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*-ZERO DIVISORS AND *-PRIME IDEALS

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Abstract

Throughout this note we introduce the concept of *-zero divisors in rings with involution and its correlation with the concept of zero divisors in rings without involution. Moreover, some related definitions; such as *-completely prime ideals and rings and *-cancellation laws are introduced. Nevertheless, we characterize *-prime and *-completely prime ideals using *-zero divisors.

By a ring we mean an associative ring. A ring A is said to be an *involution ring* if on A there is defined a unary operation (called *involution*) $*$ subject to the identities $a^{**}=a, (a+b)^*=a^*+b^*$ and $(ab)^*=b^*a^*$, for all $a,b\in A$. In other words, the involution is an anti-isomorphism of order 2 on A. For a commutative ring A, it is evident that the identity mapping of A onto A is an involution on A (see [1]-[4]). Considering the category of involution rings, all morphisms (and also embeddings) must preserve involution. So we are looking here for a paricular concept for zero divisors that works in the category of involution rings.

If the ideal I of A is closed under involution; that is $I^{(*)}=\{a^*\in A \mid a\in I\}\subseteq I$, then it is called a **-ideal* of A and will be denoted by $I \lhd^* A$.

We start by defining *-zero divisors for an involution ring A.

Definition 1 A nonzero element $a\in A$ is said to be a *-zero divisor if there exists a nonzero element $b\in A$ such that $ab=0$ and $a^*b=0$.

Remark 2 If we start by defining left *-zero divisor as in definition 1, we get $b^*a^*=0$ and $b^*a=0$ which mean that a is a right *-zero divisor, too. By reversing the roles, a right *-zero divisor is also a left *-zero divisor. Thus we have only the concept of *-zero divisor, as one expects in the category of involution rings.

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Clearly, a *-zero divisor is a zero divisor, but the converse is not always true as it is obvious from the following example.

Example 3 Consider the diret sum $R = A \oplus A^{op}$, where A is an integral domain and A^{op} is its opposite domain. R is an involution ring under the exchange involution given by $(a,b)^* = (b,a)$ for all $(a,b) \in R$. For any $0 \neq a \in A$, the element $(a,0)$ of R is a zero divisor since $(a,0)(0,b) = 0 = (0,b)(a,0)$ for every $0 \neq b \in A$. Because neither a nor b are zero divisors, from $(0,a)(0,b) \neq (0,0)$, we conclude that $(a,0)$ is not a *-zero divisor.

In particular, if a is a symmetric ($a^* = a$) or a skew symmetric ($a^* = -a$) element, then a is a zero divisor if and only if it is a *-zero divisor. Moreover, we can construct symmetric or skew symmetric *-zero devisors from given *-zero devisors as in the following result.

Proposition 4 *Let A be an involution ring and $a \in A$. If a is a *-zero divisor, then there exists a (nonzero) symmetric or skew symmetric *-zero divisor in A.*

Proof If a is a symmetric or skew symmetric element, then we are done. If a is not symmetric, then $a - a^* \neq 0$ is a skew symmetric element in A such that $(a - a^*)b = ab - a^*b = 0$ and $(a - a^*)^*b = (a^* - a)b = a^*b - ab = 0$, with an appropriately chosen $b \in A$. □

Nevertheless, the next example shows that there are zero divisors which are *-zero divisors.

Example 5 In the involution ring of all 3×3 matrices over the integers Z with the transpose as involution, the element $a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is both zero and *-zero divisor. In fact the matrix $b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ satisfies $ab = ba = 0$ and $ab = a^*b = 0$.

Definition 6 A commutative involution ring without *-zero divisors is said to be a *-integral domain.

Since a commutative involution ring is an integral domain if it has no zero divisors, so it has also no *-zero divisors and consequently it is a *-integral domain. Moreover, each involution division ring is a *-integral domain.

Next, we define the *-cancellation law to work with *-zero divisors as follows.

Definition 7 We say that The *-cancellation law holds in an involution ring A if $ab = ac$ and $a^*b = a^*c$ imply $b = c$, for any $0 \neq a \in A$.

Again, if one defines *left *-cancellation law* to be hold in A as in Definition 7, we can easily show that *the right *-cancellation law* holds also in A . Therefore, we have only the *-cancellation law as one expects.

It is obvious that if the left (right) cancellation law holds in an involution ring A , then the *-cancellation law holds in A , too.

Remind that an ideal P of a ring A is called a *completely prime ideal* if $ab \in P$ implies $a \in P$ or $b \in P$ for all $a, b \in A$ (see for instance[5]).

Now, we give the involutive version of this definition.

Definition 8 An ideal P of an involution ring A is called a *-completely prime ideal if $ab \in P$ and $a^*b \in P$ imply $a \in P$ or $b \in P$ for all $a, b \in A$. The ring A is said to be a *-completely prime ring if the zero ideal is a *-completely prime ideal.

It is evident that in an involution ring A, a completely prime ideal is *-completely prime, too.

From the definition, it follows that the ring A is *- completely prime if and only if it has no *-zero divisors. We remind also that a ring A is completely prime if and only if it has no zero divisors. By this remark, a completely prime involution ring A is also *-completely prime, since A has no zero divisors implies that A has no *-zero divisors.

Following [3], an ideal P of an involution ring A is called a **-prime ideal* if $JK \subseteq P$ implies $J \subseteq P$ or $K \subseteq P$, for any J, $K \lhd^* A$. An involution ring A is a **-prime ring* if the zero ideal is a *-prime ideal. By the way, Birkenmeier and Groenewald gave in [3] the following equivalents for *-primeness of ideals.

