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GEOMETRIC LATTICES AND INDEPENDENCE SPACES

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Abstract

This paper deals with the relation between geometric lattices and independence spaces. Finally, it gets a bijective correspondence between geometric lattices and simple independence spaces. This result makes the study of geometric lattices turn into the research of independence spaces. Moreover, it generalizes the ways of study on geometric lattices, and at the same time, on independence spaces.

1 Introduction and Preliminaries

The relationship between a finite simple matroid and a finite geometric lattice has been pointed out in [1,Chapter 3]. It was shown in [3] that a finite height geometric lattice corresponds to a simple matroid of arbitrary cardinality, and vice versa. It arises a question how about the relation between a general geometric lattice and a class of infinite matroids? It has been shown in [2] that an independence space is a class of more frequently studied classes of infinite matroids. Some methods have been presented in [6] to generate an independence space with graph theory and some properties of independence spaces can be found in [7]. In this paper, we consider independence spaces as the classes of infinite matroids. In addition, following [1,p.388,Theorem 5], the closed sets of

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an independence space form a complete semimodular lattice under set inclusion, and each of its elements is a join of atoms. Besides, Welsh said in [1,p.388] that in general, we no longer have a geometric lattice for the closed sets of an independence space since even when it is infinite, a geometric lattice is defined to have only finite dimension. However, this paper will use another way to obtain that a geometric lattice corresponds to a simple independence space, and vice versa. This result completely deals with the relation between a geometric lattice and an independence space. This correspondence presents a real way to study on geometric lattices by matroid theory. The related examples can be found in Section 2.

First of all, we summarize some facts of independence spaces and geometric lattices that are needed in the present work. In what follows, we assume that S is some arbitrary–possibly infinite–set. $Y \subset\subset X$ indicates that Y is a finite subset of X. The following definition is taken from [2, p.74] and [1, p.387, 388].

Definition 1 (1) An *independence space* $M(S)$ is a set S together with $\mathcal{I} \subseteq \mathcal{P}(\mathcal{S})$(called *independent sets*) such that the following conditions:
(i1) $\mathcal{I} \neq \emptyset$;
(i2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$;
(i3) If $A, B \in \mathcal{I}$ and $|A|, |B| < \infty$ with $|A| = |B| + 1$, then $\exists a \in A \setminus B$ fits $B \cup a \in \mathcal{I}$;
(i4) If $A \subseteq S$ and every finite subset of A is a member of $\mathcal{I}$, then $A \in \mathcal{I}$.

A subset X of S is *dependent* if $X \notin \mathcal{I}$. A *circuit* of $M(S)$ is a minimal dependent set. A *basis* of $M(S)$ is a maximal independent set.

(2) The *closure operator* σ of $M(S)$ is defined by $x \in \sigma(A)$ if $x \in A$ or if there exists a circuit C with $x \in C \subseteq A \cup x$. A set X is *closed* or a *flat* if $\sigma(X) = X$.

Lemma 1 *(1) A function $\sigma : 2^S \to 2^S$ is the closure operator of an independence space on S if and only if for X, Y subsets of S, and $x, y \in S$;*
(s1) $A \subseteq \sigma(A)$;
(s2) If A, B are subsets of S, and $A \subseteq B$, then $\sigma(A) \subseteq \sigma(B)$;
(s3) For $X \subseteq S, \sigma(X) = \sigma(\sigma(X))$;
(s4) If $y \notin \sigma(X)$ and $y \in \sigma(X \cup x)$, then $x \in \sigma(X \cup y)$;
(s5) If $a \in \sigma(X)$ for some $X \subseteq S$, then $a \in \sigma(X_f)$ for some $X_f \subset\subset X$.

(2) All the circuits of an independence space are finite. If $(S, \mathcal{I})$ is an independence space, $T \subseteq X \subseteq S$ and $T \in \mathcal{I}$, then there is a maximal $\mathcal{I}$-subset of X containing T.

