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FUZZY RADICALS OF Γ-SEMIRINGS

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Abstract

In this paper we introduce the notions of fuzzy prime radical and fuzzy nil radical of a fuzzy ideal in $\Gamma-$semiring and obtain some characterizations of these radicals. We also introduce the notion of Fuzzy primary ideal of a $\Gamma-$semiring and study it using fuzzy prime radical. Among other results we prove that in a commutative $\Gamma-$semiring, the concepts of fuzzy prime radical and fuzzy nil radical of a fuzzy ideal coincide.

1 Introduction

The notion of fuzzy set was introduced by Zadeh[14] in 1965. This concept has been used in various branches of mathematics since its inception. Rosenfeld, Kuroki and Jun have contributed a lot in applying this concept to group theory, semigroup theory and $\Gamma-$ring theory respectively. Fuzzy prime radical of a fuzzy ideal was studied by Dutta et al in $\Gamma-$ring[4]. Dutta and Biswas also studied fuzzy prime radical of a fuzzy ideal in semiring[1]. The present authors have initiated the study of $\Gamma-$semiring in terms of fuzzy subsets[8],[9], [10], [11], [13]. This paper is a sequel to this study. Here we introduce the notion of a fuzzy prime radical and fuzzy nil radical of a fuzzy ideal in $\Gamma-$semiring. We also introduce the notion of Fuzzy primary ideal of a $\Gamma-$semiring and obtained some important results as mentioned in the abstract.

Key words: Fuzzy prime radical, Fuzzy nil radical, Fuzzy primary ideal,Fuzzy prime ideal, Fuzzy semiprime ideal, $\Gamma-$semiring, left (right) operator semiring.
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For preliminaries on $\Gamma-$semiring and its operator semirings we refer to [5], [6], [7]. Also for preliminaries on fuzzy ideals of a $\Gamma-$semiring we refer to [8],[11], [12], [13].

2 Fuzzy prime radical of $\Gamma-$semirings.

The set of fuzzy ideals of a $\Gamma-$semiring S, the set of fuzzy prime ideals of S, the set of fuzzy prime ideals of the left operator semiring L of S and the set of fuzzy prime ideals of the right operator semiring R of S are denoted by $FI(S), FPI(S), FPI(L)$ and $FPI(R)$ respectively.

Definition 2.1. Let μ be a non empty fuzzy subset of a $\Gamma-$semiring S. Let us define $\overline{\mu} = \{\theta : \theta \in FPI(S), \mu \subseteq \theta\}$.

By routine verification we have the following proposition.

Proposition 2.2. *Let μ_1, μ_2 be two fuzzy subsets of a $\Gamma-$semiring S. Then*
(i) $\mu_1 \subseteq \mu_2$ implies that $\overline{\mu_2} \subseteq \overline{\mu_1}$,
(ii) $\overline{\mu_1} \cup \overline{\mu_2} \subseteq \overline{\mu_1 \cap \mu_2}$,
(iii) $\overline{\mu_1} \cup \overline{\mu_2} = \overline{\mu_1 \Gamma \mu_2}$, if μ_1, μ_2 are two fuzzy ideals of S.
(iv) $\overline{\mu_1} \cup \overline{\mu_2} = \overline{\mu_1 \circ \mu_2}$, if μ_1, μ_2 are two fuzzy ideals of S.
(v) $\overline{\lambda_I} \cup \overline{\lambda_J} = \overline{\lambda_{I \cap J}}$, if I and J are two ideals of S.

Definition 2.3. Let μ be a fuzzy ideal of a $\Gamma-$semiring S. Then the fuzzy subset $PR(\mu)$ of S, defined by $PR(\mu) = \cap\overline{\mu} = \cap\{\theta \in FPI(S) : \mu \subseteq \theta\}$ is said to be the fuzzy prime radical of μ.

Proposition 2.4. *Let μ be a fuzzy ideal of a $\Gamma-$semiring S. Then $PR(\mu)$ is a fuzzy semiprime ideal of S.*

Proof. Let μ be a fuzzy ideal of a $\Gamma-$semiring S. As $\theta(0) = 1$ for $\theta \in FPI(S)$, so $PR(\mu)(0) = 1$ (cf. Theorem 3.6[12]). Again if $\theta \in FPI(S)$ then θ is non-constant fuzzy ideal of S (cf. Definition 3.1[11]). Let $x \in S$. Then, $\theta(x) \neq \theta(0) = 1$ for some $x \in S$. i.e., $\theta(x) < 1$ for some $x \in S$. Thus $PR(\mu)(x) \neq 1$ for some $x \in S$. Hence $PR(\mu)$ is non-constant fuzzy subset of S. Now for any $x, y \in S$, $PR(\mu)(x + y) = \cap\overline{\mu}(x + y) = \inf\{\theta(x + y) : \theta \in FPI(S) \mid \mu \subseteq \theta\}$ $\geq \inf\{\min[\theta(x), \theta(y)] : \theta \in FPI(S) \mid \mu \subseteq \theta\} = \min[\inf\{\theta(x) : \theta \in FPI(S) \mid \mu \subseteq \theta\}, \inf\{\theta(y) : \theta \in FPI(S) \mid \mu \subseteq \theta\}] = \min[\cap\overline{\mu}(x), \cap\overline{\mu}(y)] = \min[PR(\mu)(x), PR(\mu)(y)]$. Again $PR(\mu)(x\gamma y) = \cap\overline{\mu}(x\gamma y) = \inf\{\theta(x\gamma y) : \theta \in FPI(S) \mid \mu \subseteq \theta\} \geq \inf\{\theta(y) : \theta \in FPI(S) \mid \mu \subseteq \theta\} = (\cap\overline{\mu})(y) = PR(\mu)(y)$. Similarly we can show that $PR(\mu)(x\gamma y) \geq PR(\mu)(x)$. Thus $PR(\mu)$ is a non-constant fuzzy ideal of S. Now $\inf[PR(\mu)(x\gamma_1 s\gamma_2 x) : s \in S, \gamma_1, \gamma_2 \in \Gamma] = \inf[\cap\overline{\mu}(x\gamma_1 s\gamma_2 x) : s \in S, \gamma_1, \gamma_2 \in \Gamma] = \inf[\inf[\theta(x\gamma_1 s\gamma_2 x) : \theta \in FPI(S) \mid \mu \subseteq \theta] : s \in S, \gamma_1, \gamma_2 \in \Gamma] = \inf[\theta(x) : \theta \in FPI(S) \mid \mu \subseteq \theta]$ (cf. Proposition 3.6 and Proposition 3.2 of [13])$= \cap\overline{\mu}(x) = PR(\mu)(x)$. Hence $PR(\mu)$ is a fuzzy semiprime ideal of S. □

