ON FRACTIONAL ANNEALING PROCESS

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Abstract

The paper deals with a full annealing process, for example for a metal alloy perturbed by some fractional noise caused by conditions in furnace such as initial temperature, initial structure of material itself, ... The energy of the heating matter system depending on its states at each time, we consider some steady state at full annealing range by introducing a fractional model for it. And we determine states of the system related to this steady state.

1 Introduction

As we know, full anneal is a special stage of annealing process where a condensed matter (a steel for example) is heated to slightly above the critical temperature (name austenitic temperature) and maintaining the temperature for a special period of time to allow the material to fully form austenite or austenite cementite grain structure (see [1], [2], [4], [6]).

Denote by $X_t (t \geq 0), X_t \in \mathbb{R}^d$, a stochastic process presenting the state of the material under full annealing stage at time t and $g(X_t)$ is the system energy corresponding to state X_t, where g is a Borel function on $\mathbb{R}^n$.

It is well-know that a classical stochastic model for annealing states X_t is given by the following equation:

$$dX_t = -\bigtriangledown g(X_t)dt + \sqrt{2T}dW_t \qquad (1.1)$$

where W_t is a $d-$dimensional Brownian motion expressing a *no-memory* noise and T is the temperature of the system (see [3]). In our context T can be considered unchange as it is the austenitic temperature.

A problem is that full annealing treatments should be made in batch-type furnaces that are in full compliance of temperature uniformity and accuracy requirements of pyrometry AMS2750. In such type of furnace, the system state $(X_t, t \geq 0)$ is no more no-memory process because each value of X_t at t can influence upon its values some times after t. Therefore the model (1.1) is no more suitable. And we propose the following model

$$dX_t = - \bigtriangledown g(X_t)dt + \sqrt{2T}dB_t^H \qquad (1.2)$$

where:

- B_t^H is a $d-$dimensional fractional Brownian motion of Liouvill form

$$B_t^H = (B_t^{H_j}, j = 1, 2, \ldots, d),$$
$$B_t^{H_j} = \int_0^t (t - s)^{H_j - 1/2}dW_s^j, j = 1, 2, \ldots, d$$

- H_j are Hurst parameters, $0 < H_j < 1$.

- W_t^j are standard Brownian motion.

- $g = g(x_1, \ldots, x_d)$ is a Borel function $\mathbb{R}^d \to \mathbb{R}$ and $g \in C^1(\mathbb{R}^d)$.

The state process (X_t) satisfying the equation (1.2) driven by the perturbation B_t^H is a process with memory as expected.

But (1.2) is a fractional stochastic differential equation and its solution can not be found by means of traditional stochastic calculus as we can made for the equation (1.2). Many approaches have been introduced to overcome this difficulty for such kind of equation. And one of these approaches is given by Thao T.H via an approximation method. This method is presented briefly in Section 2 below.

Now let $X^* = (X_1^*, X_2^*, \ldots, X_d^*)$ be a steady state on the full annealing range that means $\frac{\partial g}{\partial x_i}(X^*) = 0, i = 1, 2, \ldots, d$. In this note we try to find the relation between a state process satisfying (1.2) and the steady state X^*.

2   Fractional Brownian motion and an approximation approach

Now we recall some facts of one-dimensional fractional Brownian motion in Mandelbrot form ([7]-[10]). It is a centered Gaussian process W_t^H such that its

covariance function is given by

$$R(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), 0 < H < 1 \tag{2.1}$$

If $H = 1/2, R(t,s) = \min(t,s)$ and W_t^H is an usual standard Brownian motion. If $H \neq 1/2, W_t^H$ is a process of long memory, and Ito calculus cannot applied to it. And an approximation approach is given as follows ([7]).

