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SEMILATTICE CONGRUENCE AND FUZZY SEMILATTICE CONGRUENCE ON PO-Γ-SEMIGROUP VIA ITS OPERATOR SEMIGROUPS

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Abstract

In this paper, semilattice congruence and fuzzy semilattice congruence on po-Γ-semigroup are studied via operator semigroups. Among other results we obtain a lattice isomorphism between fuzzy semilattice congruences of a Γ-semigroup and that of its left operator semigroup. Using this result we have shown that any sublattice of the lattice of all fuzzy semilattice congruences of a po-Γ-semigroup is modular.

1 Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation. The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics, e.g., coding and language theory, automata theory, combinatorics and mathematical analysis. In 1981, Sen and Saha [19] defined the notion of a

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Γ-semigroup as a generalization of a semigroup. Many classical notions of semigroups have been extended to Γ-semigroups.

After the introduction of fuzzy sets by Zadeh[24], reconsideration of some concepts of classical mathematics began. Many properties of semigroups have been studied in terms of fuzzy subsets. Among other references we refer the readers to Kuroki's monograph[10]. Since Γ-semigroup generalizes semigroup, it is natural to investigate Γ-semigroups in terms of fuzzy subsets. In this direction we may refer to [14, 15, 16, 17]. Fuzzy relations were defined by Zadeh[25]. Since then, fuzzy equivalence relations and fuzzy congruence relations were studied. Like congruence, fuzzy congruence plays an important role in the theory of semigroups. Readers may refer to [1, 2, 7, 8, 9, 11, 12, 13]. In this paper, as continuation of our study of po-Γ-semigroups[18], we introduce the notion of semilattice congruence and fuzzy semilattice congruence and study them using operator semigroups. For basic notions of po-Γ-semigroups and fuzzy concepts in po-Γ-semigroups we refer respectively to [21],[18]. For basic notions of semilattice congruence in semigroups and Γ-semigroups we refer respectively to [6],[23].

2 Preliminaries

Definition 2.1. [20] Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma....\}$ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S\times\Gamma\times S \rightarrow S$(images to be denoted by $a\alpha b$) satisfying (1) $x\gamma y \in S$, (2) $(x\beta y)\gamma z = x\beta(y\gamma z)$ for all $x, y, z \in S, \alpha, \beta, \gamma \in \Gamma$.

Remark 1. Definition 2.1 is the definition of one sided Γ-semigroup. It may be noted here that in 1981 *Sen*[19] introduced the notion of both sided Γ-semigroups.

Note 1. Any semigroup can be considered to be a Γ-semigroup.

Example 1. Let S be the set of all 2×3 matrices over the set of positive integers and Γ be the set of all 3×2 matrices over same set. Then S is a both-sided Γ-semigroup with respect to the usual matrix multiplication.

The following example shows that there exists a one sided Γ-semigroup which is not a both sided Γ-semigroup.

Example 2. Let S be a set of all negative rational numbers. Obviously S is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p}$: p is prime $\}$. Let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence S is a one sided Γ-semigroup. It is also clear that it is not a both sided Γ-semigroup.

Definition 2.2 (21). A Γ-semigroup S is said to be po-Γ-semigroup(partially ordered Γ-semigroup) if (1) S and Γ are posets, (2) $a \leq b$ in S implies that $a\alpha c \leq b\alpha c$, $c\alpha a \leq c\alpha b$ in S for all $c \in S$ and for all $\alpha \in \Gamma$, (3) $\alpha \leq \beta$ in Γ implies that $a\alpha b \leq a\beta b$ for all $a, b \in S$.

Remark 2. The partial order relations on S and Γ are denoted by the same symbol $\leq$.

Remark 3. Definition 2.2 is the definition of one sided po-Γ-semigroup. It may be noted here that *T.K. Dutta and N.C. Adhikari*[5] introduced the notion of both sided po-Γ-semigroup and also introduced the notions of operator semigroups of a both sided po-Γ-semigroup.

Throughout this paper unless otherwise mentioned S stands for one sided Γ-semigroup.

Example 3. [5] The Γ-semigroup in Example 1 is a po-Γ-semigroup with respect to $\leq$ defined by $(a_{ik}) \leq (b_{ik})$ if and only if $a_{ik} \leq b_{ik}$ for all i, k. Then S is a po-Γ-semigroup.

Definition 2.3. [24] A fuzzy subset μ of a non-empty set X is a function $\mu : X \to [0, 1]$.

Definition 2.4 (24). Let μ be a fuzzy subset of a non-empty set X. Then the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ for $t \in [0, 1]$, is called a level subset or the t-level subset of μ.

Definition 2.5. [3]Let S be a Γ-semigroup. Then the relation ρ on $S \times \Gamma$, defined by $(x, \alpha)\rho(y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$, is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α). Let $L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$. Then L is a semigroup with respect to the multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. This semigroup L is called the left operator semigroup of the Γ-semigroup S. Dually the right operator semigroup R of S is defined where the multiplication is defined by $[\alpha, a][\beta, b] = [\alpha, a\beta b]$.

If there exists an element $[e, \delta] \in L([\gamma, f] \in R)$ such that $e\delta s = s$(resp. $s\gamma f = s$) for all $s \in S$ then $[e, \delta]$(resp. $[\gamma, f]$) is called the left(resp. right) unity of S.

Note 2. The left(right) unity of S is the identity of L(respectively R).

From Definition 2.2 we easily obtain the following:

Proposition 2.6. *Let S be a Γ-semigroup with S, Γ are posets. Then S is a po-Γ-semigroup if and only if $a \leq b$, $\alpha \leq \beta$, $c \leq d \Rightarrow a\alpha c \leq b\beta d$, for all $a, b, c, d \in S$ and $\alpha, \beta \in \Gamma$.*

Now we easily obtain the following theorems.