Proposition 2.5. *Let μ and θ be two fuzzy ideals of a $\Gamma-$semiring S. Then*
(i) $PR(\mu))(0) = 1$,
(ii) $\mu \subseteq PR(\mu)$,
(iii) $\mu \subseteq \theta$ implies that $PR(\mu) \subseteq PR(\theta)$,
(iv) $PR(PR(\mu)) = PR(\mu)$,
(v) $PR(\mu \oplus \theta) = PR(PR(\mu) \oplus PR(\theta))$ where $\mu(0) = \theta(0) = 1$.

Proof. Proof of (i), (ii) and (iii) are simple, so we omit it.

(iv) Since $\mu \subseteq PR(\mu)$, we have from (iii),

$$PR(\mu) \subseteq PR(PR(\mu)) \tag{1}$$

Again for $\phi \in \overline{\mu}$, $PR(\mu) \subseteq \phi$ and $\phi \in FPI(S)$. So $\phi \in \overline{PR(\mu)}$ and consequently $\overline{\mu} \subseteq \overline{PR(\mu)}$. Hence $\cap\overline{PR(\mu)} \subseteq \cap\overline{\mu}$. i.e.,

$$PR(PR(\mu)) \subseteq PR(\mu) \tag{2}$$

Combining (1) and (2) we have, $PR(PR(\mu)) = PR(\mu)$.

(v) We have $\mu \subseteq PR(\mu)$ and $\theta \subseteq PR(\theta)$. So $\mu \oplus \theta \subseteq PR(\mu) \oplus PR(\theta)$ and hence

$$PR(\mu \oplus \theta) \subseteq PR(PR(\mu) \oplus PR(\theta)). \tag{3}$$

Again $\mu \subseteq \mu \oplus \theta$ and $\theta \subseteq \mu \oplus \theta$ when $\mu(0) = \theta(0) = 1$. Thus $PR(\mu) \subseteq PR(\mu \oplus \theta)$ and $PR(\theta) \subseteq PR(\mu \oplus \theta)$. So $PR(\mu) \oplus PR(\theta) \subseteq PR(\mu \oplus \theta) \oplus PR(\mu \oplus \theta) = PR(\mu \oplus \theta)$. Thus,

$$PR(PR(\mu) \oplus PR(\theta)) \subseteq PR(PR(\mu \oplus \theta)) = PR(\mu \oplus \theta). \tag{4}$$

Combining (3) and (4) we have, $PR(\mu \oplus \theta) = PR(PR(\mu) \oplus PR(\theta))$. □

Proposition 2.6. *Suppose μ is a fuzzy prime ideal of a $\Gamma-$semiring S. Then $PR(\mu) = \mu$.*

Proof follows from Definition 2.3 and Proposition 2.5(ii).

Definition 2.7. The fuzzy prime radical of a Γ-semiring S is defined as the intersection of all fuzzy prime ideals of S and is denoted by $PR(S)$.

Theorem 2.8. *If PR(L) is a fuzzy prime radical of a left operator semiring L of S, then $(PR(L))^+ = PR(S)$ and $(PR(S))^{+'} = PR(L)$.*

Proof. Let μ be a fuzzy prime ideal of S. Then $\mu^{+'}$ is a fuzzy prime ideal of L (*cf. Proposition 3.3[12]*). Let $\theta = \mu^{+'}$. Then $\theta^+ = (\mu^{+'})^+ = \mu$. Now $PR(S) = \cap\{\mu : \mu \in FPI(S)\} \subseteq \cap\{\theta^+ : \theta \in FPI(L)\} = [\cap\{\theta : \theta \in FPI(L)\}]^+ = [PR(L)]^+$. Again, $PR(S) = \cap\{\mu : \mu \in FPI(S)\} = \cap\{\theta^+ : \theta \in \Lambda$, a subcollection of FPI(L)$\} \supset \cap\{\theta^+ : \theta \in FPI(L)\} = [\cap\{\theta : \theta \in FPI(L)\}]^+ = [PR(L)]^+$. Thus $PR(S) = [PR(L)]^+$. Similarly we can prove that $[PR(S)]^{+'} = [PR(L)]$. □

Corollary 2.9. *If PR(L) is the fuzzy prime radical of L, then* $[[PR(S)]^{+'}]^{+} = PR(S)$ *and* $[[PR(L)]^{+}]^{+'} = PR(L)$.

Similarly, we can prove that $(PR(S))^{*'} = PR(R)$, $(PR(R))^{*} = PR(S)$, $[[PR(S)]^{*'}]^{*} = PR(S)$ and $[[PR(R)]^{*}]^{*'} = PR(R)$ where PR(R) is the fuzzy prime radical of the right operator semiring R of S.

Theorem 2.10. *For a* Γ*-semiring S,* $[PR(R)]^{*} = [PR(L)]^{+}$.

The proof follows from the fact that $[PR(R)]^{*} = PR(S) = [PR(R)]^{+}$.

3 Fuzzy primary ideal of a $\Gamma-$semiring.

Throughout this section S denotes a commutative $\Gamma-$semiring with unities.