It is known that W_t^H can be decomposed as

$$W_t^H = \frac{1}{\Gamma(\alpha+1)} \left[Z_t + \int_0^t (t-s)^\alpha dW_s \right], \alpha = H - 1/2 \tag{2.2}$$

where Z_t is a process having absolutely continuous paths and the long memory property focus at the term

$$B_t^H = \int_0^t (t-s)^\alpha dW_s, \alpha = H - 1/2 \tag{2.3}$$

which is called the fractional Brownian motion of the Liouville form. Since B_t^H is not a semimartingale we introduce a new process for each $\epsilon > 0$

$$B_t^{H,\epsilon} = \int_0^t (t-s+\epsilon)^\alpha dW_s \tag{2.4}$$

And the following two important facts have been proved in [7] and [10].

(i) $B_t^{H,\epsilon}$ is a semimartingale:

$$dB^{H,\epsilon} = \alpha \varphi_t^\epsilon dt + \epsilon^\alpha dW_t$$
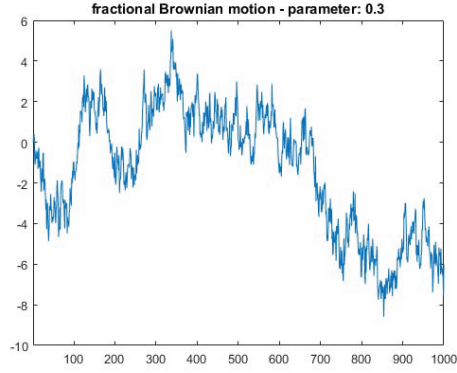
where

$$\varphi_t^\epsilon = \int_0^t (t-s+\epsilon)^{\alpha-1} dW_s$$

(ii) $B_t^{H,\epsilon}$ converges to B_t^H in $L^2(\Omega)$ as ϵ tends to 0. And this convergence is uniform with respect to t belonging to any finite interval I of time, and we have also
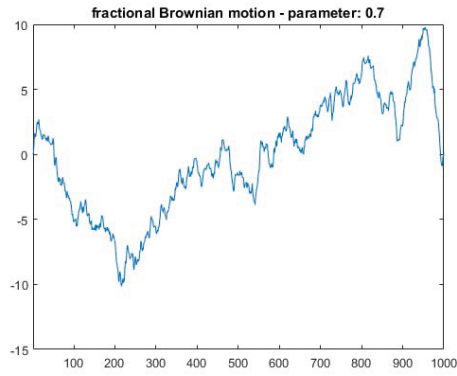
$$\sup_{t \in I} \left\| B_t^{H,\epsilon} - B_t^H \right\| \leq C(\alpha)\epsilon^{2H} \tag{2.5}$$

where $C(\alpha)$ is a positive constant depending only to $\alpha = H - 1/2$.

Apart from this, a study of fractional stochastic differential equations is presented in [8], [9] and [10].

Fractional Brownian motion with H =0.3 and d = 1

Fractional Brownian motion with H =0.7 and d = 1

3   Fractional annealing process

For the sake of the simplicity we consider the case of two dimensions $d = 2$. The case of general dimension d can be naturally extended. So we are supposed to consider the following model

$$dX_t = - \bigtriangledown g(X_t)dt + \sqrt{2T}dB_t^H \qquad (3.0)$$

where

$$X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix}, g = g(x_1, x_2), B_t^H = \begin{pmatrix} B_t^{H_1} \\ B_t^{H_2} \end{pmatrix}$$

$$B_t^{H_j} = \int_0^t (t-s)^{\alpha_j} dW_j, \alpha_j = H_j - 1/2, 0 < H_j < 1, j = 1, 2$$

Let $X^* = (X_1^*, X_2^*)$ be a steady state on full annealing range where $\frac{\partial g}{\partial x_1}(X^*) = 0, \frac{\partial g}{\partial x_2}(X^*) = 0$. We have to calculate state satisfying (1.6) and relating to X^*.