Theorem 2.7. *Let S be a po-Γ-semigroup. Then the left operator semigroup L and the right operator semigroup R of S are po-semigroups where $[a,\alpha] \leq [b,\beta]$ in L if and only if $\forall s \in S$, $a\alpha s \leq b\beta s$ in S and $[\alpha,a] \leq [\beta,b]$ in R if and only if $\forall s \in S$, $s\alpha a \leq s\beta b$ in S.*

Theorem 2.8. *Let S be a Γ-semigroup with unities and L and R be po-semigroups. Then S is a po-Γ-semigroup where $a \leq b$ in S if and only if $\forall \alpha \in \Gamma$ $[a,\alpha] \leq [b,\alpha]$ in L and $[\alpha,a] \leq [\alpha,b]$ in R.*

Definition 2.9. [14] For a fuzzy subset μ of R we define a fuzzy subset μ^* of S by $\mu^*(a) = \inf\limits_{\gamma\in\Gamma} \mu([\gamma,a])$, where $a \in S$. For a fuzzy subset η of S we define a fuzzy subset $\eta^{*'}$ of R by $\eta^{*'}([\alpha,a]) = \inf\limits_{s\in S} \eta(s\alpha a)$, where $[\alpha,a] \in R$. For a fuzzy subset δ of L, we define a fuzzy subset δ^+ of S by $\delta^+(a) = \inf\limits_{\gamma\in\Gamma} \delta([a,\gamma])$, where $a \in S$. For a fuzzy subset ν of S we define a fuzzy subset $\nu^{+'}$ of L by $\nu^{+'}([a,\alpha]) = \inf\limits_{s\in S} \nu(a\alpha s)$, where $[a,\alpha] \in L$.

For the corresponding notions in crisp set we refer to [3] and [4].

3 Semilattice congruence on po-Γ-semigroup

Definition 3.1. Let R be a relation on a po-Γ-semigroup S. Then
(i) R is called compatible if xRy implies $(a\gamma x)R(a\gamma y)$ and $(x\gamma a)R(y\gamma a)$ for all $x,y,a \in S$ and for all $\gamma \in \Gamma$, where xRy means $(x,y) \in R$.
(ii) R is called a congruence relation on S, if it is a compatible equivalence relation on S.

Generally the set of all compatible relations on S is denoted by $Com(S)$ and the set of all congruence relations on S is denoted by $Con(S)$.

Definition 3.2. Let ρ be a relation on S. Then $\rho^{+'}$, defined by $[x,\alpha]\rho^{+'}[y,\beta]$ if and only if $(x\alpha s)\rho(y\beta s)$ $\forall s \in S$, is a relation on L.

Similarly, for a relation σ on L, σ^+ defined by $x\sigma^+ y$ if and only if $[x,\alpha]\sigma[y,\alpha]$ $\forall \alpha \in \Gamma$ is a relation on S.

Definition 3.3. [6]Let S be a po-semigroup. An equivalence relation σ on S is a ordered semilattice congruence if

(1) σ is a congruence:

$$(a,b) \in \sigma \Rightarrow (ac,bc) \in \sigma, (ca,cb) \in \sigma \forall a,b,c \in S,$$

(2) $(a^2,a) \in \sigma$ and $(ab,ba) \in \sigma$ $\forall a,b \in S$,

(3) For any $a,b \in S$ with $a \leq b \Rightarrow (a,ab) \in \sigma$.

Definition 3.4. [23]Let S be a po-Γ-semigroup. An equivalence relation σ on S is a ordered semilattice congruence if

(1) σ is a congruence:

$$(a,b) \in \sigma \Rightarrow (a\gamma c, b\gamma c) \in \sigma, (c\gamma a, c\gamma b) \in \sigma \forall a,b,c \in S, \forall \gamma \in \Gamma,$$

(2) $(a\gamma a, a) \in \sigma$ and $(a\gamma b, b\gamma a) \in \sigma \ \forall a,b \in S, \forall \gamma \in \Gamma$,

(3) For any $\gamma \in \Gamma$ and $a,b \in S$ with $a \leq b \Rightarrow (a, a\gamma b) \in \sigma$.

The set of all ordered semilattice congruences on S is generally denoted by $OSC(S)$.

In what follows the po-Γ-semigroup S is assumed to be with left and right unities.

Proposition 3.5. *Let S be a po-Γ-semigroup and $\sigma \in OSC(S)$. Then $\sigma^{+'} \in OSC(L)$.*