Definition 3.1. An ideal I of a $\Gamma-$semiring S is called a primary ideal of S if for any two ideals A and B, $A\Gamma B \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq PR(I)$ where PR(I) is the prime radical of I defined by $PR(I) = \cap\{P : \ P$ is a prime ideal of S such that $I \subseteq P\}$.

Definition 3.2. A fuzzy ideal μ of a $\Gamma-$semiring S is called a fuzzy primary ideal of S if μ is non-constant and for any two fuzzy ideals $\sigma,\ \theta$ of S, $\sigma\Gamma\theta \subseteq \mu$ implies $\sigma \subseteq \mu$ or $\theta \subseteq PR(\mu)$.

Theorem 3.3. *Let* $\mu \in FI(S)$. *Then* μ *is a fuzzy primary ideal of S if and only if* μ *is non-constant and* $\sigma \circ \theta \subseteq \mu$ *where* $\sigma, \theta \in FI(S)$ *implies that either* $\sigma \subseteq \mu$ *or* $\theta \subseteq PR(\mu)$.

Proof. The proof follows from Proposition 2.8[13]. □

Lemma 3.4. *If* $\mu \in FI(S)$ *such that* $\mu(0) = 1$ *then* $PR(\mu_0) \subseteq (PR(\mu))_0$.

Proof. Let $x \in PR(\mu_0)$. Then $x \in P$ for all prime ideals P of S such that $\mu_0 \subseteq P$. Let $\theta \in FPI(S)$ such that $\mu \subseteq \theta$. Let $s \in \mu_0$. Then $\mu(s) = \mu(0) = 1 = \theta(s)$. Thus $s \in \theta_0$. Hence $\mu_0 \subseteq \theta_0$. Also θ_0 is a prime ideal of S (*cf. Theorem 3.6[12]*), so $x \in \theta_0$. Therefore $\theta(x) = \theta(0) = 1$. Now $(PR(\mu))(x) = (\cap\overline{\mu})(x) = \inf[\theta(x) : \theta \in FPI(S),\ \mu \subseteq \theta] = 1 = (PR(\mu))(0)$. Thus $x \in (PR(\mu))_0$. Hence $PR(\mu_0) \subseteq (PR(\mu))_0$. □

Lemma 3.5. *An ideal Q of S is primary if and only if for any* $a, b \in S$, $(a)\Gamma(b) \subseteq Q$ *implies that* $a \in Q$ *or* $b \in PR(Q)$.

Proof. The only if part follows from the definition of a primary ideal (cf. Definition 3.1). Next, let $(a)\Gamma(b) \subseteq Q$ implies that $a \in Q$ or $b \in PR(Q)$. Also let A and B be two ideals of S such that $A\Gamma B \subseteq Q$ and $A \nsubseteq Q$. Then there exists $x \in A \cap Q^c$. Now for any $y \in B$ we have $(x)\Gamma(y) \subseteq Q$ and hence $y \in PR(Q)$. Consequently, $B \subseteq PR(Q)$ and so Q is primary. □

Theorem 3.6. *An ideal Q of S is primary if and only if $a\Gamma S\Gamma b \subseteq Q$ implies that $a \in Q$ or $b \in PR(Q)$.*

Proof. Suppose Q is primary. Let $a, b \in S$ such that $a\Gamma S\Gamma b \subseteq Q$ and $b \notin PR(Q)$. Then any element of $(a)\Gamma(b)$ is a finite sum of elements of the form $(na + c\alpha a + a\beta d + e\gamma a\delta f)\rho(mb + g\mu b + b\nu h + j\xi b\eta k)$, each of which is in Q, hence $(a)\Gamma(b) \subseteq Q$ and hence by Lemma 3.5, $a \in Q$.

Conversely, suppose $a\Gamma S\Gamma b \subseteq Q$ implies that $a \in Q$ or $b \in PR(Q)$. Also let A and B be two ideals of S such that $A\Gamma B \subseteq Q$ and $A \nsubseteq Q$. Then there exists $x \in A \cap Q^c$. Now for any $y \in B$ we have $x\Gamma S\Gamma y \subseteq Q$ and hence $y \in PR(Q)$. Consequently, $B \subseteq PR(Q)$ and so Q is primary. □

Theorem 3.7. *Let μ be a fuzzy subset of a $\Gamma-$semiring S. If (i) $\mu(0) = 1$, (ii) μ_0 is a primary ideal of S and (iii) $\mu(S) = \{1, t\}$ where $t \in [0, 1)$ then μ is a fuzzy primary ideal of S.*

Proof. From the condition (iii), μ is non-constant. Also μ is a fuzzy ideal of S as μ_0 is an ideal of S. Let $\sigma, \theta \in FI(S)$ such that $\sigma\Gamma\theta \subseteq \mu$. Let $\sigma \nsubseteq \mu$ and $\theta \nsubseteq PR(\mu)$. Then there exist $x, y \in S$ such that $\sigma(x) > \mu(x)$ and $\theta(y) > (PR(\mu))(y)$. Since $\mu(0) = 1 = (PR(\mu))(0)$, $x \notin \mu_0$ and $y \notin (PR(\mu))_0$. So by Lemma 3.4, $y \notin PR(\mu_0)$. Hence $x\Gamma S\Gamma y \nsubseteq \mu_0$ as μ_0 is a primary ideal of S (cf. Theorem 3.6). Hence $\mu(x\gamma_1 s\gamma_2 y) = t \neq 1$, for some $\gamma_1, \gamma_2 \in \Gamma,\ s \in S$. Again $\mu(x) \neq 1$. So $\mu(x) = t$, by condition (ii). Hence $\sigma(x) > \mu(x) = t$. Again since $\mu(y) \leq (PR(\mu))(y) < \theta(y)$, $\mu(y) \neq 1$. So $t = \mu(y) < \theta(y)$. Now $t = \mu(x\gamma_1 s\gamma_2 y) \geq (\sigma\Gamma\theta)(x\gamma_1 s\gamma_2 y) \geq \min[\sigma(x), \theta(y)] > t$ which is a contradiction. Hence μ is a fuzzy primary ideal of S. □

