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ON THE CONVOLUTION WITH WEIGHT-FUNCTION FOR THE FOURIER SIN INTEGRAL TRANSFORM

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Abstract

The convolution with a weight-function for the Fourier sine integral transform is formulated and its properties are studied. A Titchmarch type theorem, non-existence of the unit element of the convolution are proved. The application to solve some particular cases of the Toeplitz plus Hankel integral equations is outlined.

1 Introduction

The convolutions for integral transforms were studied at the beginning of 20 th century, at first the convolution for the Fourier transform (see, e.g. [17]), for the Fourier cosine transform, for the Laplace transform (see [16, 17]) and the references therein for the Mellin transform [15] and after that the convolution for the Hilbert transform [15, 17], the Hankel transform [8], the Kontorovich - Lebedev transform [8], the Stieltjes transform [15] after-wards.

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In 1967, V. A. Kakichev [8] proposed a constructive method for defining the convolution with a weight-function which is more general than the convolution. And as by-products, convolutions of many integral transforms such as the Meijer, Hankel, Fourier sine, Sommerfeld were found [9]. For instance, the convolution with the weight-function $\gamma(y) = \sin y$ of the functions f and g for the Fourier sine integral transform (F_s) was studied in [8], [11]

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty}\int_0^{+\infty} \big[e^{-|x+u-v|}\,\mathrm{sign}\,(x+u-v) + e^{-|x-u+v|}\,\mathrm{sign}\,(x-u+v) - e^{-(x+u+v)} - e^{-|x-u-v|}\,\mathrm{sign}\,(x-u-v)\big] f(u)g(v)dudv, \quad x>0.$$

The convolutions for many integral transforms have numerous applications in several contexts of sciences and mathematics [3, 4, 6, 7, 16, 17, 20]. Specially, studying convolutions may shed light on how to solve the integral equation with the Toeplitz plus Hankel kernel [18]

$$f(x) + \int_0^{+\infty} [k_1(x+u) + k_2(x-u)] f(u)du = g(x), \tag{1.1}$$

in closed form. The general case of this integral equation is still open.

In this paper we introduce another convolution with the weight - function $\gamma(y) = \dfrac{y}{1+y^2}$ for the Fourier sine transform. We obtain some properties for the new convolution. Also we will show that there does not exist the unit element for the calculus of this convolution as well as there is not aliquote of zero. In applications, we apply this notion to solve some special cases of the Toeplitz plus Hankel integral equations.

2 The convolution with a weight-function for the Fourier sine integral transform

Definition 1. The convolution with the weight-function $\gamma(y) = \dfrac{y}{1+y^2}$ of two functions f, g for the Fourier sine integral transform is defined by

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty}\int_0^{+\infty} \big[e^{-|x+u-v|}\mathrm{sign}\,(x+u-v) + e^{-|x-u+v|}\mathrm{sign}\,(x-u+v) - e^{-(x+u+v)} - e^{-|x-u-v|}\,\mathrm{sign}\,(x-u-v)\big] f(u)g(v)dudv, \quad x>0. \tag{2.1}$$

Theorem 1. *Let $f, g \in L(\mathbb{R}_+)$, then their convolution with the weight-function $\gamma(y) = \dfrac{y}{1+y^2}$ for the Fourier sine integral transform* (2.1) *belongs to $L(\mathbb{R}_+)$ and the following factorization property holds*

$$F_s(f \overset{\gamma}{*} g)(y) = \frac{y}{1+y^2}(F_s f)(y)(F_s g)(y), \quad \forall y > 0. \tag{2.2}$$

Here, the Fourier sine integral transform is defined by [16]

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} f(x)\sin(yx)dx, \quad y \in \mathbb{R}_+.$$

Proof We have

$$\left|e^{-|x+u-v|}\operatorname{sign}(x+u-v)\right| = \frac{1}{e^{|x+u-v|}} \leqslant 1, \quad \forall x, u, v \in \mathbb{R}_+.$$