Proof. Clearly $\mu^{+'}$ is an equivalence relation on L. Now let $([x,\alpha],[y,\beta]) \in \sigma^{+'}$. Then $(x\alpha s, y\beta s) \in \sigma \ \forall s \in S$(cf. Def 3.2). Since σ is congruence in S, $(x\alpha s\gamma z, y\beta s\gamma z) \in \sigma \ \forall s,z \in S, \forall \gamma \in \Gamma$. So $([x\alpha s,\gamma],[y\beta s,\gamma]) \in \sigma^{+'}$ whence $([x,\alpha][s,\gamma],[y,\beta][s,\gamma]) \in \sigma^{+'} \ \forall s \in S, \forall \gamma \in \Gamma$. Similarly $([s,\gamma][x,\alpha],[s,\gamma][y,\beta]) \in \sigma^{+'} \ \forall s \in S, \forall \gamma \in \Gamma$. Hence $\sigma^{+'}$ is a congruence in L. By Definition 3.4, $(x\alpha x, x) \in \sigma \ \forall x \in S, \forall \alpha \in \Gamma$ whence $(x\alpha x\alpha s, x\alpha s) \in \sigma \ \forall x,s \in S, \forall \alpha \in \Gamma$ i.e., $([x,\alpha][x,\alpha],[x,\alpha]) \in \sigma^{+'} \ \forall x \in S, \alpha \in \Gamma$. Now let $[x,\alpha],[y,\beta] \in L$. By Definition 3.4, $(s\alpha x, x\alpha s) \in \sigma \Rightarrow (y\beta s\alpha x, y\beta x\alpha s) \in \sigma \ \forall s \in S$. Again $(x\alpha y\beta s, y\beta s\alpha x) \in \sigma$ $\forall s \in S$. So by transitivity of σ we get $(x\alpha y\beta s, y\beta x\alpha s) \in \sigma \ \forall s \in S$. Hence $([x,\alpha][y,\beta],[y,\beta][x,\alpha]) \in \sigma^{+'}$. Lastly let $[x,\alpha] \leq [y,\beta]$ in L. Then $x\alpha s \leq y\beta s$ in $S \ \forall s \in S$ whence $(x\alpha s, x\alpha s\gamma y\beta s) \in \sigma \ \forall s \in S, \forall \gamma \in \Gamma$(cf. Def 3.4). So in paticular, $(x\alpha e, x\alpha e\delta y\beta e) \in \sigma \Rightarrow (x\alpha e, x\alpha y\beta e) \in \sigma \Rightarrow (x\alpha e\gamma s, x\alpha y\beta e\gamma s) \in \sigma$ $\forall s \in S, \forall \gamma \in \Gamma$(cf. Def 3.4) where $[e,\delta]$ is the left unity of S. Now taking $\gamma = \delta$ we get $(x\alpha s, x\alpha y\beta s) \in \sigma \ \forall s \in S$. So $([x,\alpha],[x,\alpha][y,\beta]) \in \sigma^{+'}$. Hence $\sigma^{+'} \in OSC(L)$. □

Proposition 3.6. *Let S be a po-Γ-semigroup and $\sigma \in OSC(L)$. Then $\sigma^{+} \in OSC(S)$.*

Proof. It is easy to see that σ^{+} is an equivalence relation on S. Let $(x,y) \in \sigma^{+}$. Then $([x,\alpha],[y,\alpha]) \in \sigma \ \forall \alpha \in \Gamma$(cf. Def 3.2). Since σ is a congruence in L, $([x,\alpha][s,\gamma],[y,\alpha]\ [s,\gamma]) \in \sigma \ \forall s \in S, \forall \alpha,\gamma \in \Gamma$(cf. Def 3.3). So $([x\alpha s,\gamma],[y\alpha s,\gamma]) \in \sigma \ \forall \gamma \in \Gamma$ whence $(x\alpha s, y\alpha s) \in \sigma^{+} \ \forall s \in S, \forall \alpha \in \Gamma$. Similarly $(s\alpha x, s\alpha y) \in \sigma^{+} \ \forall s \in S, \forall \alpha \in \Gamma$. Hence σ^{+} is a congruence in S. By Definition 3.3, $([x,\alpha][x,\alpha],[x,\alpha]) \in \sigma$ i.e., $([x\alpha x,\alpha],[x,\alpha]) \in \sigma \ \forall x \in S, \forall \alpha \in \Gamma$ whence $(x\alpha x, x) \in \sigma^{+} \ \forall x \in S, \forall \alpha \in \Gamma$; $([x,\alpha][y,\alpha],[y,\alpha][x,\alpha]) \in \sigma$

i.e., $([x\alpha y, \alpha], [y\alpha x, \alpha]) \in \sigma\ \forall x, y \in S, \forall \alpha \in \Gamma$ whence $(x\alpha y, y\alpha x) \in \sigma^+$ $\forall x, y \in S, \forall \alpha \in \Gamma$. Now let $x \leq y$ in S. Then $[x, \alpha] \leq [y, \alpha]$ in $L\ \forall \alpha \in \Gamma$ whence $([x, \alpha], [x, \alpha][y, \alpha]) \in \sigma$(cf. Def.3.3) i.e., $([x, \alpha], [x\alpha y, \alpha]) \in \sigma\ \forall \alpha \in \Gamma$. So $(x, x\alpha y) \in \sigma^+$. Hence we conclude that $\sigma^+ \in OSC(S)$. □

Theorem 3.7. *Let S be a po-Γ-semigroup with the left operator semigroup L. Then there exists an inclusion preserving bijection $\sigma \to \sigma^{+'}$ between $OSC(S)$ and $OSC(L)$.*