Corollary 3.8. *If Q is a primary ideal of S, then λ_Q is a fuzzy primary ideal of S.*

Proposition 3.9. *If μ be a non-constant fuzzy ideal of S then $\overline{\mu} \neq \phi$.*

Proof. Since μ is not constant, there exists $s \in S$ such that $\mu(s) \neq \mu(0)$. Let $\mu(s) < t < \mu(0)$. Then $\mu_t \neq S$. Again μ_t is an ideal of S (*cf. Proposition 2.8[8]*). So there exists a prime ideal P of S such that $\mu_t \subseteq P \subset S$ *(cf. [7])*. Let σ be a fuzzy subset of S defined by

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in P \\ t & \text{if } x \notin P \end{cases}$$

Then σ is a fuzzy prime ideal of S *(cf. Theorem 3.4[11])*. Let $x \in S$. Then either $\mu(x) \geq t$ or $\mu(x) < t$. If $\mu(x) < t$ then $x \notin \mu_t \subseteq P$ which implies that $\sigma(x) = t$. So $\mu(x) < \sigma(x)$. Again if $\mu(x) \geq t$ then $x \in \mu_t \subseteq P$ whence $\sigma(x) = 1$. Then $\mu(x) \leq \sigma(x)$. Hence $\mu(x) \leq \sigma(x)$ for all $x \in S$. Thus $\mu \subseteq \sigma$ and consequently, $\sigma \in \overline{\mu}$. Hence $\overline{\mu} \neq \phi$. □

Proposition 3.10. *Let* $\sum_{i=1}^{n}[\delta_i, e_i],\ \delta_i \in \Gamma,\ e_i \in S\ (i = 1, 2, ..., n)$ *be the right unity of S and* μ *be a non-constant fuzzy ideal of S. Let* $s \in S$ *be such that* $\min_i\{\mu(e_i)\} < \mu(s)$*. Then there exists* $e \in \{e_i : i = 1, 2, ..., n\}$ *such that* $(PR(\mu))(e) < \mu(s)$*.*

Proof. Let $\mu(s) = p$ and $\min_i\{\mu(e_i)\} = t = \mu(e^{'})$ where $e^{'} \in \{e_i : i = 1, 2, ..., n\}$. Let $t < r < p$. Then μ_r is a proper ideal of S as $e^{'} \notin \mu_r$. Let P be a prime ideal of S such that $\mu_r \subseteq P \subset S$. Let θ be a fuzzy subset of S defined by

$$\theta(s) = \begin{cases} 1 & \text{if } s \in P \\ r & \text{if } s \notin P \end{cases}$$

Then as in Proposition 3.9 we can prove $\theta \in \overline{\mu}$. Now since P is a proper ideal of S, there exists atleast one $e \in \{e_i : i = 1, 2, ..., n\}$ such that $e \notin P$. Otherwise if $e \in P$ for all $i = 1, 2, ..., n$ then $x = \sum_i x\delta_i e_i \in P$, for all $x \in S$ and then $P = S$, a contradiction. Hence $\theta(e) = r$. Again $\theta \in \overline{\mu}$, so $PR(\mu) \subseteq \theta$. Therefore $(PR(\mu))(e) \leq \theta(e) = r < p < \mu(s)$. □

Lemma 3.11. *If* $\mu \in FI(S)$ *such that* $Im\ \mu = \{1, t\}$ *where* $t \in [0, 1)$ *then* $(PR(\mu))_0 = PR(\mu_0)$*.*

Proof. Let $x \in (PR(\mu))_0$. Then $(PR(\mu))(x) = (PR(\mu))(0) = 1$. So for $\theta \in \overline{\mu}$, $\theta(x) = 1$. Thus $x \in \theta_0$ for every $\theta \in \overline{\mu}$. Let P be a prime ideal of S such that $\mu_0 \subseteq P$. Now let us define a fuzzy subset σ of S defined by

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in P \\ s & \text{if } x \notin P \end{cases}$$

where $s \in [0, 1),\ s > t$. Then σ is a fuzzy prime ideal of S *(cf. Theorem 3.4[11])* such that $\mu \subseteq \sigma$. Hence $x \in \sigma_0 = P$. Thus $x \in \cap\{P : P$ is a prime ideal of S and $\mu_0 \subseteq P\}$. i.e., $x \in PR(\mu_0)$. Thus we have $(PR(\mu))_0 \subseteq PR(\mu_0)$. Again by Lemma 3.4, $PR(\mu_0) \subseteq (PR(\mu))_0$. Hence $(PR(\mu))_0 = PR(\mu_0)$. □

Theorem 3.12. *Let* μ *be a fuzzy primary ideal of S. Then (i)* $\mu(0) = 1$*, (ii)* $\mid \mu(S) \mid = 2$ *and (iii)* μ_0 *is a primary ideal of S.*

Proof. (i) Let $\mu(0) = s < 1$ and $\min_i \mu(e_i) = r$ where $\sum_{i=1}^{n}[\delta_i, e_i]$ is the right unity of S. Then by Proposition 3.10 there exists $e \in \{e_i : i = 1, 2, ..., n\}$ such that $(PR(\mu))(e) = t < \mu(0) = s$. Let $s < q \leq 1$. Again $r = \min_i \mu(e_i) \leq \mu(e) \leq$

$(PR(\mu))(e) = t$ (*cf. Proposition 2.5*). So we have $r \leq t < s < q \leq 1$. Let σ, θ be two fuzzy subsets of S defined by $\sigma(x) = s$ for all $x \in S$ and

$$\theta(x) = \begin{cases} q & \text{if } x \in \mu_0 \\ r & \text{if } x \notin \mu_0 \end{cases}$$