Proof The assertion (1) is from [1, p. 388]; the assertion (2) is from [2, p. 74 and p.80]. □

Corollary 1 *Let $M(S)$ be an independence space with σ as its closure operator. Then, $\bigcap_{\alpha \in \mathcal{A}} \sigma(X_\alpha)$ is a flat for flats X_α of $M(S), (\alpha \in \mathcal{A})$.*

Proof Applying (s1), (s2) and (1), $\bigcap\limits_{\alpha\in\mathcal{A}} X_\alpha \subseteq \sigma(\bigcap\limits_{\alpha\in\mathcal{A}} X_\alpha) \subseteq \bigcap\limits_{\alpha\in\mathcal{A}} \sigma(X_\alpha) = \bigcap\limits_{\alpha\in\mathcal{A}} X_\alpha$ for flats X_α of $M(S)$, $(\alpha \in \mathcal{A})$, so by Definition 1, $\bigcap\limits_{\alpha\in\mathcal{A}} X_\alpha = \bigcap\limits_{\alpha\in\mathcal{A}} \sigma(X_\alpha) = \sigma(\bigcap\limits_{\alpha\in\mathcal{A}} X_\alpha)$ is a flat.

According to Lemma 1 and Definition 1, for an independence space $M = (S, \mathcal{I})$ with σ as its closure operator, one has $\sigma(X) = X \cup \{x \in S | I \cup x \notin \mathcal{I}$ for some $I \subseteq X$ such that $I \in \mathcal{I}\}$. In addition, if an operator σ satisfies (s1)-(s5), then the corresponding independence space is $(S, \mathcal{I}(\sigma))$ where $\mathcal{I}(\sigma) = \{I \subseteq S | x \in I \Rightarrow x \notin \sigma(I \setminus x)\}$.

Furthermore, for an independence space $M(S)$, we have $\sigma(X) = \sigma(I)$ for a maximal independent set I contained in X. Especially, $\sigma(B) = S$ for any basis B of $M(S)$.

Definition 2 (1) (see [4,p.234]) A lattice L is called *geometric* if and only if L is semimodular, L is algebraic, and the compact elements of L are exactly the finite joins of atoms of L.

Equivalently, L is complete, L is atomistic, all atoms are compact, and L is semimodular.

(2) (see [4,p.240]) A *geometry* $(A, ^-)$ is a set A and a function $X \mapsto \overline{X}$ of $\mathcal{P}(\mathcal{A})$ into itself satisfying the following conditions:

(i) $^-$ is a closure relation, that is,

(i_1) $X \subseteq \overline{X}$;

(i_2) If $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$;

(i_3) $\overline{\overline{X}} = \overline{X}$.

(ii) $\overline{\emptyset} = \emptyset$, and $\overline{\{x\}} = \{x\}$, for $x \in A$;

(iii) If $x \in \overline{X \cup y}$, but $x \notin \overline{X}$, then $y \in \overline{X \cup x}$;

(iv) If $x \in \overline{X}$, then $x \in \overline{X_1}$, for some $X_1 \subset\subset X$.

(3) (see [4,p.229]) Let A be a set of atoms of a lattice with the least element 0. Then $G \subseteq A$ *spans* A if and only if, for every $a \in A$, there is a finite $G_1 \subseteq G$ such that $a \leq \vee G_1$.

The following result is in [4,p.241].

Lemma 2 *Let $(A, ^-)$ be a geometry. Then $L(A, ^-) = \{\overline{X} | X \subseteq A\}$ is a geometric lattice. Conversely, if L is a geometric lattice, A is the set of atoms of L, and for $X \subseteq A, \overline{X}$ is the set of atoms spanned by X, then $(A, ^-)$ is a geometry and $L \cong L(A, ^-)$.*

We give the definition of a simple independence space as follows.

Definition 3 Let $M(S)$ be an independence space. We define a *loop* of $M(S)$ to be an element x of S such that $\{x\}$ is a dependent set; and define two elements x, y of S to be *parallel* if they are not loops but $\{x, y\}$ is a dependent set.

A *simple independence space* is an independence space with no loops or parallel elements.

By Definition 3, we see that in an independence space:

x is a loop if and only if $x \in \sigma(\emptyset)$.