3.1   A change of variables

Denote by $F_1(x_1, x_2)$ and $F_2(x_1, x_2)$ be the partial derivatives of $g(x_1, x_2)$:

$$F_1(x) = F_1(x_1, x_2) = -\frac{\partial}{\partial x_1} g(x_1, x_2)$$
$$F_2(x) = F_2(x_1, x_2) = -\frac{\partial}{\partial x_2} g(x_1, x_2)$$

and

$$F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}$$

The equation (1.2) can be rewritten as following

$$dX_t = F(X_t)dt + \sqrt{2T}dB_t^H \tag{3.1.1}$$

or

$$\begin{cases} dX_t^1 = F_1(X_t)dt + \sqrt{2T}dB_t^{H_1} \\ dX_t^2 = F_2(X_t)dt + \sqrt{2T}dB_t^{H_2} \end{cases}$$

We have now

$$F_1(X^*) = 0 \text{ and } F_2(X^*) = 0$$

Define a change of variable

$$X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} \rightarrow U_t = \begin{pmatrix} U_t^1 \\ U_t^2 \end{pmatrix} = X_t - X^* = \begin{pmatrix} X_t^1 - X^{1*} \\ X_t^2 - X^{2*} \end{pmatrix} \tag{3.1.2}$$

We can see that

$$\begin{pmatrix} dU_t^1 \\ dU_t^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(X^*) & \frac{\partial F_1}{\partial x_2}(X^*) \\ \frac{\partial F_1}{\partial x_2}(X^*) & \frac{\partial F_2}{\partial x_2}(X^*) \end{pmatrix} \begin{pmatrix} U_t^1 \\ U_t^2 \end{pmatrix} dt + \sqrt{2T} \begin{pmatrix} dB_t^{H_1} \\ dB_t^{H_2} \end{pmatrix} \tag{3.1.3}$$

or

$$dU_t = AU_t dt + \sqrt{2T}dB_t^H \tag{3.1.4}$$

where

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} := \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(X^*) & \frac{\partial F_1}{\partial x_2}(X^*) \\ \frac{\partial F_1}{\partial x_2}(X^*) & \frac{\partial F_2}{\partial x_2}(X^*) \end{pmatrix} \tag{3.1.5}$$

Then we can write (3.1.3) as follows

$$\begin{cases} dU_t^1 = (a_1 U_t^1 + b_1 U_t^2)dt + \sqrt{2T}dB_t^{H_1} \\ dU_t^2 = (a_2 U_t^1 + b_2 U_t^2)dt + \sqrt{2T}dB_t^{H_2} \end{cases} \tag{3.1.6}$$

3.2   Approximate model

Now we can consider an approximate model corresponding to (3.1.4) or (3.1.6) by replacing $B_t^H = \begin{pmatrix} B_t^{H_1} \\ B_t^{H_2} \end{pmatrix}$ by $B_t^{H,\epsilon} = \begin{pmatrix} B_t^{H_1,\epsilon_1} \\ B_t^{H_2,\epsilon_2} \end{pmatrix}$ as introduced in section 1.

$$dU_t^\epsilon = AU_t^\epsilon dt + \sqrt{2T}dB_t^{H,\epsilon} \tag{3.2.1}$$

and

$$\begin{cases} dU_t^{1,\epsilon_1} = (a_1 U_t^{1,\epsilon_1} + b_1 U_t^{2,\epsilon_2})dt + \sqrt{2T}dB_t^{H_1,\epsilon_1} \\ dU_t^{2,\epsilon_2} = (a_2 U_t^{1,\epsilon_1} + b_2 U_t^{2,\epsilon_2})dt + \sqrt{2T}dB_t^{H_2,\epsilon_2} \end{cases} \tag{3.2.2}$$

where $B_t^{H_j,\epsilon_j} \to B_t^{H_j}$ in the space $L^2 = L^2(\Omega)$ of square integrable processes, and

$$\left\| B_t^{H_j,\epsilon_j} - B_t^{H_j} \right\|_{L^2} \le C_j(\alpha_j)\epsilon_j^{2H_j}, \alpha_j = H_j - 1/2, j = 1, 2 \tag{3.2.3}$$

with $C(\alpha_j)$ depending only on α_j (see [7]).

3.3   Convergence

Before finding the solution U_t^ϵ of (3.2.1) we will prove a convergence theorem.