Similarly, we get

$$\begin{aligned}
&\left|e^{-|x-u+v|}\operatorname{sign}(x-u+v)\right| \leqslant 1, \quad \forall x, u, v \in \mathbb{R}_+,\\
&\left|e^{-|x+u+v|}\right| \leqslant 1, \quad \forall x, u, v \in \mathbb{R}_+,\\
&\left|e^{-|x-u-v|}\operatorname{sign}(x-u-v)\right| \leqslant 1, \quad \forall x, u, v \in \mathbb{R}_+.
\end{aligned}$$

Hence, we have

$$\left|(f \overset{\gamma}{*} g)(x)\right| \leqslant \sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty}\int\limits_0^{+\infty} |f(u)|\,|g(v)|dudv < +\infty.$$

It shows that the convolution (2.2) is well defined in $L(\mathbb{R}_+)$. Moreover, from (2.1) we obtain

$$\begin{aligned}
\int\limits_0^{+\infty} \left|(f \overset{\gamma}{*} g)(x)\right|dx \leqslant \frac{1}{2\sqrt{2\pi}} \int\limits_0^{+\infty}\int\limits_0^{+\infty}\int\limits_0^{+\infty} &\big\{e^{-|x+u-v|} + e^{-|x-u+v|}\\
&+ e^{-(x+u+v)} + e^{-|x-u-v|}\big\}|f(u)|\,|g(v)|dudvdx.
\end{aligned}$$

On the other hand, with the substitution $x+u-v=t$, we have

$$\begin{aligned}
\int\limits_0^{+\infty} \frac{dx}{e^{|x+u-v|}} &= \int\limits_{u-v}^{+\infty} \frac{dt}{e^{|t|}} = \int\limits_{u-v}^{0} \frac{dt}{e^{-t}} + \int\limits_0^{+\infty} \frac{dt}{e^t}\\
&= e^t\Big|_{u-v}^{0} - e^{-t}\Big|_0^{+\infty}\\
&= 2 - e^{u-v} < 2, \quad \forall u, v > 0.
\end{aligned}$$

Similarly

$$\int_0^{+\infty} \frac{dx}{e^{|x-u+v|}} < 2, \quad \forall u, v > 0,$$

$$\int_0^{+\infty} \frac{dx}{e^{x+u+v}} < 2, \quad \forall u, v > 0,$$

$$\int_0^{+\infty} \frac{dx}{e^{|x-u-v|}} < 2, \quad \forall u, v > 0.$$

Hence

$$\frac{1}{2\sqrt{2\pi}} \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \{e^{-|x+u-v|} + e^{-|x-u+v|} + e^{-(x+u+v)} + e^{-|x-u-v|}\} |f(u)||g(v)| du dv dx$$
$$< 2\sqrt{\frac{2}{\pi}} \int_0^{+\infty}\int_0^{+\infty} |f(u)|\, |g(v)| du dv < +\infty.$$

So $(f*g)(x) \in L(\mathbb{R}_+)$. We now prove the factorization property (2.2). Applying formula (2.2.15, page 65) in [2] we obtain

$$\sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{y}{1+y^2} \sin(yx)\sin(yu)\sin(yv) dy$$
$$= \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} \frac{y}{1+y^2} \big[\sin y(x+u-v) + \sin y(x-u+v) - \sin y(x+u+v)$$
$$- \sin y(x-u-v) \big] dy$$
$$= \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} \frac{y}{1+y^2} \big[\operatorname{sign}(x+u-v) \sin y|x+u-v|$$
$$+ \operatorname{sign}(x-u+v) \sin y|x-u+v| - \sin y(x+u+v)$$
$$+ \operatorname{sign}(x-u-v) \sin y|x-u-v| \big] dy$$
$$= \frac{\sqrt{2\pi}}{8} \big[e^{-(x+u-v)} \operatorname{sign}(x+u-v) + e^{-(x-u+v)} \operatorname{sign}(x-u+v)$$
$$- e^{-(x+u+v)} - e^{-(x-u-v)} \operatorname{sign}(x-u-v) \big].$$

From that, we have

$$\frac{y}{1+y^2}\sin(yu)\sin(yv) = \sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty}\sin(yx)\frac{\sqrt{2\pi}}{8}\big[e^{-|x+u-v|}\,\mathrm{sign}\,(x+u-v)$$
$$+\,e^{-|x-u+v|}\,\mathrm{sign}\,(x-u+v) - e^{-(x+u+v)} - e^{-|x-u-v|}\,\mathrm{sign}\,(x-u-v)\big]dx.$$