Then σ, θ are fuzzy subsets of S. Let $x \in S$. If $x \in \mu_0$. Then $\mu(x) = s$ and

$$(\theta\Gamma\sigma)(x) = \begin{cases} \sup\limits_{x=u\gamma v} [\min[\theta(u), \sigma(v)] : u, v \in S; \gamma \in \Gamma] = s \\ 0 \quad \text{otherwise} \end{cases}$$

Therefore, $(\theta\Gamma\sigma)(x) \leq s = \mu(x)$. Now if $x \notin \mu_0$ then $\theta(x) = r$. In that case, $(\theta\Gamma\sigma)(x) = r = \min\limits_i \mu(e_i) \leq \mu(x)$. So $\theta\Gamma\sigma \subseteq \mu$. Now $\theta(0) = q > s = \mu(0)$ which implies that $\theta \nsubseteq \mu$. Again for some $e \in \{e_i : i = 1, 2, ..., n\}$, $\sigma(e) = s > t = (PR(\mu))(e)$. This implies that $\sigma \nsubseteq PR(\mu)$. Thus $\theta \nsubseteq \mu$ and $\sigma \nsubseteq PR(\mu)$ but $\theta\Gamma\sigma \subseteq \mu$, which is a contradiction to the assumption that μ is a fuzzy primary ideal of S. Hence $\mu(0) = 1$.

(ii) Since μ is not constant, $| \mu(S) | \geq 2$. Let us suppose that $| \mu(S) | \geq 3$. Let $\min\limits_i \mu(e_i) = r$. Then there exists $s \in \mu(S)$ such that $r < s < 1$ as $\mu(e_i) \leq \mu(x)$ for all $x \in S$ and for all $i = 1, 2, ..., n$. Let $t \in S$ be such that $\mu(t) = s$. Then there exists $e \in \{e_i : i = 1, 2, ..., n\}$ such that $(PR(\mu))(e) < \mu(t)$. Let σ, θ be two fuzzy ideals of S defined by $\sigma(x) = s$ for all $x \in S$ and

$$\theta(x) = \begin{cases} 1 & \text{if } x \in \mu_s \\ r & \text{if } x \notin \mu_s \end{cases}$$

Then σ, θ are fuzzy subsets of S and $\theta\Gamma\sigma \subseteq \mu$. Now $\theta(t) = 1 > s = \mu(t)$. Thus $\theta \nsubseteq \mu$. Also $\sigma(e) = s = \mu(t) > (PR(\mu))(e)$. Hence $\sigma \nsubseteq PR(\mu)$. Thus $\theta \nsubseteq \mu$ and $\sigma \nsubseteq PR(\mu)$ but $\theta\Gamma\sigma \subseteq \mu$, which is a contradiction. Hence $| \mu(S) | = 2$.

(iii) Let A and B be two ideals of S such that $A\Gamma B \subseteq \mu_0$. Let $\sigma = \lambda_A$ and $\theta = \lambda_B$. Then $\sigma\Gamma\theta \subseteq \mu$ implies that either $\sigma \subseteq \mu$ or $\theta \subseteq PR(\mu)$. If $\sigma \subseteq \mu$ then $A \subseteq \mu_0$. If $\theta \subseteq PR(\mu)$ then $B \subseteq (PR(\mu))_0 \subseteq PR(\mu_0)$ by Proposition 3.11. Hence μ_0 is a primary ideal of S. □

Corollary 3.13. *Let I be an ideal of S such that λ_I is a fuzzy primary ideal of S. Then I is a primary ideal of S.*

Proof. Since λ_I is a fuzzy primary ideal of S, $I = (\lambda_I)_0$ is a primary ideal of S. □

Combining Theorem 3.7 and Theorem 3.12 we have the following Theorem.

Theorem 3.14. *Let μ be a fuzzy ideal of S. Then μ is a fuzzy primary ideal of S if and only if (i) $\mu(0) = 1$, (ii) $| \mu(S) | = 2$ and (iii) μ_0 is a primary ideal of S.*

4 Fuzzy nil radical of $\Gamma-$semiring

Throughout this section we assume that S is a commutative $\Gamma-$semiring.

Definition 4.1. Let I be an ideal of a $\Gamma-$semiring S. The subset $\sqrt{I}$ of S defined by $\sqrt{I} = \{x \in S : x(\gamma x)^{n-1} \in I$, for some $n \in Z^+$, for all $\gamma \in \Gamma\}$ is called nil radical of I.

Definition 4.2. Let μ be a fuzzy ideal of a $\Gamma-$semiring S. Then the fuzzy subset $\sqrt{\mu}$ of S, defined by $\sqrt{\mu} = \sup\limits_{n\in Z^+} \inf\limits_{\gamma\in\Gamma} \mu(x(\gamma x)^{n-1})$ is said to be the fuzzy nil radical of μ.

Proposition 4.3. *Let I be an ideal of S and λ_I be its characteristic function. Then $\sqrt{\lambda_I} = \lambda_{\sqrt{I}}$.*

Proof. Let I be an ideal of S and λ_I be its characteristic function. Let $x \in S$. If $x \in \sqrt{I}$ then $x(\gamma x)^{n-1} \in I$, for some $n \in Z^+$, for all $\gamma \in \Gamma$. Then $\lambda_I(x(\gamma x)^{n-1}) = 1$, for some $n \in Z^+$, for all $\gamma \in \Gamma$. Thus $\inf\limits_{\gamma\in\Gamma} \lambda_I(x(\gamma x)^{n-1}) = 1$ for some $n \in Z^+$ and so $\sup\limits_{n\in Z^+} \inf\limits_{\gamma\in\Gamma} \lambda_I(x(\gamma x)^{n-1}) = 1 = \lambda_{\sqrt{I}}(x)$. Thus $\sqrt{\lambda_I}(x) = \lambda_{\sqrt{I}}(x)$ when $x \in \sqrt{I}$.