Distinct elements x and y are parallel if and only if $\{x, y\}$ is a circuit.

In this paper, all the knowledge of independence spaces are come from [1,2] and that of lattice theory are referred to [4].

2 Relation

In this section, we will find out the correspondent relation between a geometric lattice and a simple independence space. After that, using this correspondence, some properties of geometric lattices are solved by the way of independence space theory.

Lemma 3 *Let M be an independence space on S and $L(M)$ be the set whose elements are the flats of M. Then $(L(M), \subseteq)$ is a lattice and $A \wedge B = A \cap B, A \vee B = \cap\{X|X \in L(M), A \cup B \subseteq X\} = \sigma(A \cup B)$ for any two flats A, B.*

Proof By the definition of flats of M and Corollary 1, one has $A \wedge B = A \cap B$. In addition, $S = \sigma(S)$ and Definition 1 shows that $S \in L(M)$. Combining with Corollary 1 it follows that $A \vee B$ is well defined. Hence $(L(M), \subseteq)$ is a lattice.

It is easy to see that the least and greatest element of $(L(M), \subseteq)$ is $\sigma(\emptyset)$ and S respectively. As in [1], we often simply say $L(M)$ instead of $(L(M), \subseteq)$. $\square$

Lemma 4 *Let $M = M(S)$ be a simple independence space with σ as its closure operator. Then $L(M)$ is a geometric lattice. Conversely, if L is a geometric lattice, S is the set of atoms of L, and, for $X \subseteq A, \overline{X}$ is the set of atoms spanned by X, then $(S,^-)$ is an independence space with its closure operator σ as $^-$ and $L \cong L(S,^-)$.*

Proof Since M is simple, one has $\sigma(\emptyset) = \emptyset$ and $\sigma(x) = \{x\}$. Hence by Lemma 1 and Definition 2, (S, σ) is a geometry, and so, in light of Lemma 2, $L(M) = \{\sigma(X)|X \subseteq S\}$ is a geometric lattice.

Conversely, if L is a geometric lattice and S is the set of atoms of L. Then by Lemma 2, $(S,^-)$ is a geometry and for $X \subseteq A, \overline{X}$ is the set of atoms spanned by X. Consider $\sigma : 2^S \to 2^S$. Let $\sigma(X) = \overline{X}$ where $\overline{X}$ is the set of atoms spanned by X. By Definition 2, $\overline{X}$ is uniquely determined by X, and $\overline{X} \subseteq S$. Hence σ is actually the map $^-$, and further, according to Definition 2, Lemma

1 and Definition 3, σ is the closure operator of some simple independence space $M(L)$ on S.

By the proof of [4,p.241,Theorem 11], we see that for the geometry $(S,^-) = (S,\sigma)$, $L(S,^-) = \{\overline{X}|X \subseteq S\} = \{\sigma(X)|X \subseteq S\}$. In view of Lemma 1, $M(L)$ could be denoted as (S,σ), i.e. $(S,^-)$, and so $L(M(L)) = \{\sigma(X)|X \subseteq S\}$. However by Lemma 2, $L(S,\sigma)$ is a geometric lattice and $L \cong L(S,^-) = L(S,\sigma) = L(M(L))$, proving our Lemma. □

Based on the proofs of Lemma 4, (1) in Lemma 1 and Lemma 2, we see that the importance of simple independence spaces lies in the following theorem.

Theorem 1 *The correspondence between a geometric lattice L and the independence space $M(L)$ on the set of atoms of L is a bijection between the set of geometric lattices and the set of simple independence spaces.*

The relation between a finite simple matroid and a finite geometric lattice can be found in [1,Theorem 2, p.54], the core of [1, Chapter 3]. Recalling back the history of matroid theory, Theorem 2 in [1,p.54] is a milestone in dealing finite matroid theory with lattice theory. We believe that in infinite matroid theory, Theorem 1 is similar to that of Theorem 2 in [1, p.54].