Theorem 3.1. *The solution U_t^ϵ of (3.2.1) tend to the solution U_t of (2.7) in L^2 as $\epsilon \to 0$, that means*

$$U_t^{1,\epsilon_1} \to U_t^1 \ in \ L^2 \ as \ \epsilon_1 \to 0$$

and

$$U_t^{2,\epsilon_2} \to U_t^2 \ in \ L^2 \ as \ \epsilon_2 \to 0$$

Proof. We have

$$U_t^1 - U_t^{1,\epsilon_1} = \int_0^t \left[a_1(U_t^1 - U^{1,\epsilon_1}) + b_1(U_t^2 - U_t^{2,\epsilon_2}) \right] ds + \sqrt{2T}(B_t^{H_1} - B_t^{H_1,\epsilon_1}) \tag{3.3.1}$$

$$U_t^2 - U_t^{2,\epsilon_2} = \int_0^t \left[a_2(U_t^1 - U^{1,\epsilon_1}) + b_2(U_t^2 - U_t^{2,\epsilon_2}) \right] ds + \sqrt{2T}(B_t^{H_2} - B_t^{H_2,\epsilon_2}) \tag{3.3.2}$$

□

Put $M = \max\left(\frac{|a_1|}{2}, \frac{|a_2|}{2}, \frac{|b_1|}{2}, \frac{|b_2|}{2} \right)$. Then it follows from (3.3.1), (3.3.2) and (2.5) that

$$\left\| U_t^1 - U_t^{1,\epsilon_1} \right\| + \left\| U_t^2 - U_t^{2,\epsilon_2} \right\| \le C(\alpha, \epsilon) + M \int_0^1 \left[\left\| U_s^1 - U_s^{1,\epsilon_1} \right\| + \left\| U_t^2 - U_t^{2,\epsilon_2} \right\| \right] ds \tag{3.3.3}$$

where

$$C(\alpha, \epsilon) = \sqrt{2T} \left(C_1(\alpha_1)\epsilon_1^{1+2\alpha_1} + C_2(\alpha_2)\epsilon_2^{1+2\alpha_2} \right) \to 0$$

as $\epsilon = (\epsilon_1, \epsilon_2) \to 0$ Then an application of Gronwall inequality gives us

$$\left\| U_t^1 - U_t^{1,\epsilon_1} \right\| + \left\| U_t^2 - U_t^{2,\epsilon_2} \right\| \le C(\alpha, \epsilon) \exp(Mt) \to 0$$

as $\epsilon \to 0$. So that

$$U_t^\epsilon = \begin{pmatrix} U_t^{1,\epsilon_1} \\ U_t^{2,\epsilon_2} \end{pmatrix} \to \begin{pmatrix} U_t^1 \\ U_t^2 \end{pmatrix} = U_t \text{ in } L^2 \text{ as } \epsilon = (\epsilon_1, \epsilon_2) \to 0 \qquad (3.3.4)$$

3.4   Solution U_t^ϵ of (3.2.1)

Now we have to find the solution of U_t^ϵ in its explicit form. We have

$$dU_t^\epsilon = AU_t^\epsilon dt + \sqrt{2T} dB_t^{H,\epsilon}$$

where

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, U_t^\epsilon = \begin{pmatrix} U_t^{H_1,\epsilon_1} \\ U_t^{H_2,\epsilon_2} \end{pmatrix}, B_t^\epsilon = \begin{pmatrix} B_t^{H_1,\epsilon_1} \\ B_t^{H_2,\epsilon_2} \end{pmatrix}$$

It is easy to see that

$$U_t^\epsilon = \int_0^t e^{A(t-s)} \sqrt{2T} dB_t^{H,\epsilon} \qquad (3.4.1)$$

A method by Oksendal can be applied ([6], page 99) to obtain

$$e^{A(t-s)} = \frac{e^{-\lambda(t-s)}}{\xi} [\xi \cos \xi(t-s) + \lambda \sin \xi(t-s) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + A \sin \xi(t-s)] \qquad (3.4.2)$$

where

$$\lambda = -\frac{b_2}{2}, \xi = \left| a_2 - \frac{b_2^2}{4} \right| \qquad (3.4.3)$$