Hence

$$\frac{y}{1+y^2}(F_s f)(y)(F_s g)(y) = \frac{2}{\pi}\int\limits_0^{+\infty}\int\limits_0^{+\infty}\frac{y}{1+y^2}\sin(yu)\sin(yv)f(u)g(v)dudv$$
$$= \frac{1}{2\pi}\int\limits_0^{+\infty}\int\limits_0^{+\infty}\Big\{\int\limits_0^{+\infty}\sin yx\Big[\big(e^{-|x+u-v|}\,\mathrm{sign}\,(x+u-v)$$
$$+\,e^{-|x-u+v|}\,\mathrm{sign}\,(x-u+v) - e^{-(x+u+v)}$$
$$-\,e^{-|x-u-v|}\,\mathrm{sign}\,(x-u-v)\big)\Big]dx\Big\}dudv$$
$$= \sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty}\sin yx\Big\{\frac{1}{2\sqrt{2\pi}}\int\limits_0^{+\infty}\int\limits_0^{+\infty}\big[e^{-|x+u-v|}\,\mathrm{sign}\,(x+u-v)$$
$$+\,e^{-|x-u+v|}\,\mathrm{sign}\,(x-u+v) - e^{-(x+u+v)}$$
$$-\,e^{-|x-u-v|}\,\mathrm{sign}\,(x-u-v)\big]f(u)g(v)dudv\Big\}dx$$
$$= F_s(f \overset{\gamma}{*} g)(y).$$

The proof is complete. $\square$

Proposition 2.1. *In the space of functions belonging to $L(\mathbb{R}_+)$, the convolution with a weight-function for the Fourier sine integral transform* (2.1) *is commutative, associative and distributive.*

Proof We prove that the convolution (2.1) is associative, i.e.,

$$(f \overset{\gamma}{*} g) \overset{\gamma}{*} h = f \overset{\gamma}{*} (g \overset{\gamma}{*} h).$$

Indeed,

$$\begin{aligned} F_s\big(f \overset{\gamma}{*} h\big)(y) =& \frac{y}{1+y^2}F_s(f \overset{\gamma}{*} g)(y)(F_s h)(y) \\ =& \frac{y}{1+y^2}(F_s f)(y)\frac{y}{1+y^2}(F_s g)(y)(F_s h)(y) \\ =& \frac{y}{1+y^2}(F_s f)(y)F_s(g \overset{\gamma}{*} h)(y) \\ =& F_s\big(f \overset{\gamma}{*} (g \overset{\gamma}{*} h)\big)(y), \quad \forall y > 0. \end{aligned}$$

It implies that

$$(f \overset{\gamma}{*} g) \overset{\gamma}{*} h = f \overset{\gamma}{*} (g \overset{\gamma}{*} h).$$

The commutative, distributive properties are similarly proved. $\square$

Definition 2. *The norm in the space $L(\mathbb{R}_+)$ is defined by*

$$\|f\|_{L(\mathbb{R}_+)} = 2\sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} \big|f(x)\big| dx. \tag{2.3}$$

Proposition 2.2. *If f and g are functions belonging to $L(\mathbb{R}_+)$, then the following inequality holds*

$$\big\|(f \overset{\gamma}{*} g)\big\|_{L(\mathbb{R}_+)} \leqslant \|f\|_{L(\mathbb{R}_+)} \, \|g\|_{L(\mathbb{R}_+)}. \tag{2.4}$$

Proof From the proof of Theorem 1 we get

$$\int\limits_0^{+\infty} \big|(f \overset{\gamma}{*} g)(x)\big| dx \leqslant 2\sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} |f(u)| du \int\limits_0^{+\infty} |g(v)| dv.$$

Hence

$$2\sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} \big|(f \overset{\gamma}{*} g)(x)\big| dx \leqslant 2\sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} |f(u)| du 2\sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} |g(v)| dv.$$

Thus

$$\big\|(f \overset{\gamma}{*} g)\big\|_{L(\mathbb{R}_+)} \leqslant \|f\|_{L(\mathbb{R}_+)} \, \|g\|_{L(\mathbb{R}_+)}.$$

The proof is complete. $\square$

Theorem 2. *In the space of functions in $L(\mathbb{R}_+)$ there does not exist the unit element for the convolution operation* (2.1).

