

NOTE ON JORDAN TRIPLE $(\alpha, \beta)^*$ -DERIVATIONS IN H^* -ALGEBRAS

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Abstract

The main purpose of this paper is to prove the following result. Let R be a 2-torsion free semiprime $*$ -ring and α, β are endomorphisms of R . Then any Jordan triple $(\alpha, \beta)^*$ -derivation on R is a Jordan $(\alpha, \beta)^*$ -derivation. As an application of this result, we establish that any linear Jordan triple $(\alpha, \beta)^*$ -derivation on a semisimple H^* -algebra is a linear Jordan $(\alpha, \beta)^*$ -derivation.

1 Introduction

This research is inspired by our earlier work [1] and the work of M. Fošner and D. Ilašević [5]. Throughout, R will denote an associative ring and A will represent a $*$ -algebra over the field F . Let $n \geq 2$ be an integer. A ring R is said to be n -torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that R is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. A ring R is called semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping $x \mapsto x^*$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$ is called an involution. A ring equipped with an involution is called a $*$ -ring or a ring with involution. If R is an algebra we assume additionally that $(\lambda x)^* = \bar{\lambda}x^*$ for all $x, y \in R$ and $\lambda \in F$, where $\bar{\lambda}$ denotes the complex conjugate of λ . An algebra equipped with an involution is called a $*$ -algebra or algebra with involution. The radical of A , denoted by $rad(A)$, is the intersection of all maximal left(or right) ideals of A . An algebra

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A is called semisimple if $\text{rad}(A) = 0$. A Banach algebra is a linear associative algebra which, as a vector space, is a Banach space with norm $\|\cdot\|$ satisfying the multiplicative inequality; $\|xy\| \leq \|x\|\|y\|$ for all x and y in A . A $*$ -algebra which is also a Banach algebra is called a Banach $*$ -algebra. Let us recall that a semisimple H^* -algebra is a semisimple Banach $*$ -algebra whose norm is a Hilbert space norm such that $(x, yz^*) = (xz, y) = (z, x^*y)$ is fulfilled for all $x, y, z \in A$ (see [2] for details).

An additive mapping $d : R \rightarrow R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. An additive mapping $d : R \rightarrow R$ is called a Jordan triple derivation if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$. Of course, any derivation is a Jordan triple derivation. Moreover, if R is a 2-torsion free, one can easily prove that any Jordan derivation is a Jordan triple derivation, but converse is not true in general. A classical result due to Brešar ([3], Theorem 4.3) asserts that a Jordan triple derivation on 2-torsion free semiprime ring is a derivation.

Let R be a $*$ -ring, and let α, β be endomorphisms of R . An additive mapping $d : R \rightarrow R$ is said to be a $*$ -derivation (resp. Jordan $*$ -derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) holds for all $x, y \in R$. Following [5], an additive mapping $d : R \rightarrow R$ is called a Jordan triple $*$ -derivation if $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ holds for all $x, y \in R$. One can easily prove that every Jordan $*$ -derivation on a 2-torsion free $*$ -ring is a Jordan triple $*$ -derivation (see the proof of [[4], Lemma 2]) but not conversely. In [7], P. Šemrl has proved that if R is a real Banach $*$ -algebra with identity then the converse also holds. Further, Vukman [8] established that any Jordan triple $*$ -derivation on a 6-torsion free semiprime $*$ -ring is a Jordan $*$ -derivation. In the year 2008, M. Fošner and D. Ilišević [5] generalized this result for 2-torsion free semiprime $*$ -rings.

In [1], the notion of Jordan triple $*$ -derivation was extended as follows: an additive mapping $d : R \rightarrow R$ is called a Jordan triple $(\alpha, \beta)^*$ -derivation if $d(xyx) = d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ holds for all $x, y \in R$, where α and β are endomorphisms of R . In any $*$ -ring with automorphisms α and β , the mapping $x \mapsto \alpha\alpha(x^*) - \beta(x)a$, where a is fixed element in R , is a Jordan triple $(\alpha, \beta)^*$ -derivation on R . Note that for I_R , the identity map on R , a Jordan triple $(I_R, I_R)^*$ -derivation is just a Jordan triple $*$ -derivation. Using similar approach as in Lemma 2 of [4], it can be easily seen that any Jordan $(\alpha, \beta)^*$ -derivation on a 2-torsion free $*$ -ring is a Jordan triple $(\alpha, \beta)^*$ -derivation, but not conversely (cf.; [[1], Example 2.4]). Most recently, the author together with A. Fošner [1] proved that on a 6-torsion free semiprime $*$ -ring R , every Jordan triple $(\alpha, \beta)^*$ -derivation on R is a Jordan $(\alpha, \beta)^*$ -derivation.

