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PJECTIVE MODULES AND A GENERALISATION OF LIFTING MODULES

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Abstract

We introduce the notions of M-pjective modules. We show that relative pjectivity is necessary and sufficient for a direct sum of two lifting modules to be lifting. We also introduce the new concept of generalized lifting modules, and give some properties of such modules in analogy with the know properties for lifting modules.

1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and M will be unital right R-module and $S = End(M)$ is the ring of R-endomorphisms of M. Submodules of M will be right R-submodules, while one sided ideals for these rings will be right ideals for R and right ideal for S, respectively. We reserve the term "ideal"for the two-sided ideals in both rings. The notation $N \leq^{\oplus} M$ denotes that N is a direct summand in M; $N \ll M$ means that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). A module N is said to be *small* if $N \ll L$, for some module L. For $N, L \leq M, N$ is *supplement* of L in M if $N + L = M$ with $N \cap L \ll N$. Following [10], a module M is called supplemented if every submodule of M has a supplement in M. On the other hand, the module M is *amply supplemented* if, for any submodules A, B of M with $M = A + B$ there exists a supplement P of A in M such that $P \leq B$. Module M is called a *weakly supplemented* module if for each submodule A of M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$.

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M is called $\oplus$-*supplemented* if every submodule of M has a supplement that is a direct summand of M.

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *cosmall* submodule of A in M. A submodule A of M is called *coclosed* if A has no proper cosmall submodule.

A module M is *lifting* if for every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A \ll M$. By [7, Proposition 4.8], M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand.

Generalized relative injectivity was renamed as *ojective* by Mohamed and Muller (in [8]) in honour of Oshiro. They also dualised the consept of ojective modules in two ways, namely to cojective and *cojective modules(see [9]).

Here we introduce the concept of M-pjectivity, which is a generalization of $M-^*$cojectivity.

A module N is M*-pjective* if every supplement of M in $M \oplus N$ is a direct summand. If N is M-pjective and M is N-pjective, we say that N and M are relatively pjective. The problem of finding a satisfactory necessary and sufficient condition for a direct sum of lifting modules to be lifting is still open. We show that relative pjectivity is necessary and sufficient for a direct sum of two lifting modules to be lifting. We also introduced the concept of generalized lifting modules, and give sum properties of such modules in analogy with properties for lifting modules.

2. M-Pjective Modules

Definition 1 Let $M = A \oplus B$. Then B is called A-*pjective* if every supplement C of A in M is a direct summand.

Lemma 2.1 *Let A and B be submodules of a module M with $A + B = M$. Then A is a supplement of B in M if and only if A is a coclosed submodule of M and $A \cap B \ll M$.*

Proof It is easily checked by [6, Lemma 1.1]. □

Lemma 2.2 *Let $M = M_1 \oplus M_2$ and $N, L \leq M_1$. If N is a supplement of L in M_1, then:*

(1) *$N \oplus M_2$ is a supplement of L in M.*

(2) *N is a supplement of $L \oplus M_2$ in M.*

Proof (1) [3, Lemma 2.2]

(2) Let N be a supplement of L in M_1. Then $M_1 = N + L$ and N is minimal with this property. It is easy to see that $M = N + (L \oplus M_2)$. Let $X \leq N \leq M_1$

such that $M = X + (L \oplus M_2)$. Hence $X + L = M_1$. Since N is a supplement of L in M_1 we will have $X = N$. □

Lemma 2.3 *Let $K \leq L \leq M$. If K is coclosed in M, then K is coclosed in L and the converse is true if L is coclosed in M.*

Proof [4, Lemma 2.6]. □

Lemma 2.4 *Let $M = A \oplus B$, where B is A-pjective. If $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$. Then (for $i, j = 1, 2$);*

(1)*B_i is A-pjective;*

(2)*B is A_j-pjective;*

(3)*B_i is A_j-pjective.*

Proof For (1), write $N = A \oplus B_1$, and let X be a supplement of A in N. By Lemma 2.2, $X \oplus B_2$ is a supplement of A in M. As B is A-pjective, $M = X \oplus B_2 \oplus K$ for some $K \leq M$. Hence $N = X \oplus (N \cap (B_2 \cap K))$.