Proof. Let $\sigma \in OSC(S)$. Then by Proposition 3.5, $\sigma^{+'} \in OSC(L)$. Let $(x, y) \in \sigma$. Then σ being a congruence in S $(x\alpha s, y\alpha s) \in \sigma\ \forall s \in S, \forall \alpha \in \Gamma$ whence $([x, \alpha], [y, \alpha]) \in \sigma^{+'}\ \forall \alpha \in \Gamma$. So $(x, y) \in (\sigma^{+'})^+$. Thus $\sigma \subseteq (\sigma^{+'})^+$. On the other hand, for $x, y \in S$ let $(x, y) \in (\sigma^{+'})^+$. Then $([x, \alpha], [y, \alpha]) \in \sigma^{+'}$ $\forall \alpha \in \Gamma \Rightarrow (x\alpha s, y\alpha s) \in \sigma\ \forall s \in S, \forall \alpha \in \Gamma$(cf. Def 3.2) $\Rightarrow (x\gamma f, y\gamma f) \in \sigma$ i.e., $(x, y) \in \sigma$ where $[\gamma, f]$ is the right unity of S. Thus $(\sigma^{+'})^+ \subseteq \sigma$. Hence $(\sigma^{+'})^+ = \sigma$.
Again let $\sigma \in OSC(L)$ and $([x, \alpha], [y, \beta]) \in \sigma$. Then σ being a congruence in L, $([x, \alpha][s, \gamma], [y, \beta][s, \gamma]) \in \sigma$ i.e., $([x\alpha s, \gamma], [y\beta s, \gamma]) \in \sigma\ \forall s \in S, \forall \gamma \in \Gamma$. So $(x\alpha s, y\beta s) \in \sigma^+\ \forall s \in S$ whence $([x, \alpha], [y, \beta]) \in (\sigma^+)^{+'}$. Thus $\sigma \subseteq (\sigma^+)^{+'}$. On the other hand, for $[x, \alpha], [y, \beta] \in L$ let $([x, \alpha], [y, \beta]) \in (\sigma^+)^{+'}$. Then $(x\alpha s, y\beta s) \in \sigma^+\ \forall s \in S$ whence $([x\alpha s, \gamma], [y\beta s, \gamma]) \in \sigma$ i.e., $([x, \alpha][s, \gamma], [y, \beta][s, \gamma]) \in \sigma\ \forall s \in S, \gamma \in \Gamma$. Since $[x, \alpha][e, \delta] = [x, \alpha]\ \forall [x, \alpha] \in L$ where $[e, \delta]$ is the left unity of S, $([x, \alpha], [y, \beta]) \in \sigma$. Thus $(\sigma^+)^{+'} \subseteq \sigma$. Hence $(\sigma^+)^{+'} = \sigma$. Hence the mapping is bijective. Let $\sigma_1, \sigma_2 \in OSC(S)$ be such that $\sigma_1 \subseteq \sigma_2$. Let $([x, \alpha], [y, \beta]) \in \sigma_1^{+'}$. Then $(x\alpha s, y\beta s) \in \sigma_1 \forall s \in S$. So $(x\alpha s, y\beta s) \in \sigma_2 \forall s \in S$ whence $([x, \alpha], [y, \beta]) \in \sigma_2^{+'}$. Hence $\sigma_1^{+'} \subseteq \sigma_2^{+'}$. Therefore $\sigma_1^{+'} \subseteq \sigma_2^{+'}$. Hence the result follows. □

4 Fuzzy semilattice congruence on po-Γ-semigroup

Definition 4.1. Let S be a po-Γ-semigroup. A function C from $S \times S$ to $[0, 1]$ is called a fuzzy relation on S.

Let C and D be two fuzzy relations on S. Then (i) $C \subseteq D$ if and only if $C(x, y) \leq D(x, y)$, (ii) $C \cap D$ if and only if $(C \cap D)(x, y) = (C(x, y) \wedge D(x, y))$, (iii) $C \circ D$ if and only if $C \circ D(x, y) = \bigvee\limits_{z \in S} \big(C(x, z) \wedge D(z, y)\big)\ \forall x, y \in S$.

Definition 4.2. Let S be a po-Γ-semigroup. A fuzzy relation C on S is said to be a fuzzy equivalence relation on S if it satisfies the followings (i) Fuzzy reflexive: $C(x, x) = 1$, (ii) Fuzzy symmetric: $C(x, y) = C(y, x)$,

(iii) Fuzzy transitive: $C(x,y) \geq \bigvee_{z\in S}\big(C(x,z) \wedge D(z,y)\big)$ i.e., $C \circ C \subseteq C$ $\forall x, y, z \in S$.

Definition 4.3. Let S be a po-Γ-semigroup and C be a fuzzy relation on S. Then

(i) C is called a fuzzy compatible relation if $C(a\gamma x, a\gamma y) \geq C(x,y)$ and $C(x\gamma a, y\gamma a) \geq C(x,y)$ $\forall x, y, a \in S, \gamma \in \Gamma$.

(ii) C is called a fuzzy congruence relation on S if it is a fuzzy compatible equivalence relation on S.

Definition 4.4. Let S be a po-Γ-semigroup and μ be a fuzzy relation on S. Then $\mu^{+'}$ defined by $\mu^{+'}([x,\alpha],[y,\beta]) = \inf_{s\in S}\mu(x\alpha s, y\beta s)$ is a fuzzy relation on L. Similarly, for a fuzzy relation ν on L, ν^{+} defined by $\nu^{+}(x,y) = \inf_{\gamma\in\Gamma}\nu([x,\gamma],[y,\gamma])$ is a fuzzy relation on S.

Definition 4.5. Let S be a po-semigroup. A fuzzy equivalence relation μ on S is called an ordered fuzzy semilattice congruence if

(1) μ is a fuzzy congruence relation ($\mu(a,b) \leq \mu(ac,bc), \mu(a,b) \leq \mu(ca,cb) \forall a,b,c \in S$),

(2) $\mu(a^2,a) = 1$ and $\mu(ab,ba) = 1$ $\forall a, b \in S$,

(3) For any $a, b \in S$ with $a \leq b \Rightarrow \mu(a,ab) = 1$.

Definition 4.6. Let S be a po-Γ-semigroup. A fuzzy equivalence relation μ on S is called an ordered fuzzy semilattice congruence if

(1) μ is a fuzzy congruence relation, i.e, $\mu(a\gamma x, a\gamma y) \geq \mu(x,y)$ and $\mu(x\gamma a, y\gamma a) \geq \mu(x,y)$ $\forall x, y, a \in S$ and $\forall \gamma \in \Gamma$.