The main goal of this paper is to improve the above mentioned result by removing 3-torsion free restriction and prove that any Jordan triple $(\alpha, \beta)^*$ -derivation on 2-torsion free semiprime $*$ -ring is a Jordan $(\alpha, \beta)^*$ -derivation. As consequence of this result, it was shown that any linear Jordan triple $(\alpha, \beta)^*$ -derivation on a semisimple H^* -algebra is a linear Jordan $(\alpha, \beta)^*$ -derivation.

2 The main results

The following theorem is a generalization of [[5], Theorem 5.2] and [[8], Theorem 1].

Theorem 1. *Let R be a 2-torsion free semiprime $*$ -ring, and let α, β be surjective endomorphisms of R . Let $d : R \rightarrow R$ be an additive mapping. Then the following conditions are equivalent:*

- (i) d is a Jordan $(\alpha, \beta)^*$ -derivation;
- (ii) $d(xy) = d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ for all $x, y \in R$.

In order to prove above theorem, first we establish the following technical lemma which extended the result of [[6], Section 2, p-5].

Lemma 1. *Let R be a semiprime $*$ -ring, and let α, β be surjective endomorphisms of R . If there exists element $x \in R$ such that $\beta(y)x\alpha(y^*) = 0$ for all $y \in R$ or $\alpha(y)x\beta(y^*) = 0$ for all $y \in R$, then $x = 0$.*

Proof. We consider the relation $\beta(y)x\alpha(y^*) = 0$ for all $y \in R$. Replacing y by $y^* + z$ we obtain

$$\beta(y^*)x\alpha(y) + \beta(z)x\alpha(y) + \beta(y^*)x\alpha(z^*) + \beta(z)x\alpha(z^*) = 0 \text{ for all } y, z \in R. \quad (1)$$

This implies that

$$\beta(z)x\alpha(y) + \beta(y^*)x\alpha(z^*) = 0, \text{ for all } y, z \in R. \quad (2)$$

Replace z^* by y in (2) to get

$$\beta(z)x\alpha(y) + \beta(y^*)x\alpha(y) = 0 \text{ for all } y, z \in R. \quad (3)$$

and hence in view of our hypothesis we obtain

$$\beta(z)x\alpha(y) = 0 \text{ for all } y, z \in R.$$

Using the last relation, we find that

$$\begin{aligned}(x\beta(z)x)\alpha(y)(x\beta(z)x) &= x(\beta(z)x\alpha(y))(x\beta(z)x) \\ &= 0 \text{ for all } y, z \in R.\end{aligned}\quad (4)$$

Therefore, we find that $(x\beta(z)x)\alpha(y)(x\beta(z)x) = 0$ for all $y, z \in R$. Since α is a surjective endomorphism of R , so we have $(x\beta(z)x)R(x\beta(z)x) = \{0\}$ for all $z \in R$. The semiprimeness of R forces that $x\beta(z)x = 0$ for all $z \in R$. Since R is semiprime and β is surjective endomorphism of R , we conclude that $x = 0$.

Using similar arguments one can prove that if $\alpha(y)x\beta(y^*) = 0$ for all $y \in R$, then $x = 0$. The proof is complete. \square

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. We proceed to prove (i) implies (ii). Suppose that d is a Jordan $(\alpha, \beta)^*$ -derivation i.e.,

$$d(x^2) = d(x)\alpha(x^*) + \beta(x)d(x) \text{ for all } x \in R. \quad (5)$$

The linearization of the above relation yields that

$$\begin{aligned}d(xy + yx) &= d(x)\alpha(y^*) + \beta(x)d(y) + d(y)\alpha(x^*) \\ &\quad + \beta(y)d(x) \text{ for all } x, y \in R.\end{aligned}\quad (6)$$