For (2), write $L = A_1 \oplus B$, and let Y be a supplement of A_1 in L. By [6, Lemma 1.1], it is easy to see that A_1 is a supplement of Y in L. Then A is a supplement of Y in M by Lemma 2.2(1). Again by [6, Lemma 1.1] Y is supplement of A in M. As B is A-pjective, $M = Y \oplus K$ for some $K \leq M$. Hence $L = Y \oplus (K \cap L)$.

(3) Follows from (1) and (2). □

Definition 2 A module M is called an *absolute relative pjective* module (for short $ARPJ$-module) if M_i is M_j-pjective($i \neq j$); whenever $M = M_1 \oplus M_2$.

Clearly every lifting module is an $ARPJ$-module and any indecomposable module is obviously an $ARPJ$-module, which is not lifting. The following Proposition gives the relation between lifting modules and $ARPJ$-modules.

Proposition 2.5 *The following are equivalent for a module M:*

(1) *M is a lifting module;*

(2) *M is an amply supplemented, $\oplus$-supplemented and $ARPJ$-module.*

Proof (1)$\Rightarrow$ (2) It is trivially.

(2) $\Rightarrow$ (1) Let C be a coclosed submodule of M. Since M is $\oplus$-supplemented, C has a supplement in M which is a direct summand; i.e. M has a decomposition $M = M_1 \oplus M_2$, where $M_2 \cap C \ll M$. Since M is a $ARPJ$-module, M_2 is M_1-pjective. From Lemma 2.1, C is a supplement of M_2 in M and so $C \leq^{\oplus} M$. Therefore M is lifting. □

Proposition 2.6 *Let $M = M_1 \oplus M_2$. If M_1 is M_2-projective, then M_1 is M_2-pjective.*

Proof It is easily checked by [3, Proposition 3.3] □

Let M be a module. Recall that M is called a (D_3)-module, if $M = A + B$ where A and B are direct summands of M, then $A \cap B$ is a direct summand of M.

Let M_1, M_2 be modules. Following [6], the module M_1 is small M_2-projective if every homomorphism $f : M_1 \to M_2/A$, where A is a submodule of M_2 and $Imf \ll M_2/A$, can be lifted to a homomorphism $\phi : M_1 \to M_2$.

Theorem 2.7 *Let $M = M_1 \oplus M_2$ be an amply supplemented (D_3)-module . If M_1 is M_2-pjective, then M_1 is M_2-projective.*

Proof Let N be a submodule of M such that $(N + M_1)/N \ll M/N$. Then $M = N + M_2$. Since M is amply supplemented there exists a submodule N' of M such that $N' \leq N, M = N' + M_2$ and $N' \cap M_2 \ll N'$, that is, N' is a supplement of M_2 in M. Since M_1 is M_2-pjective, $M = N' \oplus K$ for some $K \leq M$. Since M is $(D_3), N' \cap M_2$, is a direct summand of M, and so $M = N' \oplus M_2$. By [6, Lemma 2.4], M_1 is small M_2-projective. Hence by [2, Proposition 14.17], M_1 is M_2-projective. □

Lemma 2.8 *Let $M = A \oplus B$ where A is B-pjective and B is lifting. If X is a coclosed submodule of M with $M = X + B$, then X is a summand of M.*

Proof Let $M = X + B$. Since B is lifting, there exists a direct summand B_1 of B such that $B = B_1 \oplus B_2$ and $B_1 \leq X \cap B, X \cap B_2 \ll B_2$. Now $M = A \oplus B_1 \oplus B_2$. Write $N = A \oplus B_2$. Then $X = B_1 + X_1$, where $X_1 = X \cap N$. Hence $M = X + B = X_1 + B_1 + B_2$, and so $N = X_1 + B_2$. Clearly $X_1 \cap B_2 = X \cap B_2 \ll B_2$. Then B_2 is a supplement of X_1 in N. Now X_1 is coclosed submodule of X, and X is coclosed submodule of M. It follows by Lemma 2.3 that X_1 is coclosed in N. It is easy to see that X_1 is a supplement of B_2 in N. Now A is B_2-pjective, by Lemma 2.4. Thus we get that X_1 is a direct summand of N, and therefore is a direct summand of M. □

The following is a necessary and sufficient condition of a direct sum of two lifting modules to be lifting.