(3.4.2) can be rewritten also as

$$\begin{aligned} e^{A(t-s)} &= \frac{e^{-\lambda(t-s)}}{\xi} \begin{bmatrix} \xi \cos \xi(t-s) + \lambda \sin \xi(t-s) & 0 \\ 0 & \xi \cos \xi(t-s) + \lambda \sin \xi(t-s) \end{bmatrix} + \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \sin \xi(t-s) \\ &= \frac{e^{-\lambda(t-s)}}{\xi} \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \end{aligned} \qquad (3.4.4)$$

where

$$\begin{aligned} A_1 &= \xi \cos(t-s) + (\lambda + a_1) \sin(t-s) \\ B_1 &= b_1 \sin(t-s) \\ A_2 &= a_2 \sin(t-s) \\ B_2 &= \xi \cos(t-s) + (\lambda + b_2) \sin(t-s) \end{aligned} \qquad (3.4.5)$$

Then it follow from (3.4.1), (3.4.3) and (3.4.4) that we can state now:

Theorem 3.2. *The solution of the approximate model (3.2.1) given by*

$$U_t^\epsilon = \begin{pmatrix} U_t^{1,\epsilon_1} \\ U_t^{2,\epsilon_2} \end{pmatrix} = \frac{\sqrt{2T}}{\xi} \int_0^t \begin{pmatrix} A_1 dB_s^{H_1,\epsilon_1} + B_1 dB^{H_2,\epsilon_2} \\ A_2 dB_s^{H_1,\epsilon_1} + B_2 dB^{H_2,\epsilon_2} \end{pmatrix} \qquad (3.4.6)$$

where A_1, B_1, A_2, B_2 and ξ are defined in (3.4.5), (3.4.3) and (3.1.5).

In account of the Convergence Theorem 3.1 we can obtain

Theorem 3.3. *Given a steady state X^* in the full fractional annealing range the state X_t of the system is the limit in the spase $L^2(\Omega)$ of X_t^ϵ when ϵ tends to 0:*

$$X_t = L^2 - \lim_{\epsilon \to 0} X_t^\epsilon$$

where $X_t^\epsilon = X^* + U_t^\epsilon$ and U_t^ϵ is obtained from (3.4.6).

Conclusion

Then each state X_t of the annealing system can be considered as a L^2-limit of $(X^* + U_t^\epsilon)$ as $\epsilon \to 0$ where X^* is a steady state of full annealing range.

Acknowledgments

References

[1] Y.A. Cengel, M.A. Boles, Thermodynamics - An Engineering Approach, Mc Grall Hill, 2005.

[2] R. Bowley, M. Sanchez, Introductory Statistical Mechanics, Oxford University Press, 2000.

[3] B. Gidas, Metropolis-type Monte-Carlo Simulation Algorithms and Simulated Annealing, Providence, RI Brown University, 1991.

[4] L.H. Van Vlack, Elements of Material Science and Engineering, Addition-Wesley, 1995.

[5] J.D. Verhoeven, Fundamentals of Physical Metallurgy, Wiley, New York, 1975.

[6] B. Oksendal, Stochastic Differential Equation, 6th Edition, Springer, 2003.

[7] T.H. Thao, An approximate approach to fractional analysis for finance, Nonlinear Analysis: Real W. Appl, Elsevier 7(1) (2006) 124–132.

[8] T.H. Thao, A practical approach to fractional stochastic dynamics, J. Comput., Nonlinear Dyn (American Association of Mechanical Engineers-ASME) 8,031015 (2013) 1–5.

[9] T.H. Thao, T.T. Nguyen, Fractal Langevin equation, Vietnam Journal of Mathematics 30(1) (2003) 89–96.

[10] T.H. Thao, On some classes of fractional stochastic dynamical systems, East-West J. of Math 15(1) (2013) 54–69.