From  $C$  is bounded, there exists a box contained  $C$ , denoted by  $T_0$ . The method can now be described in detail as follows:

**Algorithm 3.2.** *Initialization.* Choose a box  $T_0 := \{x \in \mathbb{R}^n \mid A_0 x \leq b_0\}$  containing bounded set  $C$ , where  $A_0$  is identity matrix, vector  $b_0 \in \mathbb{R}^n$ .

**Iteration** (Outer Iteration)  $k = 0, 1, \dots$

**Step 0.** Pick  $u^{k,0} = x^k \in T_k$  and  $j = 0$ .

**Step 1.** (Inner iteration)

Solve the strongly convex quadratic program

$$y^{k,j} = \operatorname{argmin}\{ \langle F(u^{k,j}), y - u^{k,j} \rangle + \frac{1}{c_{k,j}} D(y, u^{k,j}) \mid y \in T_k \}.$$

If  $y^{k,j} = u^{k,j}$ , then go to Step 2.

Otherwise, Find  $\lambda_{k,j} \in (0, 1)$  as the smallest number such that

$$\langle F((1 - \lambda_{k,j})u^{k,j} + \lambda_{k,j}y^{k,j}), y^{k,j} - u^{k,j} \rangle + \frac{1}{2c_{k,j}} D(y^{k,j}, u^{k,j}) \leq 0. \quad (3.1)$$

Set

$$\begin{aligned} z^{k,j} &:= (1 - \lambda_{k,j})u^{k,j} + \lambda_{k,j}y^{k,j}, \\ \delta_{k,j} &:= \gamma_{k,j} \frac{\lambda_{k,j} \langle F(z^{k,j}), z^{k,j} - y^{k,j} \rangle}{(1 - \lambda_{k,j}) \|F(z^{k,j})\|^2}, \\ x^{k+1,j} &:= P_C(x^{k,j} - \delta_{k,j} F(z^{k,j})). \end{aligned}$$

$j := j + 1$  and return Step 1.

**Step 2.**  $x^{k+1} := u^{k,j}$ .

If  $x^{k+1} \in C$ , then stop.

If  $x^{k+1} \notin C$ , then construct a hyperplane  $l_{k+1}$  such that

$$l_{k+1}(x) \leq 0 \quad \forall x \in C, l_{k+1}(x^{k+1}) > 0.$$

Set

$$T_{k+1} := \{x \in T_k \mid l_{k+1}(x) \leq 0\}.$$

Increase  $k$  by 1 and go to Iteration  $k$ .

Recall that  $P_C(x)$  denotes the projection of  $x$  on  $C$ .

**Remark 3.3.** The smallest number  $\lambda_k \in (0, 1)$  of Algorithm 3.2 can be replaced by the following: With  $\beta \in (0, 1)$ , we find  $n$  as the smallest natural number such that

$$\langle F(\beta^n u^{k,j} + (1 - \beta^n) y^{k,j}), y^{k,j} - u^{k,j} \rangle + \frac{1}{2c_{k,j}} D(y^{k,j}, u^{k,j}) \leq 0.$$

then set  $\lambda_{k,j} := 1 - \beta^n$ .

Convergence of the sequence  $\{x^k\}$  defined by Algorithm 3.2 is ensured by the following theorem.

**Theorem 3.4.** Suppose that the sequences  $\gamma_k \in (0, 2)$ ,  $c_{k,j} \rightarrow \bar{c} > 0$  as  $k, j \rightarrow \infty$ , and function  $F$  satisfies the following conditions:

(a)  $\liminf \gamma_k(2 - \gamma_k) > 0$ .

(b)  $f$  is pseudomonotone on  $C$ .

Then, if  $x^k \in C$  then  $x^k$  is a solution to problems (VIP) and if Algorithm 3.2 doesn't terminate then the sequence  $\{x^k\}$  converges to  $x^*$  which is a solution to problems (VIP).

*Proof.* In [1] we have showed that there always exists  $\lambda_{k,j} \in (0, 1)$  as the smallest number which satisfies (3.1).

We suppose  $x^{k+1} \in C$ , Lemma 3.1 shows that  $x^{k+1}$  is a solution to problems (VIP<sub>k</sub>). It means that

$$\langle F(x^{k+1}), x - x^{k+1} \rangle \geq 0 \quad \forall x \in T_k. \quad (3.2)$$

Otherwise outer approximation method shows that  $C \subseteq T_k$ . Combine this and 3.2, we have

$$\langle F(x^{k+1}), x - x^{k+1} \rangle \geq 0 \quad \forall x \in C.$$

It shows that  $x^{k+1}$  is a solution to problems (VIP).