(2) $\mu(a\gamma a, a) = 1$ and $\mu(a\gamma b, b\gamma a) = 1$ $\forall a, b \in S, \forall \gamma \in \Gamma$,

(3) For any $\gamma \in \Gamma$ and $a, b \in S$ with $a \leq b \Rightarrow \mu(a, a\gamma b) = 1$.

The set of all ordered fuzzy semilattice congruences on S is generally denoted by $OFSC(S)$.

It is a matter of routine verification that the definition of ordered fuzzy semilattice congruence satisfies characteristic function criterion and level subset criterion.

Theorem 4.7. *Let R be a relation on a po-Γ-semigroup S and χ_R be the characteristic function of R. Then $R \in OSC(S)$ if and only if $\chi_R \in OFSC(S)$.*

Theorem 4.8. *Let S be a po-Γ-semigroup. Then $\mu \in OFSC(S)$ if and only if $R_\mu(t) \in OSC(S)$ for all $t \in [0,1]$, where $R_\mu(t) = \{(a,b) \in S \times S : \mu(a,b) \geq t\}$ is the t-level subset of μ in S.*

In what follows a po-Γ-semigroup S is assumed to be with left and right unities.

Proposition 4.9. *Let S be a po-Γ-semigroup and $\mu \in OFSC(S)$. Then $\mu^{+'} \in OFSC(L)$.*

Proof. Clearly $\sigma^{+'}$ is a fuzzy equivalence relation on L. Now for $x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$, $\mu^{+'}([x,\alpha][z,\gamma],[y,\beta][z,\gamma]) = \inf\limits_{s\in S}\mu(x\alpha z\gamma s, y\beta z\gamma s) \geq \mu(x\alpha z, y\beta z)$ (cf. Def 4.6) $\geq \inf\limits_{z\in S}\mu(x\alpha z, y\beta z) = \mu^{+'}([x,\alpha],[y,\beta])$. Similarly $\mu^{+'}([s,\gamma][x,\alpha],$ $[s,\gamma][y,\beta]) \geq \mu^{+'}([x,\alpha],[y,\beta])\ \forall s \in S, \forall\gamma \in \Gamma$. Hence $\mu^{+'}$ is a fuzzy congruence in L. Now let $[x,\alpha] \in L$. Then $\mu(x\alpha x, x) = 1\ \forall x \in S, \forall\alpha \in \Gamma$. Since μ is fuzzy congruence, $\mu(x\alpha x\alpha s, x\alpha s) \geq \mu(x\alpha x, x) = 1\ \forall s \in S$, whence $\mu(x\alpha x\alpha s, x\alpha s) = 1\ \forall s \in S$. So $\inf\limits_{s\in S}\mu(x\alpha x\alpha s, x\alpha s) = 1$ i.e., $\mu^{+'}([x,\alpha][x,\alpha],[x,\alpha]) = 1$. Now let $[x,\alpha],[y,\beta] \in L$. By Definition 4.6, $\mu(s\alpha x, x\alpha s) = 1 \Rightarrow \mu(y\beta s\alpha x, y\beta x\alpha s) = 1$ $\forall s \in S$. Again $\mu(x\alpha y\beta s, y\beta s\alpha x) = 1\ \forall s \in S$. So by transitivity of μ we get $\mu(x\alpha y\beta s,\ y\beta x\alpha s) \geq \min\{\mu(x\alpha y\beta s, y\beta s\alpha x), \mu(y\beta s\alpha x, y\beta x\alpha s)\} = 1\ \forall s \in S$. So $\inf\limits_{s\in S}\mu(x\alpha y\beta s,\ y\beta x\alpha s) = 1$. Hence $\mu^{+'}([x,\alpha][y,\beta],[y,\beta][x,\alpha]) = 1$. Lastly let $[x,\alpha] \leq [y,\beta]$ in L. So $x\alpha s \leq y\beta s$ in S $\forall s \in S$ whence $\mu(x\alpha s, x\alpha s\gamma y\beta s) = 1$ $\forall s \in S, \forall\gamma \in \Gamma$(cf. Def 4.6). So $\mu(x\alpha e, x\alpha e\delta y\beta e) = 1 \Rightarrow \mu(x\alpha e, x\alpha y\beta e) = 1 \Rightarrow$ $\mu(x\alpha e\gamma s, x\alpha y\beta e\gamma s) = 1$ $(\forall s \in S, \gamma \in \Gamma) \Rightarrow \mu(x\alpha s, x\alpha y\beta s) = 1\ \forall s \in S$ where $[e,\delta]$ is the left unity of S. So $\inf\limits_{s\in S}\mu(x\alpha s, x\alpha y\beta s) = 1$ i.e., $\mu^{+'}([x,\alpha],[x,\alpha][y,\beta]) =$ 1. Hence $\mu^{+'} \in OFSC(L)$. $\square$

Proposition 4.10. *Let S be a po-Γ-semigroup and $\mu \in OFSC(L)$. Then $\mu^{+} \in OFSC(S)$.*