Proposition 9 *([3], Proposition 5.4) Let A be an involution ring and $P \lhd^* A$. Then the following conditions are equivalent:*

*(i) P is a *-prime *-ideal of A.*

(ii) If a, $b \in A$ such that $aAb \subseteq P$ and $a^*Ab \subseteq P$, then $a \in P$ or $b \in P$.

(iii) If $I \lhd A$ and $K \lhd^* A$ such that $IK \subseteq P$, then $I \subseteq P$ or $K \subseteq P$.

We start our results by a classical one which gives the relation between the *-cancellation law and *-zero divisors.

Proposition 10 *Let A be an involution ring. Then the *-cancellation law holds in A if and only if A has no *-zero divisors.*

Proof Suppose that the *-cancellation law hold in A . If $0 \neq a \in A$ is such that $ab = 0, a^*b = 0$,then $b = 0$ follows and consequently A has no *-zero divisors. Conversely, let A have no *-zero divisors. For $0 \neq a \in A$, if $ab = ac$ and $a^*b = a^*c$, then $a(b-c) = 0$ and $a^*(b-c) = 0$ which forces $b-c = 0$. Thus $b = c$ and the *-cancellation law holds in A. □

For *-prime rings without nonzero nilpotent elements, we claim that they have no *-zero divisors.

Proposition 11 *If A is a *-prime ring having no nonzero nilpotent elements, then A has no *-zero divisors.*

Proof Let $0 \neq a, b \in A$ be such that $ab = 0, a^*b = 0$. Then $(ba)^2 = b(ab)a = 0$. Since A has no nonzero nilpotent elements, it follows that $ba = 0$. Thus for all $x \in A$, we have $(axb)^2 = ax(ba)xb = 0$, whence $axb = 0$ and consequently $aAb = 0$. Similarly, we get $a^*Ab = 0$. Because A is *-prime, we deduce from Proposition 9 that $b = 0$, from which A has no *-zero divisors. □

From the definitions, it is easy to check that a *-completely pirme *-ideal of A is also a *-prime *-ideal. The converse is true only in particular cases; for instance if A possesses identity. For commutative involution rings, we have the following equivalences.

Theorem 12 *Let A be a commutative ring with involution and $P \lhd^* A$. Then the following conditions are equivalent:*

*(i) P is a *-prime *-ideal.*

*(ii) P is a *-completely prime *-ideal.*

*(iii) The factor ring A/P is a *-integral domain.*

Proof (i)→(ii). Let $a, b \in A$ such that $ab \in P$ and $a^*b \in P$. Then $aAb \subseteq P$ and $a^*Ab \subseteq P$. Hence, by Proposition 9, $a \in P$ or $b \in P$ and consequently P is a *-completely prime *-ideal.

(ii)→(iii). A/P is commutative because A is commutative. Since P is a *-completely prime *-ideal, then $ab \in P$ and $a^*b \in P$ imply $a \in P$ or $b \in P$ for all $a, b \in A$. In other words, $(a + P)(b + P) = P$ and $(a + P)^*(b + P) = P$ imply $a + P = P$ or $b + P = P$, whence A/P is a *-integral domain.

(iii)→(i). Suppose that $aAb \subseteq P$ and $a^*Ab \subseteq P$. By the commutativity of A, we get $(ab)b \in P$, $(ab)^*b \in P$ and $(a^*b)b \in P$, $(a^*b)^*b \in P$. Since A/P has no *-zero divisors, it follwos that $ab \in P$ or $b \in P$ and $a^*b \in P$ or $b \in P$. If $b \notin P$, then $ab \in P$ and $a^*b \in P$, from which $a \in P$. Thus P is a *-prime *-ideal, by Proposition 9. □

Proposition 13 *For a commutative ring A with involution, the following are true:*

*(i) The set $K = \{all\ \text{*-zero divisors of } A\} \cup \{0\}$ is a *- ideal of A.*

*(ii) The factor ring A/K is a *-integral domain.*

Proof (i) Let $a, b \in K$ and $r \in A$, then there exist $c, d \in A$ such that $ac = a^*c = 0$ and $bd = b^*d = 0$. Hence $(a-b)cd = 0$, $(a-b)^*cd = (a^*-b^*)cd = 0$ and $rac = 0$, $(ra)^*c = 0$. Thus $a - b, ra \in K$. Moreover $a^* \in K$, since $a^*c = a^{**}c = ac = 0$.

(ii) Since A/K is commutative and has no *-zero divisors, it is a *-integral domain. □

The following proposition gives a necessary condition for an element in the center of a *-ideal to be in the center of the ring.

Proposition 14 *Let N be a *-ideal of an involution ring A and $c \in C(N)$; the center of N. If c is not a *-zero divisor, then $c \in C(A)$.*

Proof $C(N) = \{n \in N \mid nx = xn$, for all $x \in N\}$ is a *-subring of A, since for $n \in C(N)$, $x \in N$, we have $nx^* = x^*n$. Hence $n^*x = xn^*$ and $n^* \in C(N)$. Now for any $y \in A$, we have $cy, yc, c^*, c^*y, yc^* \in N$. Hence

$$c(cy - yc) = c(cy) - cyc = cyc - cyc = o$$

and

$$c^*(cy - yc) = c^*(cy) - c^*yc = (c^*c)y - c^*yc = c(c^*y) - c^*yc = c^*yc - c^*yc = 0.$$

But c is not a *-zero divisor, whence $cy - yc = 0$ and $c \in C(A)$ follows. □

Finally, since an involution ring without zero divisors having no *-zero divisors, we have the following immediate result from Proposition 3 in [1].

Proposition 15 *Every involution ring A without zero divisors is embeddable as a *-ideal (up to isomorphism)into one and only one involution ring $\overline{A^1}$ with identity and without *-zero divisors such that $\overline{A^1}$ is a minimal *-extension of A possessing identity.*

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