Proof From Proposition 2.2 we have

$$\big\|f \overset{\gamma}{*} g\big\|_{L(\mathbb{R}_+)} \leqslant \|f\|_{L(\mathbb{R}_+)} \|g\|_{L(\mathbb{R}_+)}.$$

The remaining properties of the ring is clear. The commutative property of the ring can be easily obtained from Proposition 2.2. Now, we prove this normed ring no having the unit element. Suppose that there exists the unit element e of the convolution operation in the space of functions in $L(\mathbb{R}_+)$, it means $(e \overset{\gamma}{*} g) = (g \overset{\gamma}{*} e) = g$, for any function g belonging to $L(\mathbb{R}_+)$. Then we have

$$F_s(e \overset{\gamma}{*} g)(y) = (F_s g)(y), \quad \forall y > 0.$$

Hence
$$\frac{y}{1+y^2}(F_s e)(y)(F_s g)(y) = (F_s g)(y), \;\; \forall y > 0.$$
The last is equivalent to the equality
$$(F_s g)(y)\Big[\frac{y}{1+y^2}(F_s e)(y) - 1\Big] = 0, \;\; \forall y > 0,$$
for any function $g(y)$ belongs to $L(\mathbb{R}_+)$.

It is posibility to choose g so that $(F_s g)(y) \neq 0, \;\; \forall y > 0$, therefore
$$\frac{y}{1+y^2}(F_s e)(y) - 1 = 0, \; \forall y > 0. \tag{2.5}$$
If $y > 0$ and tends to 0, then $\frac{y}{1+y^2}$ tends to 0. On the other hand, since $e \in L(\mathbb{R}_+)$, it follows that $\frac{y}{1+y^2}(F_s e)(y) \to 0$ as $y \to 0$. This is a contradiction to formula (2.5). Therefore, this normed ring does not have the unit element. The theorem is proved. $\square$ Let $L(\mathbb{R}_+, e^x)$ be denoted the space of functions f such that
$$\int\limits_0^{+\infty} e^x |f(x)| dx < +\infty.$$

Theorem 3 (A Titchmarch type theorem). *Let $f, g \in L(\mathbb{R}_+, e^x)$. If $(f \overset{\gamma}{*} g)(x) \equiv 0$, then either $f(x) \equiv 0$ or $g(x) \equiv 0$.*

Proof Under the hypothesis $(f \overset{\gamma}{*} g)(y) \equiv 0, \quad \forall x > 0$, it follows that $F_s(f \overset{\gamma}{*} g)(y) = 0, \quad \forall y > 0$. Due to Theorem 1 we have
$$\frac{y}{1+y^2}(F_s f)(y)(F_s g)(y) = 0, \quad \forall y > 0. \tag{2.6}$$
Since
$$\begin{aligned}&\Big|\frac{d^n}{dy^n}\big(\sin(yx)f(x)\big)\Big| = \Big|f(x)x^n \sin\Big(yx + \frac{n\pi}{2}\Big)\Big| \\ \leqslant &\big|f(x)x^n\big| = \big|e^{-x}x^n e^x f(x)\big| = |e^{-x}x^n| \, . \, |e^x f(x)| \leqslant C|e^x f(x)|,\end{aligned}$$
for x large enough, due to Weierstrass criterion, the integral
$$\int\limits_0^{+\infty} \frac{d^n}{dy^n}\big[\sin(yx)f(x)\big] dx$$
uniformly converges on $\mathbb{R}_+$. Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(F_s f)(y)$ is analytic for $y > 0$. Similarly, $(F_s g)(y)$ analytic for $y > 0$. So from (2.6) we have $(F_s f)(y) = 0, \quad \forall y > 0$ or $(F_s g)(y) = 0, \forall y > 0$. If follows that either $f(x) = 0$, $\forall x > 0$ or $g(x) = 0 \; \forall x > 0$.