Theorem 2.9 *Let $M = M_1 \oplus M_2$ be an amply supplemented module. Then M is lifting if and only if the M_i is lifting, and is M_j-pjective, $i \neq j (= 1, 2)$.*

Proof It follows from Lemma 2.8, and [6, Theorem 2 .1]. □

3. Generalized Lifting Modules

Definition 3 A module M is called a *generalized* lifting module(for short a GL-module) if the following condition satisfied: If $M = M_1 \oplus M_2$ and $A \leq M$, then there exist $C_i \leq^{\oplus} M_i (i = 1, 2)$ such that $C_1 \oplus C_2$ is a supplement of A in M.

Observe that from the definition of GL-modules, every GL-module is $\oplus$-supplemented module.

In the following we are going to show that every lifting module is a GL-module, and also give the relations between $\oplus$-supplemented modules and GL-modules.

Theorem 3.1 (cf. [5]) *For any ring R, any finite direct sum of $\oplus$-supplemented R-modules is $\oplus$-supplemented.*

Remark 3.2 In the proof of the Theorem 3.1 we obtain that for any submodule L of $M = M_1 \oplus M_2$, there exists a supplement $K \oplus H$ of L, where $K \leq^{\oplus} M_1$ and $H \leq^{\oplus} M_2$. Therefore we have the following Corollary.

Corollary 3.3 *The following are equivalent for a module M:*

(1) *M is a GL-module.*

(2) *Every direct summand of M is a $\oplus$-supplemented module.*

Proof (1) $\Rightarrow$ (2) Let $M = M_1 \oplus M_2$ and $A \leq M_1$. Since M is GL-module, there exist $C_i \leq^{\oplus} M_i (i = 1, 2)$, such that $C_1 \oplus C_2$ is supplement of A in M. Therefore $A + (C_1 \oplus C_2) = M$, hence $A + C_1 = M_1$. Since $C_1 \cap A \leq (C_1 \oplus C_2) \cap A \ll M$, $C_1 \cap A \ll C_1$. Therefore M_1 is $\oplus$-supplemented.

(2) $\Rightarrow$ (1) It is easily checked by Remark 3.2. □

Corollary 3.4 *Let $M = \oplus_{i=1}^{n} M_i$ be a finite direct sum of modules. If each M_i is $\oplus$-supplemented, then M is GL-module.*

Corollary 3.5 *Direct summands of a GL-module are GL-modules.*

Proof It is an immediate consequence of Corollary 3.3. □

Corollary 3.6 *Every lifting module is a GL-module.*

Proof Since every direct summand of lifting module is lifting, hence is $\oplus$-supplemented. □

Example 3.7 Let p be any prime integer. $\mathbb{Z}$-Module $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not lifting(see [7, Proposition A.7]), but is GL-module.

We say that a module M has *hollow dimension* n, if there exists a small epimorphism from M to a direct sum of n hollow modules.

Corollary 3.8 *Let M be $\oplus$-supplemented and finite hollow dimensional module. Then M is a GL-module.*

Proof We prove by induction on the hollow dimension of M. It is clear that every hollow module is a GL-module. Now let M be a module of hollow dimension n. Since every nonzero proper summand submodule of M has hollow dimension less than n, by induction it is a GL-module; and hence is $\oplus$-supplemented. Therefore by Corollary 3.3 M is a GL-module □

The following implications are now clear for a module M:

Lifting module $\Rightarrow$ GL-module $\Rightarrow$ $\oplus$-Supplemented module.

