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COUPLING THE LQ REGULARIZATION METHOD AND THE OUTER APPROXIMATION METHOD FOR SOLVING PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract

In our recent papers [2, 3] we have proposed a modified logarithmic-quadratic method for solving monotone generalized variational inequalities and quasimonotone multivalued variational inequalities on polyhedral. The method is based on the special logarithmic quadratic function which replaces the usual quadratic. In this paper we combine this result with the outer approximation method to obtain a new interior approximation algorithm for solving pseudomonone variational inequalities satisfying a certain Lipschitz condition on a closed convex set. Next, to avoid the Lipschitz condition we combine this technique with linesearch technique to obtain a convergent algorithm for pseudomonone variational inequalities.

Keywords: Variational inequalities, pseudomonotone, interior-quadratic function, linesearch algorithm, outer approximation method.
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1 Introduction

Classical variational inequalities, denoted by (VIP), are to find a vector $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C,$$

where C is a nonempty closed convex subset of $\mathbb{R}^n$ and F is a continuous mapping from $\mathbb{R}^n$ into itself. Problems (VIP) include nonlinear complementarity problems (shortly (NCP), when $C = \mathbb{R}^n_+$) and system of nonlinear equations (when $C = \mathbb{R}^n$). Throughout this paper we assume that C is bounded, that $F(x)$ is continuous, pseudomonone on C and the solution set of (VIP), denoted by S^*, is nonempty. Variational inequalities have many important applications in economics, operation researches and nonlinear analysis, and have been studied by many researchers (see [5, 11, 12, 13, 14, 15, 18, 24]).

First, let us recall the well known concepts that will be used in the sequel (see [19, 20]).

Definition 1.1. *Let $C \subseteq \mathbb{R}^n$ and $F : C \to \mathbb{R}^n$. The function F is said to be (a) pseudomonotone on C if for each $x, y \in C$ the inequality*

$$\langle F(y), x - y \rangle \geq 0$$

implies

$$\langle F(x), x - y \rangle \geq 0.$$

(b) Lipschitz on C with constant L (shortly $L-$Lipschitz) if for each $x, y \in C$

$$||F(x) - F(y)|| \leq L||x - y||.$$

Among powerful approaches to (NCP) is the logarithmic-quadratic proximal method (shortly (LQ)) presented originally by Auslender et al. in Ref. [7, 8, 9] under that the operator is monotone on $C := \mathbb{R}^n_+$, which is starting with any point $x^0 \in C$ and $\lambda_k \geq \lambda > 0$, iteratively updates x^{k+1} conforming the following problem:

$$0 \in \lambda_k F(x) + \nabla_x d_\phi(x, x^k),$$

where

$$d_\phi(x, y) = \sum_{i=1}^n y_i^2 \phi(y_i^{-1} x_i)$$

and

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu(t - \log t - 1) & \text{if } t > 0 \\ +\infty & \text{otherwise,} \end{cases} \tag{1.1}$$

with $\nu > \mu > 0$. Then the term d_ϕ forces the iteratives $\{x^{k+1}\}$ to stay in the interior of C.

In our recently papers [1, 2, 3, 4], we have proposed a new type of proximal interior method for solving (VIP) and equilibrium problems (shortly (EQ)) on polyhedral $C := \{x \in \mathbb{R}^n |\ Ax \leq b\}$, where A is an $p \times n$ matrix, $b \in \mathbb{R}^p$ through replacing function $d_\phi(x, y)$ by $D(x, y)$ which is defined as $D(x, y) = d(l(x), l(y))$, where

$$l_i(x) = b_i - \langle a_i, x\rangle \quad i = 1, 2, \cdots, p,$$
$$l(x) = \big(l_1(x), l_2(x), \cdots, l_p(x)\big)^T,$$

and

$$d(x,y) = \begin{cases} \frac{1}{2}||x-y||^2 + \mu \sum\limits_{i=1}^{n} y_i^2 (\frac{x_i}{y_i} \log \frac{x_i}{y_i} - \frac{x_i}{y_i} + 1) & \text{if } x > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.2}$$

with $\mu \in (0, 1)$.