Proof. Clearly μ^+ is a fuzzy equivalence relation on S. Now for $x, y, z \in S$, and $\alpha \in \Gamma$, $\mu^+(x\alpha z, y\alpha z) = \inf\limits_{\gamma\in\Gamma}\mu([x\alpha z,\gamma],[y\alpha z,\gamma]) = \inf\limits_{\gamma\in\Gamma}\mu([x,\alpha][z,\gamma],[y,\alpha][z,\gamma])$ $\geq \mu([x,\alpha],[y,\alpha])$ (cf. Def 4.5) $\geq \inf\limits_{\alpha\in\Gamma}\mu([x,\alpha],[y,\alpha]) = \mu^+(x,y)$. Similarly $\mu^+(z\alpha x, z\alpha y) \geq \mu^+(x,y)\forall z \in S, \alpha \in L$. Hence μ^+ is fuzzy congruence in S. By Definition 4.5, $\mu([x,\alpha][x,\alpha],\ [x,\alpha]) = 1$ i.e., $\mu([x\alpha x,\alpha],[x,\alpha]) = 1$ $\forall x \in S, \forall\alpha \in \Gamma$. So $\inf\limits_{\alpha\in\Gamma}\mu([x\alpha x,\alpha],[x,\alpha]) = 1$ whence $\mu^+(x\alpha x, x) = 1\ \forall x \in$ $S, \forall\alpha \in \Gamma$(cf. Def 4.4). Again by Definition 4.5, $\mu([x,\alpha][y,\alpha],\ [y,\alpha][x,\alpha]) = 1$ i.e., $\mu([x\alpha y,\alpha],[y\alpha x,\alpha]) = 1\ \forall x, y \in S, \forall\alpha \in \Gamma$. So $\inf\limits_{\alpha\in\Gamma}\mu([x\alpha y,\alpha],\ [y\alpha x,\alpha]) = 1$ whence $\mu^+(x\alpha y, y\alpha x) = 1\ \forall x, y \in S, \forall\alpha \in \Gamma$. Lastly let $x \leq y$ in S. So $[x,\alpha] \leq [y,\alpha]$ in L $\forall\alpha \in \Gamma$ whence $\mu([x,\alpha],[x,\alpha][y,\alpha]) = 1$(cf. Def4.5) i.e., $\mu([x,\alpha],[x\alpha y,\alpha]) = 1\ \forall\alpha \in \Gamma$. So $\inf\limits_{\alpha\in\Gamma}\mu([x,\alpha],[x\alpha y,\alpha]) = 1$ whence $\mu^+(x, x\alpha y) =$ 1. Hence $\mu^+ \in OFSC(S)$. $\square$

Theorem 4.11. *Let S be a po-Γ-semigroup with the left operator semigroup L. Then there exists an inclusion preserving bijection $\mu \to \mu^{+'}$ between $OFSC(S)$ and $OFSC(L)$.*

Proof. Let $\mu \in OFSC(S)$. Then by Proposition 4.9, $\mu^{+'} \in OFSC(L)$. Let $x, y \in S$. Then μ being a fuzzy congruence in S $\mu(x,y) \leq \mu(x\alpha s, y\alpha s)$ $\forall s \in S, \forall \alpha \in \Gamma$. Hence $\mu(x,y) \leq \mu^{+'}([x,\alpha],[y,\alpha])$ $\forall \alpha \in \Gamma$ whence $\mu(x,y) \leq (\mu^{+'})^{+}(x,y)$(cf. Def 4.4). Thus $\mu \subseteq (\mu^{+'})^{+}$. On the other hand, let $x, y \in S$. $(\mu^{+'})^{+}(x,y) = \inf\limits_{\alpha\in\Gamma}\mu^{+'}([x,\alpha],[y,\alpha]) = \inf\limits_{\alpha\in\Gamma}\inf\limits_{s\in S}\mu(x\alpha s, y\alpha s) \leq \mu(x,y)$ (since S has right unity). Thus $(\mu^{+'})^{+} \subseteq \mu$. Hence $(\mu^{+'})^{+} = \mu$.

Again let $\mu \in OFSC(L)$ and $[x,\alpha],[y,\beta] \in L$. Then by Definition 4.5, $\mu([x,\alpha],[y,\beta]) \leq \mu([x,\alpha][s,\gamma],[y,\beta][s,\gamma]) = \mu([x\alpha s,\gamma],[y\beta s,\gamma])$ $\forall s \in S, \forall \gamma \in \Gamma$. So $\mu([x,\alpha],[y,\beta]) \leq \mu^{+}(x\alpha s, y\beta s)$ $\forall s \in S$ whence $\mu([x,\alpha],[y,\beta]) \leq (\mu^{+})^{+'}([x,\alpha],[y,\beta])$(cf. Def 4.4). Thus $\mu \subseteq (\mu^{+})^{+'}$. On the other hand, for $[x,\alpha],[y,\beta] \in L$, $(\mu^{+})^{+'}([x,\alpha],[y,\beta]) = \inf\limits_{s\in S}\mu^{+}(x\alpha s, y\beta s) = \inf\limits_{s\in S}\inf\limits_{\gamma\in\Gamma}\mu([x\alpha s,\gamma],[y\beta s,\gamma]) = \inf\limits_{s\in S}\inf\limits_{\gamma\in\Gamma}\mu([x,\alpha][s,\gamma],[y,\beta][s,\gamma]) \leq \mu([x,\alpha],[y,\beta])$ (since S has left unity and so L has identity). Thus $(\mu^{+})^{+'} \subseteq \mu$. Hence $(\mu^{+})^{+'} = \mu$. Hence the mapping is bijective. Let $\mu_1, \mu_2 \in OFSC(S)$ be such that $\mu_1 \subseteq \mu_2$. Let $([x,\alpha],[y,\beta]) \in L$. Then $\mu_1^{+'}([x,\alpha],[y,\beta]) = \inf\limits_{s\in S}\mu_1(x\alpha s, y\beta s) \leq \inf\limits_{s\in S}\mu_2(x\alpha s, y\alpha s) = \mu_2^{+'}([x,\alpha],[y,\beta])$. Therefore $\mu_1^{+'} \subseteq \mu_2^{+'}$. Hence the theorem.
□

Now following the terminology of [10] we define the following notion in po-semigroup and po-Γ-semigroup.