Now if $x \notin \sqrt{I}$ then for some $\gamma \in \Gamma$, $x(\gamma x)^{n-1} \notin I$ for all $n \in Z^+$. Therefore $\lambda_I(x(\gamma x)^{n-1}) = 0$ for some $\gamma \in \Gamma$ and for all $n \in Z^+$. Thus $\inf\limits_{\gamma\in\Gamma} \lambda_I(x(\gamma x)^{n-1}) = 0$ for all $n \in Z^+$. So $\sqrt{\lambda_I}(x) = \sup\limits_{n\in Z^+} \inf\limits_{\gamma\in\Gamma} \lambda_I(x(\gamma x)^{n-1}) = 0 = \lambda_{\sqrt{I}}(x)$. Thus $\sqrt{\lambda_I}(x) = \lambda_{\sqrt{I}}(x)$ for all $x \in S$. Hence $\sqrt{\lambda_I} = \lambda_{\sqrt{I}}$. □

Proposition 4.4. *Let S be a commutative $\Gamma-$semiring with identity. If μ is a fuzzy ideal of S then $\sqrt{\mu}$ is a fuzzy ideal of S.*

Proof. Let $x, y \in S$ and $\gamma \in \Gamma$. Since S is a commutative $\Gamma-$semiring with identity for $m, n \in Z^+$ we have

$$(x+y)(\gamma(x+y)^{m+n-1}) = x(\gamma x)^{m-1}(\gamma \sum_{i=0}^{n} \binom{m+n}{i} x(\gamma x)^{n-i-1} y(\gamma y)^{i-1})$$

$$+ \ y(\gamma y)^{n-1})(\gamma \sum_{i=n+1}^{m+n} \binom{m+n}{i} x(\gamma x)^{m+n-i-1} y(\gamma y)^{i-n-1}).$$ Therefore

$$\mu((x+y)(\gamma(x+y)^{m+n-1})) \geq \min[\mu(x(\gamma x)^{m-1}(\gamma \sum_{i=0}^{n} \binom{m+n}{i} x(\gamma x)^{n-i-1} y(\gamma y)^{i-1})),$$

$$\mu(y(\gamma y)^{n-1})(\gamma \sum_{i=n+1}^{m+n} \binom{m+n}{i} x(\gamma x)^{m+n-i-1} y(\gamma y)^{i-n-1})]$$

$$\geq \min[\mu(x(\gamma x)^{m-1}), \mu(y(\gamma y)^{n-1})], \text{ for all } m, n \in Z^+.$$

Now $\sqrt{\mu}(x+y) = \sup_{k\in Z^+}\inf_{\gamma\in\Gamma} \mu[(x+y)(\gamma(x+y))^{k-1}] \geq \sup_{m,n\in Z^+}\inf_{\gamma\in\Gamma} \mu[(x+y)(\gamma(x+y))^{m+n-1}] \geq \sup_{m,n\in Z^+}\inf_{\gamma\in\Gamma} \min[\mu(x(\gamma x)^{m-1}), \mu(y(\gamma y)^{n-1})] = \min[\sup_{m\in Z^+}\inf_{\gamma\in\Gamma} \mu(x(\gamma x)^{m-1}), \sup_{n\in Z^+}\inf_{\gamma\in\Gamma} \mu(y(\gamma y)^{n-1})] = \min[\sqrt{\mu}(x), \sqrt{\mu}(y)]$. Again $\sqrt{\mu}(x\gamma y) = \sup_{n\in Z^+}\inf_{\delta\in\Gamma} \mu(x\gamma y)(\delta(x\gamma y))^{n-1} \geq \sup_{n\in Z^+}\inf_{\delta\in\Gamma} \mu[y(\delta(x\gamma y))^{n-1}] \geq \sup_{n\in Z^+}\inf_{\delta\in\Gamma} \mu[y(\delta y)^{n-1}]$ (since S is commutative) $= \sqrt{\mu}(y)$. Similarly $\sqrt{\mu}(x\gamma y) \geq \sqrt{\mu}(x)$. Hence $\sqrt{\mu}$ is a fuzzy ideal of S. □

Proposition 4.5. *Let $\mu, \theta \in FI(S)$. Then the following are hold:*
(i) $\mu \subseteq \sqrt{\mu}$,
(ii) $\mu \subseteq \theta$ *implies that* $\sqrt{\mu} \subseteq \sqrt{\theta}$,
(iii) $\sqrt{\sqrt{\mu}} = \sqrt{\mu}$,
(iv) $\sqrt{\mu_t} \subseteq (\sqrt{\mu})_t$,
(v) $\sqrt{\mu} \cap \sqrt{\theta} = \sqrt{\mu\cap\theta} = \sqrt{\mu\circ\theta}$,
(vi) $\sqrt{\mu\oplus\theta} = \sqrt{\sqrt{\mu}\oplus\sqrt{\theta}}$, *provided* $\mu(0) = \theta(0) = 1$,
(vii) $\sqrt{\mu_0} = (\sqrt{\mu})_0$.

Proof. (i) $\mu(x(\gamma x)^{n-1}) \geq \mu(x)$ for all $n \in Z^+$ and for all $\gamma \in \Gamma$. Thus $\inf_{\gamma\in\Gamma}\mu(x(\gamma x)^{n-1}) \geq \mu(x)$ for all $n \in Z^+$, implies that
$\sup_{n\in Z^+}\inf_{\gamma\in\Gamma} \mu(x(\gamma x)^{n-1}) \geq \mu(x)$. i.e., $\sqrt{\mu}(x) \geq \mu(x)$ for all $x \in S$. So, $\mu \subseteq \sqrt{\mu}$.

(ii) $\sqrt{\mu}(x) = \sup_{n\in Z^+}\inf_{\gamma\in\Gamma} \mu(x(\gamma x)^{n-1}) \leq \sup_{n\in Z^+}\inf_{\gamma\in\Gamma} \theta(x(\gamma x)^{n-1}) = \sqrt{\theta}(x)$ for all $x \in S$. Thus $\sqrt{\mu} \subseteq \sqrt{\theta}$.