By Theorem 1, the study of simple independence spaces is just the study of geometric lattices. Many of the interesting properties of independence spaces are preserved if we just confine attention to simple independence spaces. We will make free use of this close relationship between geometric lattices and independence spaces. It is also useful to "translate" some of the results about geometric lattices to an independence space framework.

Let M be an independence space. Using the language of geometric lattices, we get the following (I1)-(I3).
(I1) "M is *semimodular*" means that for two flats A, B of M, if $A \subset B$ and $A \subset C \subset B$ for no flats C of M, then "$\sigma(A \cup C) \subset \sigma(B \cup C)$ and $\sigma(A \cup C) \subset D \subset \sigma(B \cup C)$ for no flats D of M" or "$\sigma(A \cup C) = \sigma(B \cup C)$".
(I2) A flat A is called *compact* if and only if $A \subseteq \sigma(\cup X_\alpha)$ for some flats $X_\alpha, (\alpha \in \mathcal{A})$ implies that $A \subseteq \sigma(\cup X_\beta)$ for some $\mathcal{B} \subset\subset \mathcal{A}$ and $\beta \in \mathcal{B}$.
(I3) An *interval* $[A, B]$ of M means $[A, B] = \{X|A \subseteq X \subseteq B, X \text{ is a flat of } M\}$ for given flats $A \subseteq B$.

Following [4,p.234], an interval of a geometric lattice is a geometric lattice. It follows from [4,p.235] that any geometric lattice L is complemented, in fact, it is relatively complemented. The purely lattice-theoretic proofs for the two results are given in [4,pp.234-235& 6].

Using the relation between finite simple matroids and geometric lattices (cf.[1,p.54,Theorem 2]), Theorem 3 in [1, p.55] showed the same results as the above two lattice theoretic results for finite cases by the set-theoretic finite matroid proof. How about a general geometric lattice's proof with the application of a set-theoretic infinite matroid such as our independence space? Our Theorem 1 completely solves it by the relation between a geometric lattice and an independence space(a class of infinite matroids), henceforth, Theorem 1 presents a way to prove the above two results in set-theoretic in infinite matroids.

We sketch a set-theoretic independence space proof as the following Theorem 2.

Theorem 2 *(1) An interval of a geometric lattice L is again a geometric lattice.*

(2) Any geometric lattice L is complemented; in fact, it is relatively complemented.

Proof (1) Let $[a,b]$ be an interval of L. Let $X_\alpha \in [A,B], (\alpha \in \mathcal{A})$ where A, B are two flats of $M(L)$ and $a = \vee A, b = \vee B$. Then by Lemma 1 and Corollary 1, $A = \sigma(A) \subseteq \sigma(\bigcup_{\alpha\in\mathcal{A}} X_\alpha) \subseteq \sigma(B) = B$, $A = \sigma(A) \subseteq \sigma(\bigcap_{\alpha\in\mathcal{A}} X_\alpha) \subseteq \sigma(B) = B$, and so, $\sigma(\bigcup_{\alpha\in\mathcal{A}} X_\alpha), \sigma(\bigcap_{\alpha\in\mathcal{A}} X_\alpha) = \bigcap_{\alpha\in\mathcal{A}} X_\alpha \in [A,B]$.

Posit $E = \{\sigma(A\cup p) |\ p$ is a 1-flat of $M(L), p \notin A,$ and $p \in B\}$. Certainly, $\sigma(\sigma(A\cup p)) = \sigma(A\cup p)$. Then $E \subseteq [A,B]$ and by the born of $M(L)$, one has that E is the collection of 1-flats of $M(L)$, i.e. all 1-flats of $[A,B]$ are compact.

For $X \in [A,B]$, it is evident that X is the join of $\mathcal{T}_X$, where $\mathcal{T}_X$ is some 1-flats of $M(L)$, i.e. $X = \bigcup_{e\in\mathcal{T}_X} e$. Since $X \in [A,B]$, it has $(\mathcal{T}_X \setminus A) \subseteq B$ and $X = A \cup (\bigcup_{e\in\mathcal{T}_X\setminus A} e) = \bigcup_{e\in\mathcal{T}_X\setminus A} \sigma(A\cup e)$, i.e., X is a join of some members of E.