Applying to (VIP), in this paper we consider function $D(x, y)$ for every $x, y \in T_k \ \ \forall k = 0, 1, \cdots$, where T_k is a polyhedral sequence defined by

$$T_k = \{x \in \mathbb{R}^n |\ b_i - a_i x \geq 0 \ i = 1, 2, \cdots, n + k\},$$

a_i $(i = 1, 2, \cdots, n + k)$ are the rows of matrix $A_k, l_i(x) = b_i - \langle a_i, x\rangle \quad i = 1, 2, , \cdots, n+k$ and $l(x) = (l_1(x), l_2(x), \cdots, l_{n+k}(x))^T$. For given $x^{k,j} \in T_k$ $(j = 0, 1, \cdots)$, we denote by $\nabla_x D(x, x^{k,j})$ the gradient of $D(., x^{k,j})$ at x. Then we have

$$\nabla_x D(x, x^{k,j}) = -A_k^T \big(l(x) - l(x^{k,j}) + \mu X_{k,j} \log \frac{l(x)}{l(x^{k,j})}\big),$$

where

$$X_{k,j} = diag\big(l_1(x^{k,j}), \cdots, l_{n+k}(x^{k,j})\big),$$
$$\log \frac{l(x)}{l(x^{k,j})} = \Big(\log \frac{l_1(x)}{l_1(x^{k,j})}, \cdots, \log \frac{l_{n+k}(x)}{l_{n+k}(x^{k,j})}\Big).$$

2 An interior proximal algorithm

To solve the generalized variational inequalities:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^*\rangle + \varphi(x) - \varphi(x^*) \geq 0 \ \ \forall x \in C,$$

where $C \subset \mathbb{R}^n$ is a polyhedral defined $C := \{x \in \mathbb{R}^n |\ Ax \leq b\}, \varphi : C \to \mathbb{R}$ is convex, $F : C \to \mathbb{R}^n$ and the matrix $A := (a_{ij})_{p\times n}$ such that $\text{rank}A = n$, the LQ regularization algorithm is described in our paper (see [2]) as the following.

Algorithm 2.1. Step 0. *Choose* $x^0 \in C, k := 0,$ *a positive sequence* $\{c_k\}$ *such that* $c_k \to c > 0$ *as* $k \to +\infty$.
Step 1. *Solve the strongly convex program:*

$$\min\{\langle F(x^k), y - x^k\rangle + \varphi(y) + \frac{1}{c_k} D(y, x^k)|\ y \in C\} \tag{2.1}$$

to obtain the unique solution y^k.
If $y^k = x^k$, *then terminate:* x^k *is a solution to problems* (VIP).
Otherwise go to Step 2.
Step 2. *Find* x^{k+1} *which is the unique solution to the strongly convex program:*

$$\min\{\langle F(y^k), y - y^k\rangle + \varphi(y) + \frac{1}{c_k} D(y, x^k)|\ y \in C\}.$$

Step 3. *Set* $k := k + 1$, *and return to Step 1.*

The following lemma establishes convergence of the algorithm.

Lemma 2.2. ([2], Theorem 2.7) *Suppose that the function* F *is pseudomonone and* $L-$*Lipschitz on* C. *Choose* ϵ, μ *and sequence* $\{c_k\}$ *such that*

$$0 < \epsilon, 0 < \mu < \min\{\frac{1 - \epsilon - c_k||\bar{A}^{-1}||^2}{3}, \frac{1 - \epsilon - c_k\bar{L}^2}{5}\} \quad \forall k = 1, 2, \cdots,$$

where $\bar{A} := (a_{ij})_{n\times n}$ *is a submatrix of* A *such that rank* $\bar{A}$=n *and*

$$||\bar{A}^{-1}|| = \sup_{||x||=1} ||\bar{A}^{-1}x||.$$

Then
(i) If Algorithm 2.1 terminates at Step 1, then x^k *is a solution to* (VIP).
(ii) If the algorithm does not terminate, then the sequence $\{x^k\}$ *converges to a solution to problems* (VIP).

Note that auxiliary problems in Algorithm 2.1 can be solved efficiently by using available softwares.