The theorem is proved. $\square$

3 Application to solving integral equations

Consider the integral equation

$$f(x) + \lambda \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(u)\psi(x,u)du = h(x), \quad x > 0. \tag{3.1}$$

Here

$$\psi(x,u) = \int_0^{+\infty} \big[e^{-|x+u-v|} \operatorname{sign}(x+u-v) + e^{-|x-u+v|} \operatorname{sign}(x-u+v)$$
$$- e^{-(x+u+v)} - e^{-|x-u-v|} \operatorname{sign}(x-u-v)\big] g(v)dv,$$

and $\lambda \in \mathbb{R}$, g and h are functions in $L(\mathbb{R}_+)$, f is the unknown function.

Theorem 4. *With the condition*

$$1 + \frac{\lambda y}{1+y^2}(F_s g)(y) \neq 0, \quad \forall y > 0$$

there exists a unique solution in $L(\mathbb{R}_+)$ of (3.1) *which is defined by*

$$f(x) = h(x) - \lambda(h \overset{\gamma}{*} \varphi)(x).$$

Here, $\varphi(x) \in L(\mathbb{R}_+)$ and it is defined by

$$(F_s\varphi)(y) = \frac{(F_s g)(y)}{1 + \frac{\lambda y}{1+y^2}(F_s g)(y)}.$$

Proof The equation (3.1) can be rewritten in the from

$$f(x) + \lambda(f \overset{\gamma}{*} g)(x) = h(x).$$

Suppose that equation (3.1) exists solution $f \in (\mathbb{R}_+)$. Due to Theorem 1, we have

$$(F_s f)(y) + \lambda \frac{y}{1+y^2}(F_s f)(y).(F_s g)(y) = (F_s h)(y), \quad \forall y > 0,$$

Since

$$(F_s f)(y)\Big[1 + \frac{\lambda y}{1+y^2}(F_s g)(y)\Big] = (F_s h)(y), \quad \forall y > 0.$$

Therefore

$$(F_s f)(y) = (F_s h)(y) \frac{1}{1 + \frac{\lambda y}{1+y^2}(F_s g)(y)}.$$

Due to Wiener-Levy's theorem [1, 14] there exists a function $\varphi \in L(\mathbb{R}_+)$ such that

$$(F_s\varphi)(y) = \frac{(F_sg)(y)}{1 + \frac{\lambda y}{1+y^2}(F_sg)(y)}.$$

It follows that

$$(F_sf)(y) = (F_sh)(y)\Big[1 - \frac{\lambda y}{1+y^2}(F_s\varphi)(y)\Big] = (F_sh)(y) - \lambda F_s(h \overset{\gamma}{*} \varphi)(y).$$

Hence

$$f(x) = h(x) - \lambda(h \overset{\gamma}{*} \varphi)(x).$$

By Theorem 1, $f \in L(\mathbb{R}_+)$. We can easily check that $f(x) = h(x) - \lambda F_s(h \overset{\gamma}{*} \varphi)(x)$ is the unique solution of equation (3.1) in $L(\mathbb{R}_+)$. The theorem is proved. □

remark 1. *Theorem 4 shows that the Toeplitz plus Hankel integral equation* (1.1) *with*

$$k_1(t) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} [e^{-|t-v|}\, sign\,(t-v) - e^{-(t+v)}]g(v)dv,$$

$$k_2(t) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} [e^{-|t+v|}\, sign\,(t+v) - e^{-|t-v|}\, sign\,(t-v)]g(v)dv,$$

has a unique solution in $L_1(\mathbb{R}_+)$ *which is defined by*

$$f(x) = h(x) - \lambda(h \overset{\gamma}{*} \varphi)(x).$$

For solving the new class of integral equations we will used the following known generalized convolutions.
The generalized convolution for the Fourier sine and cosine transforms is of the form [15]

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u)[g(|x-u|) - g(x+u)]du. \tag{3.2}$$

