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THE HAMILTONIAN NUMBER OF SOME CLASSES OF CUBIC GRAPHS

Narong Punnim* and **Sermsri Thaithae**†

Department of Mathematics, Srinakharinwirot University,
Sukhumvit 23, Bangkok 10110, Thailand
narongp@swu.ac.th, sermsri@swu.ac.th

Abstract

A Hamiltonian walk in a graph G is a closed spanning walk of minimum length. The length of a Hamiltonian walk in G will be denoted by $h(G)$. Thus if G is a connected graph of order $n \geq 3$, then $h(G) = n$ if and only if G is Hamiltonian. Thus h may be considered as a measure of how far a given graph is from being Hamiltonian. Let G be a connected graph of order n. The Hamiltonian coefficient of G, denoted by $hc(G)$, is defined as $hc(G) = \frac{h(G)}{n}$. It has been shown in [6] that for every graph G of order n, $hc(G) \leq \frac{2n-2}{n} < 2$, and $hc(G) = \frac{2n-2}{n}$ if and only if G is a tree. Let $\mathcal{CR}(3^n)$ be the class of connected cubic graphs of order n. By putting

$$h(3^n) = \{h(G) : G \in \mathcal{CR}(3^n)\},$$

we obtained in [10] that if G is a 2-connected cubic graph of order $n \geq 10$ and $h(G) \geq n + 2$, then there exists a connected cubic graph G' of order n containing a cut edge such that $h(G) \leq h(G')$. We obtained in the same paper concerning the results on Hamiltonian number in the class of connected cubic graphs as follows. For an even integer $n \geq 4$ and $n \neq 14$. There exists an integer b such that $h(3^n) = \{k \in \mathbb{Z} : n \leq k \leq b\}$. Moreover, an explicit formula for the integer b is given by the following.

1. $b = n$ if and only if $n = 4, 6, 8$.
2. $b = n + 2$ if and only if $n = 10, 12$.
3. If $n = 14 + 2i$ and $i \geq 0$, then $b = 18 + 3i$.

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It should be noted that a cubic graph G_i of order $14 + 2i$ with $h(G_i) = 18 + 3i$ is a graph containing as many cut edges as possible. Furthermore, $\frac{h(G_i)}{|v(G_i)|} = \frac{18+3i}{14+2i} < \frac{3}{2}$ and

$$\lim_{i \to \infty} hc(G_i) = \frac{3}{2}.$$

The problem of finding the maximum value of $hc(G)$ in the class of 2-connected cubic graphs of order n is not easy. We introduce three classes of 2-connected cubic graphs with relatively small circumference and obtain several significant results on their Hamiltonian numbers and their Hamiltonian coefficients.

1 Introduction

Robinson and Wormald proved in [9] that if H is the number of Hamiltonian cycles in a cubic graph chosen uniformly at random from all labeled cubic graphs on $2n$ vertices, then

$$\lim_{n \to \infty} Pr(H > 0) = 1.$$

This means that almost all cubic graphs are Hamiltonian. In contrast of their result, we are seeking for non-Hamiltonian cubic graphs of order $2n$.

We limit our discussion to graphs that are simple and finite. For the most part, our notation and terminology follows that of Chartrand and Lesniak [4]. A *walk* W in a graph G is a sequence $x_1, x_2, \ldots, x_t$ of vertices of G in which $x_i x_{i+1} \in E(G)$ for all $i = 1, 2, \ldots, t-1$. If $x_1 = x_t$, then W is called a closed walk. A walk in G which contains all vertices of G is called a *spanning walk* of G and a closed walk in G which contains all vertices is called a *closed spanning walk* of G. For a walk W of G the *length* of W, denoted by $|W|$, is the number of edges used in W. It is clear that for given a connected graph G, it is possible to start at an arbitrary vertex u of G, walk in some sequence along the edges of G and return to the starting vertex u having passed through every vertex in G at least once. In general such a walk might pass through some vertices, and traverse some edges, more than once. We call such a walk a *closed spanning walk* in G. A *Hamiltonian walk* in G is a closed spanning walk of minimum length. The length of a Hamiltonian walk in G will be denoted by $h(G)$. Thus if G is a connected graph of order n, then $h(G) = n$ if and only if G is Hamiltonian. Thus h may be considered as a measure of how far a given graph is from being Hamiltonian.