When K is not polyhedral, we suggest approximating C by polyhedral convex sets. Polyhedral outer approximation methods of a convex set are based upon the fact that any nonempty closed convex set can be approximated by polyhedral convex sets. This technique has been widely used in convex programming and variational inequalities (see [16, 23]). In this section, we embed upper results in a polyhedral outer approximation procedure in order to solve problems (VIP). To this end, we suppose, as usual, that the closed convex set C is given as

$$C := \{x \in \mathbb{R}^n|\ g_j(x) \le 0, j \in J\},$$

where J is a finite index set and the functions g_j $(j \in J)$ are convex and subdifferentiable on $\mathbb{R}^n$. By taking $g(x) := \max_{j \in J} g_j(x)$, we can write $C = \{x \in \mathbb{R}^n |\ g(x) \le 0\}$. Suppose now that Slater's condition is satisfied, i.e., that there exists v^0 such that $g(v^0) < 0$.

For getting the convergence of the polyhedral approximation algorithm described below, we need the following result.

Lemma 2.3. *([23] Theorem 6.1, p.180) Let $C \subseteq \mathbb{R}^n$, $\{x^j\} \subset \mathbb{R}^n \setminus C$ be a bounded sequence, and $v^0 \in int C$, $y^j \in [v^0, x^j] \setminus int C$, $p^j \in \partial g(y^j)$ and $0 \le \alpha_j \le g(y^j)$ such that $\alpha_j - g(y^j) \to 0$ as $j \to +\infty$. If, for every j, the affine functions $l_j(x) := \langle p^j, x - y^j \rangle + \alpha_j$ satisfy*

$$l_j(x^j) > 0, \quad l_j(x^{j+1}) \le 0, \quad l_j(x) \le 0 \quad \forall x \in C,$$

then every accumulation point of the sequence $\{x^k\}$ belongs to C.

Now we are in a position to describe the polyhedral approximation algorithm.

Algorithm 2.4. *Initialization. Choose a box $T_0 := \{x \in \mathbb{R}^n |\ A_0 x \le b_0\}$ containing bounded set C, where A_0 is indentity matrix, vector $b_0 \in \mathbb{R}^n$.*
Iteration *(Outer Iteration) $k = 0, 1, \cdots$*
Step 0. Pick $u^{k,0} = x^k \in T_k$ and $j = 0$.
Step 1. (Inner iteration)
Solve the strongly convex quadratic program

$$y^{k,j} = argmin\{\langle F(u^{k,j}), y - u^{k,j} \rangle + \frac{1}{c_k} D(y, u^{k,j})|\ y \in T_k\}.$$

If $y^{k,j} = u^{k,j}$, then go to Step 2.
Otherwise, solve the strongly convex quadratic program

$$y^{k,j+1} = argmin\{\langle F(y^{k,j}), y - y^{k,j} \rangle + \frac{1}{c_k} D(y, u^{k,j})|\ y \in T_k\},$$

$j := j + 1$ and return to Step 1.
Step 2. $x^{k+1} := u^{k,j}$.
If $x^{k+1} \in C$, then stop.
If $x^{k+1} \notin C$, then construct a hyperplane l_{k+1} such that

$$l_{k+1}(x) \le 0 \ \ \forall x \in C, l_{k+1}(x^{k+1}) > 0.$$

Set

$$T_{k+1} := \{x \in T_k |\ l_{k+1}(x) \le 0\}.$$

Increase k by 1 and go to Iteration k.

Convergence of Algorithm 2.4 is ensured by the following theorem.

Theorem 2.5. *Suppose that the function F is pseudomonone and $L-$Lipschitz on C. We choose ϵ, μ and the sequence $c_k > 0$ such that*

$$0 < \epsilon, 0 < \mu < \min\{\frac{1-\epsilon-c_k||\bar{A}_k^{-1}||^2}{3}, \frac{1-\epsilon-c_k\bar{L}_k^2}{5}\} \quad \forall k = 0, 1, \cdots,$$

where polyhedral $T_k := \{x \in \mathbb{R}^n | \ A_k x \leq b_k\}$, $||\bar{A}_k^{-1}|| = \sup_{||x||=1} ||\bar{A}_k^{-1}x||$ and $\bar{L}_k = L||\bar{A}_k{}^{-1}||$. Then,
(i) If $x^k \in C$, then x^k is a solution to problems (VIP).
(ii) If the algorithm does not terminate, then the sequence $\{x^k\}$ converges to a solution to problems (VIP).