Now we are in a position to prove that the sequence  $\{x^k\}$  converges to  $x^*$  which is a solution to problems (VIP), if Algorithm 3.2 doesn't terminate. Using Lemma 3.1, we have  $x^{k+1}$  is a solution to (VIP<sub>k</sub>). Algorithm 3.2 also shows that the sequence  $\{x^k\}$  must converge to  $x^* \in C$  and

$$T_0 \supseteq T_1 \supseteq \cdots \supseteq T_k \supseteq \cdots \supseteq C.$$

By the continuity of  $F$ , we have  $x^*$  as a solution to problems (VIP). The proof is complete.  $\square$

## References

- [1] Anh P. N., *An LQ regularization method for solving pseudomonotone equilibrium problems on polyhedral*, Vietnam Journal of Mathematica, 36 (2008), 209-228.
- [2] Anh P. N., *An interior proximal method for solving monotone generalized variational inequalities*, East-West J. of Mathematics, 10 (2008), pp. 81-100.
- [3] Anh P. N., *An interior proximal method for solving pseudomonotone non-lipschitzian multivalued variational inequalities*, Nonlinear Analysis Forum, 14 (2009), pp. 27-42.
- [4] Anh P. N., and Muu L. D., *Coupling the Banach contraction mapping principle and the proximal point algorithm for solving monotone variational inequalities*, Acta Mathematica Vietnamica, 29 (2004) 119-133.
- [5] Anh P. N., Muu L. D., and Strodiot J. J., *Generalized Projection Method for Non-Lipschitz Multivalued Monotone Variational Inequalities*, Acta Mathematica Vietnamica, (2009), 67-79.
- [6] Anh P. N., Muu L.D., Nguyen V. H., and Strodiot J. J., *On the Contraction and Nonexpensiveness Properties of the Marginal Mappings in Generalized Variational Inequalities Involving Co-coercive Operators*, In Generalized Convexity and Monotonicity, Edited by A. Eberhard, N. Hadjisavvas and D.T. Luc Springer , Chap. 5, (2005) 89-111.
- [7] Aubin J.P., and Ekeland I., "Applied Nonlinear Analysis", Wiley, New York (1984).
- [8] Auslender A., and Teboulle M., *Interior projection-like methods for monotone variational inequalities*, Mathematical Programming, 104 (2005) 39-68.

- [9] Auslender A., Teboulle M., and Bentiba S., *A logarithmic-quadratic proximal method for variational inequalities*, J. of Computational Optimization and Applications, 12 (1999) 31-40.
- [10] Auslender A., Teboulle M., and Bentiba S., *Iterior proximal and multiplier methods based on second order homogeneous kernels*, Mathematical Operation Research, 24 (1999) 646-668.
- [11] Blum E., and Oettli W., *From optimization and variational inequality to equilibrium problems*, The Mathematics Student, 63 (1994) 127-149.
- [12] Cohen G., *Auxiliary problem principle and decomposition of optimization problems*, J. of Optimization Theory and Applications, 32 (1980) 277-305.
- [13] Facchinei F., and Pang J.S., *Finite-Dimensional Variational Inequalities and Complementary Problems*, Springer-Verlag, NewYork (2003).
- [14] Fang S.C., Petersen P.L., *Generalized Variational Inequalities*, J. of Optimization Theory and Applications, 38 (1982) 363-383.
- [15] Farouq N. El., *Pseudomonotone variational inequalities: convergence of auxiliary problem method*, J. of Optimization Theory and Applications, 111 (2001) 305- 325.
- [16] Fukushima M., *Equivalent Differentiable Optimization Problems and Descent Methods for Asymmetric Variational Inequality Problems*, Mathematical Programming, 53 (1992) 99-110.
- [17] Han D., *A modified alternating direction method for variational inequality problems*, Applied Mathematics Optimization, 45 (2002) 63-74.
- [18] Konnov I. V., *Combined Relaxation Methods for Variational Inequalities*, Springer-Verlag, Berlin (2000).
- [19] Noor M. A., *Iterative schemes for quasimonotone mixed variational inequalities*, SIAM J. Optimization, 50 (2001) 29-44.
- [20] Rockafellar R. T., "Convex Analysis", Princeton University Press (1970).
- [21] Rockafellar R. T., *Monotone operators and the proximal point algorithm*, SIAM J. Control Optimization, 14 (1976) 877-898.
- [22] Taji K., Fukushima M., and Ibaraki T., *A globally convergent Newton method for solving strongly monotone variational inequalities*, Mathematical Programming, 58 (1993) 369-383.
- [23] Tuy H., "Convex Analysis and Global Optimization", Kluwer Academic Publishers, Dordrecht, Holland, 1998.
- [24] Zhu D. L., and Marcotte. P., *Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities*, SIAM J. Optimization, 6 (1996) 714-726.