Put $X, Y \in [A,B], X \subset Y$ and $X \subset T \subset Y$ for no $T \in [A,B]$. Let $Z \in [A,B]$. Then by the semimodularity of $M(L)$, we have $X \cup Z = Y \cup Z$ or "$(X\cup Z) \subset (Y \cup Z)$ with $(X \cup Z) \subset T \subset (Y\cup Z)$ for no flat T of $M(L)$, especially, for no $T \in [A,B]$". Hence $([A,B], \subseteq)$ is semimodular.

Summing up, using Theorem 1 and Definition 2, we have that $[a,b]$ is geometric.

(2) Let A be a flat of $M = M(L) = (S,\mathcal{I})$ and I_A be a maximal independent set of M contained in A and let $K = S \setminus A$. Then there is a basis I of M satisfying $I_A \subseteq I \subseteq A \cup K = S$. Set $B = \sigma(I \setminus I_A)$. Since $\sigma(A\cup B) \supseteq \sigma(I_A \cup (I \setminus I_A)) = \sigma(I) = S$, one obtains $\sigma(A \cup B) = S$. Posit $C = A \cap B$. If $C \neq \sigma(\emptyset)$, then there is an member $p \in S$ satisfying $p \in C$. Since $p \in \sigma(I) = S$ and $p \in B = \sigma(I \setminus I_A)$, by the (s5), there exist $I_1 \subset\subset I$ and $I_2 \subset\subset I \setminus I_A$ such that $p \in \sigma(I_1)$ and $p \in \sigma(I_2)$. Because $p \in \sigma(I_1) \cap \sigma(I_2) = \sigma(I_1 \cap I_2) = \sigma(\emptyset)$. But M is simple, it has $\sigma(\emptyset) = \{\emptyset\}$, a contradiction. Hence, $A \cap B = \emptyset$, proving that L is complemented by Theorem 1. In view of (1) and the above result, the second statement follows. $\square$

About some other applications of Theorem 1, we will talk as follows.

(α) Suppose $M(S)$ is an independence space with σ as its closure operator and has 1-flats $T = \{F_\alpha : \alpha \in \mathcal{A}_1\}$. Let $f: F_\alpha \to a_\alpha, (\alpha \in \mathcal{A}_1, S_1 = \{a_\alpha | \alpha \in \mathcal{A}_1\})$. Then obviously, f is a bijection from T to S_1. Define $\sigma(\bigcup_{\alpha\in\mathcal{T}} F_\alpha) = \sigma'(\{a_\alpha | \alpha \in \mathcal{T}\})$ where $\mathcal{T} \subseteq \mathcal{A}_1$. We can easily verify that σ' is the closure operator of an simple independence space M_1 on S_1. We call M_1 the *simple independence space determined by* $M(S)$.

Similar to the above, for a matroid of arbitrary cardinality M_a, we can get the simple matroid of arbitrary cardinality determined by M_a. (The knowledge about a matroid of arbitrary cardinality is cf.[3,5,8].)

(β) [3] points out the correspondence between a finite length geometric lattice L_f and the matroid of arbitrary cardinality $M(L_f)$ on the set of atoms of L_f is a bijection between the set of finite length geometric lattices and the set of simple matroids of arbitrary cardinality. Therefore, consider this result with Theorem 1 and (α), we earn that every matroid of arbitrary cardinality is an independence space, but not vice versa.

By [1,p.54,Theorem 2], Theorem 1 and (α), we induce that a finite matroid is an independence space, but not vice versa.

(γ) Here we only give some instances to show that the application of Theorem 1 to study on the properties of geometric lattices by the way of independence spaces. Actually, it could be used to deal with the properties of independence spaces by the method of geometric lattice theory. For example, by [5], one sees that using the correspondence between a finite length geometric lattice and a matroid of arbitrary cardinality, some results are obtained for matroids of arbitrary cardinality by the help of geometric lattice theory. Under similar thoughts, the minors of independence spaces and so on will be seen in the near future.

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