Proof. (i) We suppose $x^k \in C$, Lemma 2.3 shows that x^k is a solution to the following variational inequality problems, denoted by (VIP_k):

Find a vector $\bar{x} \in T_k$ such that

$$\langle F(\bar{x}), x - \bar{x}\rangle \geq 0 \quad \forall x \in T_k.$$

From $C \subseteq T_k$ and $x^k \in C$ is a solution to (VIP_k), it follows that x^k is a solution to problems(VIP).
(ii) Since $x^k \in T_k$ for all $k = 0, 1, \cdots$ and the sequence $\{T_k\}$ satisfies all of assumptions of Lemma 2.3, outer proximal method shows that the sequence $\{x^k\}$ must converge to $x^* \in C$ and

$$T_0 \supseteq T_1 \supseteq \cdots \supseteq T_k \supseteq \cdots \supseteq C,$$

where x^* belongs to T_k for all $k = 0, 1, \cdots$. Then x^* is a solution to problems (VIP).
The proof is complete. □

3 The interior-outer proximal linesearch method

Convergence of Algorithm 2.4 requires that the function F satisfies the Lipschitz condition on C. This condition depends on positive constant L and in cases, it is unknown or difficult to approximate. So in this section, in order to avoid this assumption, we combine the interior proximal method, the outer proximal method and the linesearch technique. The interior-linesearch technique has been used widely in descent method for solving variational inequalities (VIP) on $C := \mathbb{R}^n_+$ (see [13, 18]).

In case C is a polyhedral, we construct iteratively a sequence converging to a solution to (VIP) without assumping Lipschitz continuity of F. The sequence

$\{y^k\}$ is described in Algorithm 2.1, the iterate x^k is defined as the following Amijo-type linesearch technique:

Find $\lambda_k \in (0,1)$ as the smallest number such that

$$\langle F((1-\lambda_k)x^k + \lambda_k y^k), y^k - x^k\rangle + \frac{1}{2c_k}D(y^k, x^k) \leq 0.$$

Set

$$\begin{aligned} z^k &:= (1-\lambda_k)x^k + \lambda_k y^k, \\ \delta_k &:= \gamma_k \frac{\lambda_k \langle F(z^k), z^k - y^k\rangle}{(1-\lambda_k)||F(z^k)||^2}, \\ x^{k+1} &:= P_C(x^k - \delta_k F(z^k)). \end{aligned}$$

The convergent of the sequence $\{x^k\}$ is defined as the following.

Lemma 3.1. *([2], Theorem 3.5) Suppose that the sequences $\gamma_k \in (0,2)$ such that* $\liminf \gamma_k(2-\gamma_k) > 0$, *$c_k \to \bar{c}$ as $k \to \infty$, and function F is pseudomonotone on C. Then,*
(i) If $x^k = y^k$, then x^k is a solution to problems (VIP).
(ii) if $x^k \neq y^k$ for all $k = 0, 1, \cdots$, then the iterate x^k converges to x^* which is a solution to problems (VIP).

Now we are in a position to consider C which is a convex subset of $\mathbb{R}^n$. We embed Amijo-type linesearch technique in a polyhedral outer approximation procedure in order to solve problems (VIP). From C is bounded, there exists a box contained C, denoted by T_0. The method can now be described in detail as follows:

Algorithm 3.2. *Initialization. Choose a box $T_0 := \{x \in \mathbb{R}^n |\ A_0 x \leq b_0\}$ containing bounded set C, where A_0 is indentity matrix, vector $b_0 \in \mathbb{R}^n$.*
Iteration *(Outer Iteration) $k = 0, 1, \cdots$*
Step 0. *Pick $u^{k,0} = x^k \in T_k$ and $j = 0$.*
Step 1. *(Inner iteration)*
Solve the strongly convex quadratic program

$$y^{k,j} = argmin\{\langle F(u^{k,j}), y - u^{k,j}\rangle + \frac{1}{c_{k,j}}D(y, u^{k,j})|\ y \in T_k\}.$$