Corollary 3.9 *The following are equivalent for a module $M = \oplus_{i=1}^{n} M_i$:*

(1) *The $M_i(i = 1, 2, ..., n)$ is $\oplus$-supplemented;*

(2) *Each submodule of M has a supplement in M of the form $\oplus_{i=1}^{n} N_i$, where $N_i \leq^{\oplus} M_i(i = 1, 2, ..., n)$.*

Proof (1) $\Rightarrow$ (2) By induction on the number n of the summands M_i of M, for $n = 1$ it is clear. If $n = 2$, it follows by Remark 3.2.

(2) $\Rightarrow$ (1) Let $L \leq M_i \leq M$. By assumption L has a supplement in M of the form $\oplus_{i=1}^{n} N_i$, where $N_i \leq^{\oplus} M_i(i = 1, 2, ..., n)$. Then $L + \oplus_{i=1}^{n} N_i = M$. Hence for each i, $L + N_i = M_i$, $L \cap N_i \leq L \cap (\oplus_{i=1}^{n} N_i) \ll M$. Therefore $L \cap N_i \ll M_i$. Hence M_i is $\oplus$-supplemented. □

Proposition 3.10 *Let M be a $\oplus$-supplemented module. Then it has a decomposition $M = M_1 \oplus M_2$, where M_2 is a cosmall submodule of $Rad(M)$ in M.*

Proof Since M is $\oplus$-supplemented, there exists a decomposition $M = M_1 \oplus M_2$, such that $Rad(M) \cap M_1 \ll M$ and $Rad(M) + M_1 = M$. We have $Rad(M) = Rad(M_1) \oplus Rad(M_2)$. Then $M = M_1 \oplus Rad(M_2)$. Hence $M_2 = Rad(M_2)$, therefore $Rad(M) = Rad(M_1) \oplus M_2$. Now we show that $Rad(M)/M_2 \ll M/M_2$. Let $Rad(M)/M_2 + L/M_2 = M/M_2$ for some $L \supseteq M_2$. Then $M = Rad(M) + L = (Rad(M_1) \oplus M_2) + L = Rad(M_1) + L$. Since $Rad(M_1) = Rad(M) \cap M_1 \ll M$, hence $L = M$. □

Proposition 3.11 *If M is a GL-module with finite hollow dimension, then M is a finite direct sum of hollow submodules.*

Proof Since M has a finite hollow dimension, then M is a direct sum of indecomposable submodules. By Corollary 3.5, the indecomposable summand of M are GL-modules, and hence, by [5, Lemma 2.14] are hollow modules. □

Remark 3.12 Consider a direct sum of hollow modules, which contains an indecomposable and not hollow summand submodule. This module is $\oplus$-supplemented [5, Corollary 1.6], which is not a GL-module (by [5, Lemma 2.14] and Corollary 3.3). This also shows that direct summands of $\oplus$-supplemented modules need not be $\oplus$-supplemented.

Lemma 3.13 *Let $A \leq B \leq^{\oplus} M$. If C is a supplement of A in M, then $C \cap B$ is a supplement of A in B.*

Proof Since C is supplement of A in M, $C + A = M$ and $C \cap A \ll M$. Then

$(C \cap B) + A = B$, and $C \cap B \cap A = C \cap A \ll M$. $\square$

Theorem 3.14 *If M is a $\oplus$-supplemented, and satisfies in this condition that for every two direct summands N_1 and N_2 of M such that $N_1 \cap N_2$ is coclosed in M, implies that $N_1 \cap N_2$ is a direct summand of M. Then M is a GL-module.*

Proof Let $B \leq^{\oplus} M$ and $A \leq B$. Since M is a $\oplus$-supplemented, there exists a supplement K of A in M such that $K \leq^{\oplus} M$. By Lemma 3.13, $K \cap B$ is a supplement of A in B; and hence a coclosed submodule of B. By assumption, $K \cap B \leq^{\oplus} M$. This shows that any summand B of M is $\oplus$-supplemented. Therefore M is GL-module. $\square$

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