This convolution satisfies the following factorization property

$$F_s(f \underset{1}{*} g)(y) = (F_sf)(y)(F_cg)(y). \tag{3.3}$$

The generalized convolution for the Fourier cosine and sine integral transforms is defined by [13]

$$(f \underset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u)[g(|x-u|)\,\mathrm{sign}\,(x-u) + g(x+u)]du, \tag{3.4}$$

for which the following factorization equality holds

$$F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y). \tag{3.5}$$

We now consider the integral equation

$$f(x) + \lambda_1 \int_0^{+\infty} f(u)\theta_1(x,u)du + \lambda_2 \int_0^{+\infty} f(u)\theta_2(x,u)du = h(x), \quad x > 0. \tag{3.6}$$

Here

$$\begin{aligned}
\theta_1(x,u) =& \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} \big[e^{-|x+u-v|} \operatorname{sign}(x+u-v) + e^{-|x-u+v|} \operatorname{sign}(x-u+v) \\
& - e^{-(x+u+v)} - e^{-|x-u-v|} \operatorname{sign}(x-u-v)\big] g(v)dv, \\
\theta_2(x,u) =& \frac{1}{\sqrt{2\pi}} [k(|x-u|) - k(x+u)],
\end{aligned}$$

and $\lambda_1, \lambda_2 \in \mathbb{R}$, g, k and h are given functions of $L(\mathbb{R}_+)$, f is unknown function.

Theorem 5. *With the condition*

$$1 + \lambda_1 F_s\Big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}} e^{-x}\Big)(y)(F_s g)(y) + \lambda_2 (F_c k)(y) \neq 0, \ \forall y > 0,$$

there exist a unique solution in $L(\mathbb{R}_+)$ of (3.6) which is defined by

$$f(x) = h(x) - (h \underset{1}{*} l)(x).$$

Proof The equation (3.6) can be rewriten in the form

$$f(x) + \lambda_1 \frac{y}{1+y^2} (F_s f)(y)(F_s g)(y) + \lambda_2 (F_s f)(y)(F_c k)(y) = (F_s h)(y).$$

Due to Theorem 1 and the factorization equalities (3.3), (3.5) we have

$$(F_s f)(y) + \frac{y}{1+y^2} (F_s f)(y)(F_s g)(y) + \lambda_2 (F_s f)(y)(F_c k)(y) = (F_s h)(y).$$

By the formula (2.2.15, p. 65) in [5] we obtain

$$\frac{y}{1+y^2} = F_s\Big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}} e^{-x}\Big)(y).$$

It follows that

$$(F_s f)(y)[1 + \lambda_1 F_s\Big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}} e^{-x}\Big)(y)(F_s g)(y) + \lambda_2 (F_c k)(y) = (F_s h)(y).$$

Since $1+\lambda_1 F_s\Big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\Big)(y)(F_s g)(y)+\lambda_2(F_c k)(y)\neq 0,\ \forall y>0$, we have

$$\begin{aligned}(F_s f)(y) =&(F_s h)(y)\Big(1-\frac{\lambda_1 F_s\Big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\Big)(y)(F_s g)(y)+\lambda_2(F_c k)(y)}{1+\lambda_1 F_s\Big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\Big)(y)(F_s g)(y)+\lambda_2(F_c k)(y)}\Big)\\ =&(F_s h)(y)\Big(1-\frac{\lambda_1 F_c(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\underset{2}{*}g)(y)+\lambda_2(F_c k)(y)}{1+\lambda_1 F_c(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\underset{2}{*}g)(y)+\lambda_2(F_c k)(y)}\Big)\end{aligned}$$

Due to the Wiener-Levi's Theorem [1, 14], there exists a function $l\in L_1(\mathbb{R}_+)$ such that

$$(F_c l)(y)=\frac{\lambda_1 F_c(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\underset{2}{*}g)(y)+\lambda_2(F_c k)(y)}{1+\lambda_1 F_c(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\underset{2}{*}g)(y)+\lambda_2(F_c k)(y)}.$$

It follows that

$$\begin{aligned}(F_s f)(y) =&(F_s h)(y)[1-(F_c l)(y)]\\ =&(F_s h)(y)-F_s(h\underset{1}{*}l)(y).\end{aligned}$$