Definition 4.12. Let μ be a fuzzy equivalence on a po-semigroup(po-Γ-semigroup) S. For each $a \in S$, we define a fuzzy subset μ_a of S as follows:

$$\mu_a(x) = \mu(a,x) \forall x \in S.$$

The fuzzy subset μ_a of S is called the fuzzy equivalence class of μ containing a.

Note 3. It is easy to verify that $\mu_a = \mu_b$ if and only if $\mu(a,b) = 1$ $\forall a, b \in S$ and if $\mu \in OFSC(S)$ then μ_a is a fuzzy subsemigroup of S $\forall a \in S$.

Proposition 4.13. *Let $\mu \in OFSC(S)$ where S be a po-Γ-semigroup S. Then $S/\mu = \{\mu_a : a \in S\}$ is a commutative po-Γ-semigroup with respect to the map from $S/\mu \times \Gamma \times S/\mu \to S/\mu$ by $(\mu_a, \gamma, \mu_b) \to \mu_{a\gamma b}$ and with respect to the usual partial order(fuzzy set inclusion).*

Proof. Let $\mu_a = \mu_{a_1}$ and $\mu_b = \mu_{b_1}$ where $a, b, a_1, b_1 \in S$. Then $\mu(a, a_1) = 1 = \mu_(b, b_1)$. Then $\mu(a\gamma b, a_1\gamma b_1) \geq \mu(a\gamma b, a_1\gamma b) \wedge \mu(a_1\gamma b, a_1\gamma b_1) \geq \mu(a, a_1) \wedge \mu(b, b_1)$ (cf. Def 4.6) $= 1$. So $\mu_{a\gamma b} = \mu_{a_1\gamma b_1}$. Hence the map is well-defined. Now for $a, b, c \in S$ and $\alpha, \gamma \in \Gamma$, $\mu_a\gamma(\mu_b\alpha\mu_c) = \mu_a\gamma\mu_{b\alpha c} = \mu_{a\gamma(b\alpha c)} = \mu_{(a\gamma b)\alpha c} = \mu_{(a\gamma b)}\alpha\mu_c = (\mu_a\gamma\mu_b)\alpha\mu_c$. Hence S/μ is a Γ-semigroup. Now let $a, b \in S$ and $\gamma \in \Gamma$. Since $\mu \in OFSC(S)$, $\mu(a\gamma b, b\gamma a) = 1$. So $\mu_{a\gamma b} = \mu_{b\gamma a}$ which means $\mu_a\gamma\mu_b = \mu_b\gamma\mu_a$. Hence S/μ is a commutative Γ-semigroup. Again S/μ is poset with usual fuzzy set inclusion. Let $a, b, c \in S$ and $\gamma \in \Gamma$ with $\mu_a \leq \mu_b$. So $\mu(a, x) \leq \mu(b, x)\ \forall x \in S$. Then $\mu(b\gamma c, x) \geq \mu(b\gamma c, a\gamma c) \wedge \mu(a\gamma c, x) \geq \mu(b, a) \wedge \mu(a\gamma c, x) \geq \mu(a, a) \wedge \mu(a\gamma c, x) = 1 \wedge \mu(a\gamma c, x) = \mu(a\gamma c, x) \forall x \in S$. Hence $\mu_a\gamma\mu_c \leq \mu_b\gamma\mu_c$. Similarly we obtain $\mu_c\gamma\mu_a \leq \mu_c\gamma\mu_b$. Hence S/μ is commutative po-Γ-semigroup. □

Applying the same argument as above we obtain the following result.

Proposition 4.14. *Let S be a po-semigroup and $\mu \in OFSC(S)$. Then S/μ is a commutative po-semigroup under multiplication $*$ defined by $\mu_a * \mu_b = \mu_{ab}$ and with respect to the partial order $\leq$ where $\mu_a \leq \mu_b$ if and only if $\mu(a, x) \leq \mu(b, x)$ $\forall x \in S$.*

If the composition of fuzzy congruences, $\circ$, is commutative in both $FSC(S)$ and $FSC(L)$ then $< FSC(S), \circ, \cap >$ and $< FSC(L), \circ, \cap >$ both become lattices.

Proposition 4.15. *Let S be a Γ-semigroup with unities and L be its operator semigroup. Then there exists a lattice isomorphism between $FSC(S)$ and $FSC(L)$ via the mapping $\sigma \to \sigma^{+'}$.*

Proof. By the proof of Theorem 4.11 we see that the map is inclusion preserving. So it is sufficient to prove that $(\mu\cap\nu)^{+'} = \mu^{+'}\cap\nu^{+'}$ and $(\mu\circ\nu)^{+'} = \mu^{+'}\circ\nu^{+'}$ $\forall \mu, \nu \in FSC(S)$. Let $[x, \alpha], [y, \beta] \in L$. Then

$$\begin{aligned}
(\mu \cap \nu)^{+'}([x, \alpha], [y, \beta]) &= \inf_{s\in S}(\mu \cap \nu)(x\alpha s, y\beta s) \\
&= \inf_{s\in S}\{\mu(x\alpha s, y\beta s) \wedge \nu(x\alpha s, y\beta s)\} \\
&= \inf_{s\in S}\mu(x\alpha s, y\beta s) \wedge \inf_{s\in S}\nu(x\alpha s, y\beta s) \\
&= \mu^{+'}([x, \alpha], [y, \beta]) \wedge \nu^{+'}([x, \alpha], [y, \beta]) \\
&= (\mu^{+'} \cap \nu^{+'})([x, \alpha], [y, \beta]).
\end{aligned}$$

$$\begin{aligned}
(\mu\circ\nu)^{+'}([x,\alpha],[y,\beta]) &= \inf_{s\in S}(\mu\circ\nu)(x\alpha s, y\beta s)\\
&= \inf_{s\in S}\sup_{z\in S}\{\mu(x\alpha s, z)\wedge\nu(z, y\beta s)\}\\
&\geq \inf_{s\in S}\sup_{z\gamma s\in S}\{\mu(x\alpha s, z\gamma s)\wedge\nu(z\gamma s, y\beta s)\}\\
&\geq \sup_{z\gamma s\in S}\{\inf_{s\in S}\mu(x\alpha s, z\gamma s)\wedge\inf_{s\in S}\nu(z\gamma s, y\beta s)\}\\
&\geq (\mu^{+'}\circ\nu^{+'})([x,\alpha],[y,\beta]).
\end{aligned}$$