In [5] an alternative way to define the length $h(G)$ of a Hamiltonian walk in a connected graph G was presented. A Hamiltonian graph G contains a spanning cycle $C : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$, where then $v_i v_{i+1} \in E(G)$ for $1 \le i \le n$. Thus Hamiltonian graphs of order $n \ge 3$ are those graphs for which there is a cyclic ordering $C : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$ such that $\sum_{i=1}^{n} d(v_i, v_{i+1}) = n$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1} for $1 \le i \le n$. For a connected graph G of order $n \ge 3$ and a cyclic ordering $s : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of the elements of $V(G)$, the number $d(s)$ is defined as $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$. Therefore,

$d(s) \geq n$ for each cyclic ordering s of the elements of $V(G)$. The *Hamiltonian number* $h(G)$ of G is defined in [5] by $h(G) = \min\{d(s)\}$, where the minimum is taken over all cyclic orderings s of elements of $V(G)$. It was shown in [5] that the Hamiltonian number of a connected graph G is, in fact, the length of a Hamiltonian walk in G.

Let G be a connected graph of order n. The *Hamiltonian coefficient* of G, denoted by $hc(G)$, is defined as $hc(G) = \frac{h(G)}{n}$. It has been shown in [6] that for every graph G of order n, $hc(G) \leq \frac{2n-2}{n} < 2$ and $hc(G) = \frac{2n-2}{n}$ if and only if G is a tree.

Let $\mathcal{CR}(3^n)$ be the class of connected cubic graphs of order n. By putting $h(3^n) = \{h(G) : G \in \mathcal{CR}(3^n)\}$, we obtained in [10] that if G is a 2-connected cubic graph of order $n \geq 10$ and $h(G) \geq n + 2$, then there exists a connected cubic graph G' of order n containing a cut edge such that $h(G) \leq h(G')$. We obtained in the same paper concerning the results on Hamiltonian number in the class of connected cubic graph as stated in the following theorem.

Theorem 1.1 [10] *For an even integer $n \geq 4$ and $n \neq 14$. There exists an integer b such that $h(3^n) = \{k \in \mathbb{Z} : n \leq k \leq b\}$. Moreover, an explicit formula for the integer b is given by the following.*

1. *$b = n$ if and only if $n = 4, 6, 8$.*
2. *$b = n + 2$ if and only if $n = 10, 12$.*
3. *If $n = 14 + 2i$ and $i \geq 0$, then $b = 18 + 3i$.*

It should be noted that a cubic graph G_i of order $14+2i$ with $h(G_i) = 18+3i$ is a graph containing as many cut edges as possible. Furthermore, $\frac{h(G_i)}{|v(G_i)|} = \frac{18+3i}{14+2i} < \frac{3}{2}$ and

$$\lim_{i \to \infty} hc(G_i) = \frac{3}{2}.$$

The problem of finding the maximum value of $hc(G)$ in the class of 2-connected cubic graphs of order n is not easy. We introduce three classes of 2-connected cubic graphs with small circumference and hence large Hamiltonian number. Hamiltonian walks were studied further by Asano, Nishizeki, and Watanabe [1, 2], Bermond [3], and Vacek [11]. Since every connected 2-regular graph is Hamiltonian, it is reasonable to investigate the Hamiltonian number in the class of cubic graphs. The following theorem was proved by Goodman and Hedetniemi [6] and will be applied throughout the paper.