(iii) $\sqrt{\sqrt{\mu}}(x) = \sup_{n\in Z^+}\inf_{\gamma\in\Gamma} \sqrt{\mu}(x(\gamma x)^{n-1}) = \sup_{n\in Z^+}\inf_{\gamma\in\Gamma} [\sup_{m\in Z^+}\inf_{\delta\in\Gamma} \mu(y(\delta y)^{m-1})]$ where $y = x(\gamma x)^{n-1}$. i.e., $\sqrt{\sqrt{\mu}}(x) = \sup_{n\in Z^+}\sup_{m\in Z^+}\inf_{\gamma\in\Gamma}\inf_{\delta\in\Gamma} \mu(y(\delta y)^{m-1})$
$\leq \sup_{p\in Z^+}\inf_{\beta\in\Gamma} \mu(x(\beta x)^{p-1}) = \sqrt{\mu}(x)$. Therefore $\sqrt{\sqrt{\mu}} \subseteq \sqrt{\mu}$. Again using (i) and (ii) we have $\sqrt{\mu} \subseteq \sqrt{\sqrt{\mu}}$ and hence $\sqrt{\sqrt{\mu}} = \sqrt{\mu}$.

(iv) Let $x \in \sqrt{\mu_t}$. Then $x(\gamma x)^{n-1} \in \mu_t$ for some $n \in Z^+$ and for all $\gamma \in \Gamma$. Thus $\mu(x(\gamma x)^{n-1}) \geq t$ for some $n \in Z^+$ and for all $\gamma \in \Gamma$. Therefore $\inf_{\gamma\in\Gamma} \mu(x(\gamma x)^{n-1}) \geq t$ for some $n \in Z^+$ and so $\sup_{n\in Z^+}\inf_{\gamma\in\Gamma} \mu(x(\gamma x)^{n-1}) \geq t$ implies that $\sqrt{\mu}(x) \geq t$ and consequently, $x \in (\sqrt{\mu})_t$. Hence $\sqrt{\mu_t} \subseteq (\sqrt{\mu})_t$.

(v) We have $\mu\circ\theta \subseteq \mu\cap\theta \subseteq \mu,\ \theta$. Thus from (ii), $\sqrt{\mu\circ\theta} \subseteq \sqrt{\mu\cap\theta} \subseteq \sqrt{\mu},\ \sqrt{\theta}$.

Therefore $\sqrt{\mu\cap\theta}\subseteq\sqrt{\mu}\cap\sqrt{\theta}$. Thus

$$\sqrt{\mu\circ\theta}\subseteq\sqrt{\mu\cap\theta}\subseteq\sqrt{\mu}\cap\sqrt{\theta} \tag{1}$$

Again for $x\in S$, $\sqrt{\mu\circ\theta}(x)=\sup\limits_{n\in Z^+}\inf\limits_{\gamma\in\Gamma}(\mu\circ\theta)(x(\gamma x)^{n-1})=$ $=\sup\limits_{n\in Z^+}\inf\limits_{\gamma\in\Gamma}[\sup[\inf\limits_{1\leq i\leq p}[\min[\mu(u_i),\theta(v_i)]]:x(\gamma x)^{n-1}=\sum\limits_{i=1}^{p}u_i\delta_iv_i, u_i,v_i\in S,\gamma\in\Gamma]]\geq\sup\limits_{s,t\in Z^+}\inf\limits_{\gamma\in\Gamma}\min[\mu(x(\gamma x)^{s-1}),\theta(x(\gamma x)^{t-1})]=\min[\sup\limits_{s\in Z^+}\inf\limits_{\gamma\in\Gamma}\mu(x(\gamma x)^{s-1}),$ $\sup\limits_{t\in Z^+}\inf\limits_{\gamma\in\Gamma}\theta(x(\gamma x)^{t-1})]=\min[\sqrt{\mu}(x),\sqrt{\theta}(x)]=(\sqrt{\mu}\cap\sqrt{\theta})(x)$. Thus

$$\sqrt{\mu\circ\theta}\supseteq(\sqrt{\mu}\cap\sqrt{\theta}) \tag{2}$$

Combining (1) and (2) we get the result.

(vi) Since $\mu,\theta\subseteq\mu\oplus\theta$ as $\mu(0)=\theta(0)=1$, it follows that $\sqrt{\mu},\sqrt{\theta}\subseteq\sqrt{\mu\oplus\theta}$ [by (ii)]. Thus $\sqrt{\mu}\oplus\sqrt{\theta}\subseteq\sqrt{\mu\oplus\theta}\oplus\sqrt{\mu\oplus\theta}=\sqrt{\mu\oplus\theta}$. Therefore, by using (iii) we get

$$\sqrt{\sqrt{\mu}\oplus\sqrt{\theta}}\subseteq\sqrt{\sqrt{\mu\oplus\theta}}=\sqrt{\mu\oplus\theta} \tag{A}$$

Again $\mu\subseteq\sqrt{\mu}$ and $\theta\subseteq\sqrt{\theta}$. Therefore, by using (ii), $\mu\oplus\theta\subseteq\sqrt{\mu}\oplus\sqrt{\theta}$. i.e.,

$$\sqrt{\mu\oplus\theta}\subseteq\sqrt{\sqrt{\mu}\oplus\sqrt{\theta}} \tag{B}$$

Combining (A) and (B) we have, $\sqrt{\mu\oplus\theta}=\sqrt{\sqrt{\mu}\oplus\sqrt{\theta}}$.