Again

$$(\mu^{+'}\circ\nu^{+'})([x,\alpha],[y,\beta]) = \sup_{[z,\gamma]\in L}\{\mu^{+'}([x,\alpha],[z,\gamma])\wedge\nu^{+'}([z,\gamma],[y,\beta])\}$$

$$\begin{aligned}
&= \sup_{[z,\gamma]\in L}\{\inf_{s\in S}\mu(x\alpha s, z\gamma s)\wedge\inf_{s\in S}\nu(z\gamma s, y\beta s)\}\\
&= \inf_{s\in S}\sup_{[z,\gamma]\in L}\{\mu(x\alpha s, z\gamma s)\wedge\nu(z\gamma s, y\beta s)\}\\
&\geq \inf_{s\in S}\sup_{z\in S}\{\mu(x\alpha s, z\delta s)\wedge\nu(z\delta s, y\beta s)\}\text{ , fixing }\gamma\text{ by }\delta\\
&\geq \inf_{s\in S}\sup_{z\in S}\{\mu(x\alpha s, z)\wedge\mu(z, z\delta s)\wedge\nu(z\delta s, z)\wedge\nu(z, y\beta s)\}\\
&= \inf_{s\in S}\{\sup_{z\in S}\{\mu(x\alpha s, z)\wedge\nu(z, y\beta s)\}\wedge\sup_{z\in S}\{\nu(z\delta s, z)\wedge\mu(z, z\delta s)\}\}\\
&\geq (\mu\circ\nu)^{+'}([x,\alpha],[y,\beta])\wedge\inf_{s\in S}\sup_{z\in S}\{\mu(z, z\delta s)\wedge\nu(z\delta s, z)\}\\
&= (\mu\circ\nu)^{+'}([x,\alpha],[y,\beta])\wedge 1,\text{ since }\mu,\nu\in FSC(S)\\
&= (\mu\circ\nu)^{+'}([x,\alpha],[y,\beta]).
\end{aligned}$$

Hence the result follows. □

Theorem 4.16. *Let S be a Γ-semigroup with unities and H be any sublattice of the lattice $< FSC(S), \circ, \cap >$. Then H is a modular lattice.*

Proof. Let $H^{+'} = \{\mu^{+'} : \mu \in FSC(S)\}$. Then $H^{+'} \subseteq FSC(L)$. Clearly $H^{+'}$ is a poset under the set inclusion $\leq$ in L. Since $(\mu\cap\nu)^{+'} = \mu^{+'}\cap\nu^{+'}$ and $(\mu\circ\nu)^{+'} = \mu^{+'}\circ\nu^{+'}$ where $\mu,\nu\in FSC(S)$(cf. Prop 4.15), $< H^{+'}, \circ, \cap >$ is a sublattice of the lattice $< FSC(L), \circ, \cap >$. Hence by Theorem 5.7.21 of [10] $H^{+'}$ is a modular lattice. Now let $\mu \leq \sigma$ in H. Then

$\mu(x\alpha s, y\beta s) \leq \sigma(x\alpha s, y\beta s)\ \forall x, y, s \in S,\ \forall\alpha,\beta\in\Gamma$.

$\Rightarrow \inf_{s\in S}\mu(x\alpha s, y\beta s) \leq \inf_{s\in S}\sigma(x\alpha s, y\beta s)\ \forall x, y \in S,\ \forall\alpha,\beta\in\Gamma$.

$\Rightarrow \mu^{+'}([x,\alpha],[y,\beta]) \leq \sigma^{+'}([x,\alpha],[y,\beta])\ \forall x, y \in S,\ \forall\alpha,\beta\in\Gamma$.

$\Rightarrow \mu^{+'} \leq \sigma^{+'}$ in $H^{+'}$.

$\Rightarrow (\mu^{+'}\circ\nu^{+'})\cap\sigma^{+'} \leq \mu^{+'}\circ(\nu^{+'}\cap\sigma^{+'})\ \forall\nu^{+'}\in H^{+'}$, since $H^{+'}$ is modular lattice

$\Rightarrow (\mu \circ \nu)^{+'} \cap \sigma^{+'} \leq \mu^{+'} \circ (\nu \cap \sigma)^{+'} \; \forall \nu \in H.$
$\Rightarrow \{(\mu \circ \nu) \cap \sigma\}^{+'} \leq \{\mu \circ (\nu \cap \sigma)\}^{+'} \; \forall \nu \in H.$
$\Rightarrow (\{(\mu \circ \nu) \cap \sigma\}^{+'})^{+} \leq (\{\mu \circ (\nu \cap \sigma)\}^{+'})^{+} \; \forall \nu \in H.$
$\Rightarrow (\mu \circ \nu) \cap \sigma \leq \mu \circ (\nu \cap \sigma) \; \forall \nu \in H$. Hence H is a modular lattice.

□

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