Theorem A *Let G be a connected graph and $B_1, B_2, \ldots, B_k$ be the blocks of G. Then $h(G) = \sum_{i=1}^{k} h(B_i)$.*

2 2-connected cubic graphs

It is clear that a Hamiltonian graph must be 2-connected. A graph G of order n is called an *almost Hamiltonian graph* if $h(G) = n + 1$. Let n be an even integer $n \geq 10$. We proved in [8] that if G is a cubic graph of order n, then G is an almost Hamiltonian graph if and only if G is 2-connected non-Hamiltonian containing a cycle of order $n - 1$ as its subgraph. We had constructed an almost Hamiltonian cubic graph of even order $n \geq 10$, recursively. As a consequence, we obtained in the same paper that if $P(k, m)$ is the generalized Petersen graph of order $2k$, then $h(P(k, m)) \leq 2k + 1$ and $h(P(k, m)) = 2k + 1$ if and only if $m = 2$ and $k \equiv 5 (\mathrm{mod}\ 6)$.

Let G be a connected graph of order n and $W : w_1, w_2, w_3, \ldots, w_\ell, w_1$ be a Hamiltonian walk of G. Let $W_1(G)$ be the set of vertices $v \in V(G)$ such that v appears in $w_1, w_2, w_3, \ldots, w_\ell$ exactly once and $W_2(G) = V(G) - W_1(G)$. Thus G is Hamiltonian if and only if $W_2(G) = \emptyset$. Furthermore, $h(G) \geq n + |W_2(G)|$. Let $e = uv \in E(G)$. Then e is said to appear in W if u and v appear as consecutive vertices on W.

Lemma 2.1 *Let G be a connected cubic graph of order n and W be a Hamiltonian walk of G. If X is the set of $v \in V(G)$ such that all edges incident with v appear in W, then $X \subseteq W_2(G)$.*

Proof. Let $W : w_1, w_2, \ldots, w_\ell, w_1$ be a Hamiltonian walk of a connected cubic graph G of order n. Since for each w_i $(1 \leq i \leq \ell)$, w_i can have at most two distinct neighbors along W, it follows that if xv, yv, zv appear in W, then v must appear at least twice on W. Thus $v \in W_2(G)$. □

Let G be a connected graph and $uv \in E(G)$. A *subdivision* of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w.

Lemma 2.2 *Let G be a connected graph. If G' is a graph obtained from G by a subdivision of an edge uv of G, then $h(G') \geq h(G) + 1$.*

Proof. Let $W' : w_1, w_2, \ldots, w_\ell, w_1$ be a Hamiltonian walk of G'. Let w be the new vertex of subdividing the edge uv of G. It is clear that there is a unique i in which $w_i = w$ and we can assume that $w_1 = w$. If $w_2 \neq w_\ell$, then $W : w_2, w_3, \ldots, w_\ell, w_2$ is a closed spanning walk of G. If $w_2 = w_\ell$, then $W : w_2, w_3, \ldots, w_\ell$ is a closed spanning walk of G. Thus $h(G) \leq h(G') - 1$. □

These results and notation will be used in subsections 2.2 and 2.3.

2.1 Petersen graphs

Let $P = P(5, 2)$ be the Petersen graph of order 10. It is well known (for example see [7]) that P has the following properties.

1. It is a vertex transitive and edge transitive graph.

2. It is not Hamiltonian and $h(P) = 11$.
3. Let $u, v \in V(P)$ and $u \neq v$. Let $W : u = u_1, u_2, \ldots, u_t = v$ be a spanning walk in P. Then $t = 10$ if $uv \notin E(P)$, otherwise $t = 11$.