(vii) For any $x\in S$, $x\in\sqrt{\mu_0}\Leftrightarrow x(\gamma x)^{n-1}\in\mu_0$ for some $n\in Z^+$, for all $\gamma\in\Gamma$ $\Leftrightarrow\mu(x(\gamma x)^{n-1})=\mu(0)$ for some $n\in Z^+$, for all $\gamma\in\Gamma\Leftrightarrow\sup\limits_{n\in Z^+}\inf\limits_{\gamma\in\Gamma}\mu(x(\gamma x)^{n-1})=$ $\mu(0)=\sqrt{\mu}(0)\Leftrightarrow\sqrt{\mu}(x)=\mu(0)=\sqrt{\mu}(0)\Leftrightarrow x\in(\sqrt{\mu})_0$. Hence $\sqrt{\mu_0}=(\sqrt{\mu})_0$.□

Proposition 4.6. *Let $t\in[0,1)$ and μ be a fuzzy ideal of S. Then $(\sqrt{\mu})_{[t]}=\sqrt{\mu_{[t]}}$.*

Proof. For $x\in\sqrt{\mu_{[t]}}\Leftrightarrow x(\gamma x)^{n-1}\in\mu_{[t]}$ for some $n\in Z^+$, for all $\gamma\in\Gamma\Leftrightarrow$ $\mu(x(\gamma x)^{n-1})>t$ for some $n\in Z^+$, for all $\gamma\in\Gamma\Leftrightarrow\inf\limits_{\gamma\in\Gamma}\mu(x(\gamma x)^{n-1})>t$ for some $n\in Z^+\Leftrightarrow\sup\limits_{n\in Z^+}\inf\limits_{\gamma\in\Gamma}\mu(x(\gamma x)^{n-1})>t\Leftrightarrow\sqrt{\mu}(x)>t\Leftrightarrow x\in(\sqrt{\mu})_{[t]}$. Hence $(\sqrt{\mu})_{[t]}=\sqrt{\mu_{[t]}}$. □

Proposition 4.7. *Let μ be a non constant fuzzy ideal of S. Then $\sqrt{\mu}$ is a fuzzy semiprime ideal of S.*

Proof. Let $x \in S$. Now $\inf_{\gamma \in \Gamma} \sqrt{\mu}(x\gamma x) = \inf_{\gamma \in \Gamma} \sup_{n \in Z^+} \inf_{\delta \in \Gamma} \mu((x\gamma x)(\delta(x\gamma x))^{n-1})$ $= \sup_{n \in Z^+} \inf_{\gamma \in \Gamma} \mu((x\gamma x)(\gamma(x\gamma x))^{n-1}) = \sup_{n \in Z^+} \inf_{\gamma \in \Gamma} \mu(x(\gamma x)^{2n-1}) \leq \sup_{m \in Z^+} \mu(x(\gamma x)^{m-1})$ $= \sqrt{\mu}(x)$. Thus $\inf_{\gamma \in \Gamma} \sqrt{\mu}(x\gamma x) \leq \sqrt{\mu}(x)$. Again $\sqrt{\mu}(x\gamma x) \geq \sqrt{\mu}(x)$ and so $\inf_{\gamma \in \Gamma} \sqrt{\mu}(x\gamma x) \geq \sqrt{\mu}(x)$. Hence $\inf_{\gamma \in \Gamma} \sqrt{\mu}(x\gamma x) = \sqrt{\mu}(x)$ and hence $\sqrt{\mu}$ is completely semiprime ideal of S and since S is commutative, we have $\sqrt{\mu}$ is fuzzy semiprime ideal of S. □

Proposition 4.8. *If μ is a fuzzy prime ideal of S then $\sqrt{\mu} = \mu$.*

Proof. Let $x \in S$. Since μ is a fuzzy prime ideal, it is fuzzy semiprime and so $\mu(x\gamma x) = \mu(x)$. Now $\mu(x(\gamma x)^2) = \mu(x\gamma(x\gamma x)) = \max[\mu(x), \mu(x\gamma x)] = \mu(x)$ as S is commutative (cf. Proposition 3.8[11]). In general we can show that $\mu(x(\gamma x)^n) = \mu(x)$. Now $\sqrt{\mu}(x) = \sup_{n \in Z^+} \inf_{\gamma \in \Gamma} \mu(x(\gamma x)^{n-1}) = \sup_{n \in Z^+} \inf_{\gamma \in \Gamma} \mu(x) = \mu(x)$. □

Proposition 4.9. *Let μ be a fuzzy ideal of S. Then $\sqrt{\mu} = PR(\mu)$.*

Proof. Let θ be a fuzzy prime ideal of S such that $\mu \subseteq \theta$. Then by Proposition 4.5, $\sqrt{\mu} \subseteq \sqrt{\theta} = \theta$. Thus $\sqrt{\mu} \subseteq \cap\{\theta : \theta \in FPI(S) \mid \mu \subseteq \theta\}$ $= PR(\mu)$. So $\sqrt{\mu} \subseteq PR(\mu)$.
If possible let $\sqrt{\mu} \neq PR(\mu)$, then there exists an element $s \in S$ such that $\sqrt{\mu}(s) < (PR(\mu))(s)$. Let $\sqrt{\mu}(s) = t$. Then $s \notin (\sqrt{\mu})_{[t]}$. i.e., $s \notin \sqrt{\mu_{[t]}}$ by Proposition 4.6. Then there exists a prime ideal P of S such that $\mu_{[t]} \subseteq P$ and $s \notin P$ (cf. Theorem 3.14[7]). Let us define a fuzzy subset ϕ of S as follows

$$\phi(x) = \begin{cases} 1 & \text{if } x \in P \\ t & \text{if } x \notin P,\ 0 \leq t < 1 \end{cases}$$

Then ϕ is a fuzzy prime ideal of S(cf. Theorem 3.4[11]). Now if $x \in P$ then $\phi(x) = 1$. So $\mu(x) \leq \phi(x)$. If $x \notin P$ then $x \notin \mu_{[t]}$. i.e., $\mu(x) \leq t = \phi(x)$. Thus $\mu(x) \leq \phi(x)$ for all $x \in S$ and so $\mu \subseteq \phi$. Now $\sqrt{\mu}(s) < (PR(\mu))(s) \leq \phi(s)$ $= t = \sqrt{\mu}(s)$, a contradiction. Hence $\sqrt{\mu} = PR(\mu)$. □

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