If $y^{k,j} = u^{k,j}$, then go to Step 2.
Otherwise, Find $\lambda_{k,j} \in (0,1)$ as the smallest number such that

$$\langle F((1-\lambda_{k,j})u^{k,j} + \lambda_{k,j}y^{k,j}), y^{k,j} - u^{k,j}\rangle + \frac{1}{2c_{k,j}}D(y^{k,j}, u^{k,j}) \leq 0. \quad (3.1)$$

Set

$$
\begin{aligned}
z^{k,j} &:= (1-\lambda_{k,j})u^{k,j} + \lambda_{k,j}y^{k,j},\\
\delta_{k,j} &:= \gamma_{k,j}\frac{\lambda_{k,j}\langle F(z^{k,j}), z^{k,j}-y^{k,j}\rangle}{(1-\lambda_{k,j})||F(z^{k,j})||^2},\\
x^{k+1,j} &:= P_C(x^{k,j} - \delta_{k,j}F(z^{k,j})).
\end{aligned}
$$

$j := j+1$ *and return Step 1.*
Step 2. $x^{k+1} := u^{k,j}$.
If $x^{k+1} \in C$*, then stop.*
If $x^{k+1} \notin C$*, then construct a hyperplane* l_{k+1} *such that*

$$l_{k+1}(x) \le 0 \quad \forall x \in C, l_{k+1}(x^{k+1}) > 0.$$

Set

$$T_{k+1} := \{x \in T_k |\ l_{k+1}(x) \le 0\}.$$

Increase k *by* 1 *and go to Iteration* k*.*

Recall that $P_C(x)$ denotes the projection of x on C.

Remark 3.3. *The smallest number* $\lambda_k \in (0,1)$ *of Aglgorithm 3.2 can be replaced by the following: With* $\beta \in (0,1)$*, we find* n *as the smallest natural number such that*

$$\langle F(\beta^n u^{k,j} + (1-\beta^n)y^{k,j}), y^{k,j} - u^{k,j}\rangle + \frac{1}{2c_{k,j}}D(y^{k,j}, u^{k,j}) \le 0.$$

then set $\lambda_{k,j} := 1-\beta^n$*.*

Convergence of the sequence $\{x^k\}$ defined by Algorithm 3.2 is ensured by the following theorem.

Theorem 3.4. *Suppose that the sequences* $\gamma_k \in (0,2), c_{k,j} \to \bar{c} > 0$ *as* $k, j \to \infty$*, and function* F *satisfies the following conditions:*
(a) $\liminf \gamma_k(2-\gamma_k) > 0$*.*
(b) f *is pseudomonotone on* C*.*
Then, if $x^k \in C$ *then* x^k *is a solution to problems* (VIP) *and if Algorithm 3.2 doesn't terminate then the sequence* $\{x^k\}$ *converges to* x^* *which is a solution to problems* (VIP)*.*

Proof. In [1] we have showed that there always exists $\lambda_{k,j} \in (0,1)$ as the smallest number which satisfies (3.1).

We suppose $x^{k+1} \in C$, Lemma 3.1 shows that x^{k+1} is a solution to problems (VIP_k). It means that

$$\langle F(x^{k+1}), x - x^{k+1}\rangle \ge 0 \quad \forall x \in T_k. \tag{3.2}$$

Otherwise outer approximation method shows that $C \subseteq T_k$. Combine this and 3.2, we have

$$\langle F(x^{k+1}), x - x^{k+1} \rangle \geq 0 \quad \forall x \in C.$$

It shows that x^{k+1} is a solution to problems (VIP).

Now we are in a position to prove that the sequence $\{x^k\}$ converges to x^* which is a solution to problems (VIP), if Algorithm 3.2 doesn't terminate. Using Lemma 3.1, we have x^{k+1} is a solution to (VIP_k). Algorithm 3.2 also shows that the sequence $\{x^k\}$ must converge to $x^* \in C$ and

$$T_0 \supseteq T_1 \supseteq \cdots \supseteq T_k \supseteq \cdots \supseteq C.$$

By the continuity of F, we have x^* as a solution to problems (VIP). The proof is complete. $\square$

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