Let $P(k)$ be a cubic graph with $10k$ vertices formed by k pairwise disjoint copies of $P(5,2) - e$ by adding k edges to link them in a ring as shown in Fig. 1. More precisely let $P_1, P_2, \ldots, P_k$ be k pairwise vertex disjoint copies of $P = P(5,2)$. Let $u_i, v_i \in V(P_i)$ and $u_i v_i \in E(P_i)$. Let $P(k)$ be the graph with $V(P(k)) = \bigcup_{j=1}^{k} V(P_j)$ and $E(P(k)) = (\bigcup_{j=1}^{k} E(P_j - u_j v_j)) + \{u_1 v_2, u_2 v_3, \ldots, u_{k-1} v_k, u_k v_1\}$. Then $P(k)$ is a 2-connected cubic graph of order $10k$.

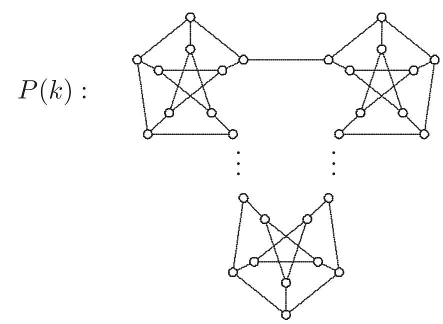

Fig. 1.

Theorem 2.3 *Let k be an integer with $k \geq 2$. Then $h(P(k)) = 11k$.*

Proof. Let $P(k)$ be the graph as described above. It is easy to obtain a closed spanning walk of length $11k$ of $P(k)$. Thus $h(P(k)) \leq 11k$. Let

$$W : w_1, w_2, \ldots, w_t, w_{t+1} = w_1$$

be a Hamiltonian walk of $P(k)$. Let $w_{i_1}, w_{i_2}, \ldots, w_{i_\ell}$ be the subsequence of W consisting of those vertices of P_i. Thus $1 \leq i_1 < i_2 < \ldots < i_\ell \leq t$ and $\ell \geq 10$. Suppose that there exists an integer $j = 1, 2, \ldots, \ell$ such that $i_{j+1} - i_j > 1$. Then $w_{i_1}, w_{i_j}, w_{i_{j+1}}, w_{i_\ell} \in \{u_i, v_i\}$. If $\ell = 10$, then $w_{i_1}, w_{i_2}, \ldots, w_{i_\ell}$ are all distinct. Thus $\ell \geq 12$, which is a contradiction. Therefore, for all $j = 1, 2, \ldots, \ell$, $i_{j+1} - i_j = 1$ and in this case $w_{i_1}, w_{i_2}, \ldots, w_{i_\ell}$ forms a u_i, v_i spanning path of P_i. Thus $\ell \geq 11$. This completes the proof. □

The result of Theorem 2.3 implies $hc(P(k)) = 1.1$ for all $k \geq 1$.

2.2 Cubic graphs with small circumference

We introduce another class of 2-connected cubic graphs which can be defined in the following way. For a positive integer $k \geq 2$, let $H(k)$ be a cubic graph as shown below.

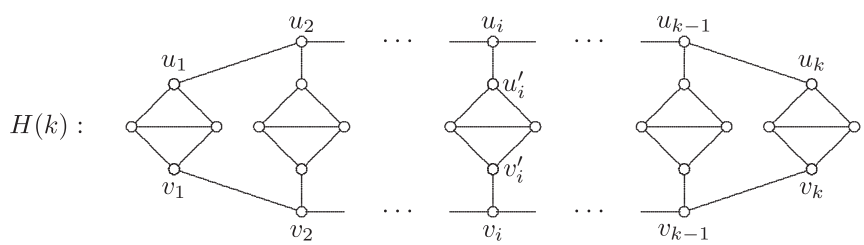

Fig. 2.

The graph $H(k)$ has the following properties.

1. $H(2)$ is Hamiltonian.

2. It was shown in [12] that $H(k)$ is of order $6k-4$ having circumference $2k+4$. Thus it is not Hamiltonian if $k \geq 3$.