Hence

$$f(x)=h(x)-(h\underset{1}{*}l)(x).$$

¿From $h,l\in L(\mathbb{R}_+)$, we have $(h\underset{1}{*}l)(x)\in L(\mathbb{R}_+)$, and therefore $f\in L(\mathbb{R}_+)$. One can easily check the $f(x)=h(x)-(h\underset{1}{*}l)(x)\in L(\mathbb{R}_+)$ is the unique solution of the equation (3.6).
The theorem is proved. $\square$

remark 2. *The equation* (3.6) *is a particular case of the Toeplitz plus Hankel integral equation* (1.1) *with*

$$\begin{aligned}k_1(t) =&\frac{\lambda_1}{2\sqrt{2\pi}}\int\limits_0^{+\infty}[e^{-|t-v|}\ sign\,(t-v)-e^{-(t+v)}]g(v)dv-\frac{\lambda_2}{2\sqrt{2\pi}}k(t),\\ k_2(t) =&\frac{1}{2\sqrt{2\pi}}\int\limits_0^{+\infty}[e^{-|t+v|}\ sign\,(t+v)-e^{-|t-v|}\ sign\,(t-v)]g(v)dv+\frac{\lambda_2}{2\sqrt{2\pi}}k(|t|).\end{aligned}$$

Finally, consider the system of integral equations

$$\begin{aligned} f(x)+\lambda_1\int\limits_0^{+\infty} g(u)\psi_1(x,u)du =&p(x)\\ \lambda_2\int\limits_0^{+\infty} f(u)\psi_2(x,u)du+g(x) =&q(x),\quad x>0. \end{aligned} \tag{3.7}$$

Here,

$$\begin{aligned} \psi_1(x,u) =&\frac{1}{2\sqrt{2\pi}}\int\limits_0^{+\infty}\big[e^{-|x+u-v|}\,\mathrm{sign}\,(x+u-v)+e^{-|x-u+v|}\,\mathrm{sign}\,(x-u+v)\\ &-e^{-(x+u+v)}-e^{-|x-u-v|}\,\mathrm{sign}\,(x-u-v)\big]h(v)dv,\\ \psi_2(x,u) =&\frac{1}{\sqrt{2\pi}}[k(|x-u|)-k(x+u)], \end{aligned}$$

and $\lambda_1,\lambda_2\in\mathbb{R}$, k,h,p,q are given functions in $L(\mathbb{R}_+)$; f,g are unknown functions.

Theorem 6. *With the condition* $1-\lambda_1\lambda_2 F_c\Big(k \underset{F_c}{*} \big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)\Big)(y)\neq 0$, *for all* $y>0$, *there exists the unique solution in* $L(\mathbb{R}_+)$ *of* (3.7), *which is defined by*

$$\begin{aligned} f(x) =&p(x)-\lambda_1\Big(q \underset{1}{*}\Big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\underset{2}{*} h\Big)\Big)(x)-(p\underset{1}{*} l)(x)-\lambda_1\Big(q\underset{1}{*}\Big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\underset{2}{*} h\Big)\underset{1}{*} l\Big)(x)\\ g(x) =&q(x)-\lambda_2(p\underset{1}{*} k)(x)+(q\underset{1}{*} l)(x)-\lambda_2\big((p\underset{1}{*} k)\underset{1}{*} l\big)(x),\ x>0. \end{aligned}$$

Here, $l\in L(\mathbb{R}_+)$ *and is defined by*

$$(F_c l)(y)=\frac{\lambda_1\lambda_2 F_c\Big(k\underset{F_c}{*}\big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\underset{2}{*} h\big)\Big)(y)}{1-\lambda_1\lambda_2 F_c\Big(k\underset{F_c}{*}\big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\underset{2}{*} h\big)\Big)(y)}.$$

Proof The system (3.7) can be rewriten in the form

$$\begin{aligned} f(x)+\lambda_1(h\overset{\gamma}{*}g)(x)=p(x),\ x>0,\\ \lambda_2(f\underset{1}{*} k)(x)+g(x)=q(x),\ x>0. \end{aligned}$$

¿From Theorem 1 and the factorization equality (3.3) we get

$$\begin{aligned} (F_s f)(y)+\lambda_1\frac{y}{1+y^2}(F_s h)(y)(F_s g)(y)=(F_s p)(y),\\ \lambda_2(F_s f)(y)(F_c k)(y)=(F_s g)(y)=(F_s q)(y),\ y>0. \end{aligned}$$