Replacing y by $xy + yx$ in (6), then on one hand we find that

$$\begin{aligned}d(x(xy + yx) + (xy + yx)x) &= d(x)\alpha(x^*y^*) + d(x)\alpha(y^*x^*) + \beta(x)d(x)\alpha(y^*) \\ &\quad + \beta(x^2)d(y) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x) \\ &\quad + d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + d(y)\alpha(x^{*2}) \\ &\quad + \beta(y)d(x)\alpha(x^*) + \beta(xy)d(x) + \beta(yx)d(x) \text{ for all } x, y \in R.\end{aligned}\quad (7)$$

On the other hand, we have

$$\begin{aligned}d(x(xy + yx) + (xy + yx)x) &= d(x^2y + yx^2) + 2d(xyx) \\ &= d(x)\alpha(x^*y^*) + \beta(x)d(x)\alpha(y^*) + \beta(x^2)d(y) \\ &\quad + d(y)\alpha(x^{*2}) + \beta(y)d(x)\alpha(x^*) + \beta(yx)d(x) \\ &\quad + 2d(xyx) \text{ for all } x, y \in R.\end{aligned}\quad (8)$$

Combining (7) and (8), we obtain

$$2d(xyx) = 2\{d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)\} \text{ for all } x, y \in R.$$

Since R is 2-torsion free, the last expression forces that

$$d(xyx) = d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x) \text{ for all } x, y \in R; \quad (9)$$

and hence d is Jordan triple $(\alpha, \beta)^*$ -derivation on R .

Let us prove the converse part *i.e.*, (ii) implies (i). Suppose relation (9) holds. Replace x by $x + z$ in (9) to get

$$\begin{aligned}
d((x+z)y(x+z)) &= d(x+z)\alpha(y^*)\alpha(x^*+z^*) + \beta(x+z)d(y)\alpha(x^*+z^*) \\
&\quad + \beta(x+z)\beta(y)d(x+z) \\
&= d(x)\alpha(y^*x^*) + d(z)\alpha(y^*x^*) + d(x)\alpha(y^*z^*) + d(z)\alpha(y^*z^*) \\
&\quad + \beta(x)d(y)\alpha(x^*) + \beta(z)d(y)\alpha(x^*) + \beta(x)d(y)\alpha(z^*) \\
&\quad + \beta(z)d(y)\alpha(z^*) + \beta(xy)d(x) + \beta(zy)d(x) + \beta(xy)d(z) \\
&\quad + \beta(zy)d(z) \quad \text{for all } x, y, z \in R. \tag{10}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d((x+z)y(x+z)) &= d(xyx) + d(zyz) + d(xyz + zyx) \\
&= d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x) \\
&\quad + d(z)\alpha(y^*z^*) + \beta(z)d(y)\alpha(z^*) + \beta(zy)d(z) \\
&\quad + d(xyz + zyx) \quad \text{for all } x, y, z \in R. \tag{11}
\end{aligned}$$

Comparing (10) and (11), we arrive at

$$\begin{aligned}
d(xyz + zyx) &= d(x)\alpha(y^*z^*) + \beta(x)d(y)\alpha(z^*) + \beta(xy)d(z) \\
&\quad + d(z)\alpha(y^*x^*) + \beta(z)d(y)\alpha(x^*) \\
&\quad + \beta(zy)d(x) \quad \text{for all } x, y, z \in R. \tag{12}
\end{aligned}$$

Since d is additive, so for any $x, y \in R$, we have

$$d((xy)^2) = d(xyxy) = d(xy(xy) + (xy)yx - xy^2x) = d(xy(xy) + (xy)yx) - d(xy^2x).$$

Application of (9) and (12) yields that

$$\begin{aligned}
d((xy)^2) &= d(x)\alpha(y^*)\alpha(y^*x^*) + \beta(x)d(y)\alpha(y^*x^*) + \beta(xy)d(xy) \\
&\quad + d(xy)\alpha(y^*x^*) + \beta(xy)d(y)\alpha(x^*) + \beta(xy)\beta(y)d(x) \\
&\quad - d(x)\alpha(y^{*2})\alpha(x^*) - \beta(x)d(y^2)\alpha(x^*) \\
&\quad - \beta(xy^2)d(x) \quad \text{for all } x, y \in R. \tag{13}
\end{aligned}$$