We proved in [8] that a cubic graph G of order $n \geq 10$ with $h(G) = n+1$ if and only if G is 2-connected having circumference $n-1$. Furthermore we proved in the same paper that for every even integer $n \geq 10$, there exists a 2-connected cubic graph G of order n with $h(G) = n+1$. Since $H(3)$ is 2-connected cubic graph of order 14 with circumference 10, it follows that $h(H(3)) \geq 16$. It is easy to produce a closed spanning walk of $H(3)$ of length 16. Thus $h(H(3)) = 16$. Similarly, the graph $H(4)$ is of order 20 with $h(H(4)) = 22$. We now suppose that $k \geq 5$. Let $H(k)$ with vertices $u_1, u_2, \ldots, u_k, v_k, v_{k-1}, \ldots, v_2, v_1$ on its circumference as shown in Fig. 2. Thus $H(k) - \{u_iu_{i+1}, v_iv_{i+1}\}$ is disconnected containing two components. Put $G_1 \cup G_2 = H(k) - \{u_iu_{i+1}, v_iv_{i+1}\}$, where G_1 contains u_i and G_2 contains u_{i+1}. Summarizing we have the following results.

1. If $i = 1$, then $G_1 = K_4 - e$, for some edge e of K_4, and G_2 is a graph obtained from $H(k-1)$ with two subdivisions. Thus $h(H(k) - u_1u_2) = h(G_1) + h(G_2) + 2 \geq 4 + h(H(k-1)) + 2 + 2 = h(H(k-1)) + 8$.

2. If $i \geq 2$ and $k - i \geq 2$, then G_1 is a graph obtained from $H(i)$ with two subdivisions and G_2 is obtained from $H(k-i)$ with two subdivisions. Thus $h(H(k)-u_iu_{i+1}) \geq h(H(i))+2+h(H(k-i))+2+2 = h(H(i))+h(H(k-i))+6$.

3. For $i \geq 2$ we have that u_i is adjacent to u_{i-1} and u_{i+1}. Put u'_i as the third vertex that is adjacent to u_i. Similarly, v'_i is the third vertex that is adjacent to v_i. Thus $h(H(k) - u_i u'_i) \geq 4 + h(H(k-1)) + 2 + 2 = h(H(k-1)) + 8$.

4. If k is even and $k \geq 4$, then, by Theorem A, $h(H(k)) \leq h(H(k) - \{u_2u_3, u_4u_5, \ldots, u_{k-2}u_{k-1}\}) = 6k - 4 + 2(k-2)/2 = 7k - 6$.

5. If k is odd and $k \geq 5$, then, by Theorem A, $h(H(k)) \leq h(H(k) - \{u_2u_3, u_4u_5, \ldots, u_{k-1}u_k\}) = 6k - 4 + 2(k-1)/2 = 7k - 5$.

6. Let W be a Hamiltonian walk of $H(k)$. If $i \geq 2$ and u_{i-1}, u_i, u_{i+1} and u'_i appear in W, then, by Lemma 2.2, u_i must appear at least twice on W. Similarly for v_i. Thus if each of vertices $u_2, u_3, \ldots, u_{k-1}, v_2, v_3, \ldots, v_{k-1}$ appears at least twice in W, then $h(H(k)) \geq 6k - 4 + 2(k-2) = 8k - 8$.

7. If $k \geq 4$, then there exists i $(1 \leq i \leq k-1)$ such that $h(H(k)) = h(H(k) - e)$ where $e \in \{u_i u_{i+1}, u_i u'_i\}$.

The following theorem can be obtained from above observation.