On the other hand, since $\dfrac{y}{1+y^2} = F_s\big(\dfrac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x}\big)(y)$, we obtain

$$\begin{aligned}(F_sf)(y) + \lambda_1 F_c\big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)(y)(F_sg)(y) = (F_sp)(y),\ y>0\\ \lambda_2(F_sf)(y)(F_ck)(y) = (F_sg)(y) = (F_sq)(y),\ y>0.\end{aligned}$$

Since

$$\begin{aligned}\Delta = &\begin{vmatrix} 1 & \lambda_1 F_c\big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)(y) \\ \lambda_2(F_ck)(y) & 1\end{vmatrix}\\ &= 1-\lambda_1\lambda_2F_c\Big(k \underset{F_c}{*} \big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)\Big)(y) \neq 0,\end{aligned}$$

we see that

$$\frac{1}{\Delta} = 1 + \frac{\lambda_1\lambda_2F_c\Big(k \underset{F_c}{*} \big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)\Big)(y)}{1-\lambda_1\lambda_2F_c\Big(k \underset{F_c}{*} \big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)\Big)(y)},$$

is well defined.

Due to Wiener-Levy's theorem [1], there exists a continuous function $l \in L(\mathbb{R}_+)$ such that

$$(F_cl)(y) = \frac{\lambda_1\lambda_2F_c\Big(k \underset{F_c}{*} \big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)\Big)(y)}{1-\lambda_1\lambda_2F_c\Big(k \underset{F_c}{*} \big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)\Big)(y)}.$$

It follows

$$\frac{1}{\Delta} = 1 + (F_cl)(y).$$

Now we have

$$\begin{aligned}(F_sf)(y) = (1+(F_cl)(y))&\begin{vmatrix}(F_sp)(y) & \lambda_1F_c\big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)(y)\\ (F_sq)(y) & 1\end{vmatrix}\\ =&(1+(F_cl)(y))\Big((F_sp)(y) - \lambda_1F_s\big(q \underset{1}{*} \big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)\big)(y)\Big)\\ =&(F_sp)(y) - \lambda_1F_s\big(q \underset{1}{*} \big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)\big)(y) + F_s(p \underset{1}{*} l)(y)\\ &-\lambda_1F_s\Big(\big(q \underset{1}{*} \big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\big)\big) \underset{1}{*} l\Big)(y).\end{aligned}$$

Hence

$$f(x) = p(x) - \lambda_1\Big(q \underset{1}{*}\Big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\Big)\Big)(x) - (p \underset{1}{*} l)(x) - \lambda_1\Big(q \underset{1}{*} \Big(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\Big) \underset{1}{*} l\Big)(x).$$

On the other hand,

$$\begin{aligned}(F_s g)(y) =&(1+(F_c l)(y))\begin{vmatrix}1 & (F_s p)(y)\\ \lambda_2(F_c k)(y) & (F_s q)(y)\end{vmatrix}\\ =&(1+(F_c l)(y))\big((F_s q)(y)-\lambda_2 F_s(p \underset{1}{*} k)(y)\big)\\ =&(F_s q)(y)-\lambda_2 F_s(p \underset{1}{*} k)(y)+F_s(q \underset{1}{*} l)(y)-\lambda_2 F_s((p \underset{1}{*} k) \underset{1}{*} l)(y).\end{aligned}$$

Hence

$$g(x)=q(x)-\lambda_2(p \underset{1}{*} k)(x)+(q \underset{1}{*} l)(x)-\lambda_2\big((p \underset{1}{*} k) \underset{1}{*} l\big)(x),\ x>0.$$

From the hypothesis, we have $(p \underset{1}{*} l)(x), \left(q \underset{1}{*} \left(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\right)\right)(x)$, $\left(q \underset{1}{*} \left(\frac{\pi\sqrt{\pi}}{2\sqrt{2}}e^{-x} \underset{2}{*} h\right) \underset{1}{*} l\right)(x) \in L(\mathbb{R}_+)$. Therefore, $f \in L(\mathbb{R}_+)$.

Similarly, we have $g \in L(\mathbb{R}_+)$. The theorem has been proved. □

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