For any $x, y \in R$, the above relation implies that

$$\begin{aligned}
d(xy)^2 - d(xy)\alpha(y^*x^*) - \beta(xy)d(xy) \\
+ \beta(x)(d(y^2) - d(y)\alpha(y^*) - \beta(y)d(y))\alpha(x^*) = 0. \tag{14}
\end{aligned}$$

This can be rewritten as

$$\delta(xy) + \beta(x)\delta(y)\alpha(x^*) = 0 \text{ for all } x, y \in R; \quad (15)$$

where $\delta(x) = d(x^2) - d(x)\alpha(x^*) - \beta(x)d(x)$ for all $x \in R$. Using equation (15) three times, we find that

$$\begin{aligned} 2\beta(z)y\delta(x)\alpha(y^*z^*) &= \beta(z)(\beta(y)\delta(x)\alpha(y^*))\alpha(z^*) + \beta(z)y\delta(x)\alpha(y^*z^*) \\ &= \beta(z)(-\delta(yx)\alpha(z^*)) - \delta((zy)x) \\ &= -\beta(z)\delta(yx)\alpha(z^*) - \delta(zyx) \\ &= -\beta(z)\delta(yx)\alpha(z^*) - \delta(zyx) \\ &= \delta(z(yx)) - \delta(zyx) \\ &= 0 \text{ for all } x, y, z \in R. \end{aligned} \quad (16)$$

This implies that

$$2\beta(zy)\delta(x)\alpha(y^*z^*) = 0 \text{ for all } x, y, z \in R. \quad (17)$$

Since R is 2-torsion free, the above expression forces that $\beta(zy)\delta(x)\alpha(y^*z^*) = 0$ for all $x, y, z \in R$ i.e., $\beta(z)(\beta(y)\delta(x)\alpha(y^*))\alpha(z^*) = 0$ for all $x, y, z \in R$. Application of Lemma 1 yields that $\beta(y)\delta(x)\alpha(y^*) = 0$ for all $x, y \in R$. Again, using Lemma 1, we obtain $\delta(x) = 0$ i.e., $d(x^2) = d(x)\alpha(x^*) + \beta(x)d(x)$ for all $x \in R$. Hence, d is a Jordan $(\alpha, \beta)^*$ -derivation on R . This completes the proof of our theorem.

Following are the immediate consequences of Theorem 1.

Corollary 1. *Let R be a 2-torsion free semisimple $*$ -ring, and let α, β be surjective endomorphism of R . Let $d : R \rightarrow R$ be an additive mapping. Then the following conditions are equivalent:*

- (i) d is a Jordan $(\alpha, \beta)^*$ -derivation;
- (ii) $d(xyx) = d(x)\alpha(y^*x^*) + \alpha(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ for all $x, y \in R$.

Proof. As a consequence of Theorem 1 and of the fact that every simple $*$ -ring is a semiprime $*$ -ring.

Corollary 2. (*[5], Theorem 5.2*) *Let R be a 2-torsion free semiprime $*$ -ring, and let $d : R \rightarrow R$ be an additive mapping. Then the following condition, are mutually equivalent:*

- (i) d is a Jordan $*$ -derivation;

(ii) $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ for all $x, y \in R$.

Corollary 3. ([8], Theorem 1). Let R be a 6-torsion free semiprime $*$ -ring and let $d : R \rightarrow R$ be an additive mapping satisfying the relation

$$d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$$

for all $x, y \in R$. Then d is a Jordan $*$ -derivation on R .

Finally, we prove another theorem in the spirit of Theorem 1, that is,

Theorem 2. Let A be a semisimple H^* -algebra. Suppose that α and β are surjective homomorphisms of A . Let $d : A \rightarrow A$ be a linear mapping. Then the following conditions are equivalent:

(i) d is a Jordan $(\alpha, \beta)^*$ -derivation;

(ii) $d(xyx) = d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ for all $x, y \in A$.

Proof. By the structure theorem of semisimple H^* -algebra (see [2]), every semisimple H^* -algebra is a semiprime $*$ -ring and hence proof is complete by Theorem 1.

Corollary 4. Let A be a semisimple H^* -algebra. Let $d : A \rightarrow A$ be a linear mapping. Then the following conditions are equivalent:

(i) d is a Jordan $*$ -derivation;

(ii) $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ for all $x, y \in A$.

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