Theorem 2.4 *Let k be an integer with $k \geq 2$. Then*

$$h(H(k)) = \begin{cases} 7k - 5 & if\ k\ is\ odd, \\ 7k - 6 & if\ k\ is\ even. \end{cases}$$

The result of Theorem 2.4 implies $hc(H(k)) = \frac{7k-5}{6k-4} < \frac{7}{6}$ for odd integers $k \geq 3$, $hc(H(k)) = \frac{7k-6}{6k-4} < \frac{7}{6}$ for even integers $k \geq 2$, and

$$\lim_{k \to \infty} hc(H(k)) = \frac{7}{6}.$$

2.3 Cubic graphs with even smaller circumference

The graph $H(k)$ as we have described is a 2-connected cubic graph of order $6k-4$ with circumference $2k+6$. Thus the circumference is about one third of its order. We will introduce in this subsection a class of 2-connected cubic graphs $G(k)$, $k \geq 2$ as follows and it was shown in [12] that $G(k)$ is of order $12k-4$ with circumference $2k+8$ which is about one sixth of the order.

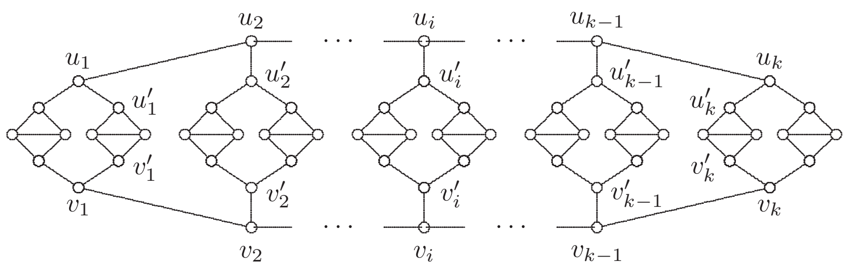

Fig. 3.

In order to obtain $h(G(k))$ we first observe the following elementary facts.

1. The graph $G(2)$ is of order 20 and of circumference 12. Thus it is not Hamiltonian. Therefore $h(G(2)) \geq 22$. It is easy to produce a closed spanning walk of $G(2)$ of length 22 and hence $h(G(2)) = 22$.
2. By Theorem A, $h(G(k)) \leq h(G(k) - \{u_1u'_1, u_2u'_2, \ldots, u_ku'_k\}) = 12k-4+2k = 14k-4$.

Let $G^*(k)$ be a graph obtained from $G(k)$ by two subdivisions, one on u_1u_2 and the other on v_1v_2. Put x and y to be the subdividing vertices on u_1u_2 and v_1v_2, respectively. It is clear that $h(G(k)+2 \leq h(G^*(k)) \leq h(G(k))+4$. We have the following result.

Lemma 2.5 *$h(G^*(2)) = 26$ and $h(G(3)) = 38$.*

Proof. Let $W : w_1, w_2, \ldots, w_\ell, w_1$ be a Hamiltonian walk of $G^*(2)$. If $u_1, u_2, v_1, v_2 \in W_2(G^*(2))$, then $h(G^*(2)) \geq 22+4 = 26$. Suppose, without loss of generality, that $u_1 \in W_1(G^*(2))$. Then $h(G^*(2)) = h(G^*(2)-u_1x)$ or $h(G^*(2)) = h(G^*(2)-u_1u'_1)$. By applying Theorem A in both cases, we can conclude that $h(G^*(2)) = 26$.

The graph $G(3)$ has order 32. Let $W : w_1, w_2, \ldots, w_\ell, w_1$ be a Hamiltonian walk of $G(3)$. If $u_1, u_2, u_3, v_1, v_2, v_3 \in W_2(G(3))$, then $h(G(3)) \geq 32+6 = 38$. Suppose, without loss of generality, that $h(G(3)) = h(G(3) - u_2u'_2)$, $h(G(3)) = h(G(3) - u_1u_2)$ or $h(G(3)) = h(G(3) - u_1u'_1)$. By using the result of Theorem A and $h(G^*(2)) = 26$, $h(G(3) - u_2u'_2) = 26 + 10 + 2 = 38$ and $h(G(3) - u_1u_2) = 26+10+2 = 38$. In order to calculate $h(G(3)-u_1u'_1)$ we may assume further that every Hamiltonian walk of $G(3) - u_1u'_1$ contains $u_1u_2, u_2u_3, u_2u'_2, v_1v_2, v_2v_3, v_2v'_2$. Thus, by Lemma 2.2, $u_2, v_2 \in W_2(G(3))$. Since $G(3) - u_1u'_1$ contains three blocks one of which is of order 2 and the other of order 4 having v'_1 as the common vertex. Thus $v_1, v'_1 \in W_2(G(3))$. There are also three possibilities for u_3, u'_3, v_3, v'_3 for W, namely $u_3, v_3 \in W_2(G(3))$, $u_3, u'_3 \in W_2(G(3))$ or $v_3, v'_3 \in W_2(G(3))$. Therefore $h(G(3)) = 38$, as required. □

Theorem 2.6 *Let k be an integer with $k \geq 3$. Then $h(G(k)) = 14k - 4$.*

Proof. We will proceed by induction on k. The result holds for $k = 3$. Suppose that the result holds for $k - 1 \geq 3$. Let $G(k)$ be the graph as shown in Fig. 3 and W be a Hamiltonian walk of $G(k)$. Suppose that for each $i = 1, 2, \dot{;} k$, $u_i, v_i \in W_2(G(k))$. Then $h(G(k)) = |W| \geq 12k - 4 + 2k = 14k - 4$. Suppose, without loss of generality, that there exists i such that $u_i \notin W_2(G(k))$. If $2 \leq i \leq k - 2$, then $h(G(k)) = h(G(k) - u_i u_i')$ or $h(G(k)) = h(G(k) - u_i u_{i+1})$. Since $G - u_i u_i'$ consists of three blocks, one of which is isomorphic to $G^*(k - 1)$. Thus, by Theorem A and induction, $h(G(k)) = h(G(k) - u_i u_i') = h(G^*(k-1)) + 10 + 2 \geq 14(k-1) - 4 + 2 + 10 + 2 = 14k - 4$. The graph $G(k) - u_i u_{i+1}$ consists of three blocks, one of which is $G^*(i)$ and the other is $G^*(k - i)$. Thus, by Lemma 2.6, Theorem A and induction, $h(G(k)) = h(G^*(i)) + h(G^*(k - i)) + 2 \geq 14i - 4 + 2 + 14(k - i) - 4 + 2 + 2 = 14k - 2 > 14k - 4$. If $u_1 \notin W_2(G(k))$, then $h(G(k)) = h(G(k) - u_1 u_2)$ or $h(G(k)) = h(G(k) - u_1 u_1')$. If $h(G(k)) = h(G(k) - u_1 u_2)$, then $h(G(k)) = h(G(k) - u_1 u_2) = 10 + h(G^*(k - 1)) + 2 \geq 10 + 14(k - 1) - 4 + 2 + 2 = 14k - 4$. If $h(G(k)) = h(G(k) - u_1 u_1')$, then, by above argument, we may assume that for each $1 \leq i \leq k - 1$, $u_i, v_i \in W_2(G(k))$. Since the graph $G(k) - u_1 u_1'$ consists of three blocks and by Theorem A, it follows that $v_1, v_1' \in W_2(G(k))$. It can be shown that $u_k, v_k \in W_2(G(k))$ or $u_k, u_k' \in W_2(G(k))$ or $v_k, v_k' \in W_2(G(k))$. Therefore, $h(G(k)) = 14k - 4$. □

The result of Theorem 2.6 implies $hc(G(k)) = \frac{14k-4}{12k-4} > \frac{7}{6}$ for integers $k \geq 3$, and

$$\lim_{k\to\infty} hc(G(k)) = \frac{7}{6}.$$

We close this paper by proposing the following conjecture.

Conjecture Let G_n be a 2-connected cubic graph of order n with $h(G_n) = \max(h, \mathcal{CR}_2(3^n))$. Then

$$\lim_{n\to\infty} hc(G_n) = \frac{7}{6}.$$

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