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THE CONVERGENCE OF GALERKIN METHOD FOR DATA ASSIMILATION PROBLEM IN THE PERTUBED OBSERVATION CONDITION

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Abstract

The purpose of the variational data assimilation problem is to compute the unknown initial condition of a process in order to forecast the next evolution process. In this paper, we prove that the solution of the discrete problem converges to the solution of the initial problem when the observation is pertubed.

1 Introduction

The variation data assimilation problem is important in many practial situations. There are a number of papers concerning this problem (cf. [1, 3, 4, 5, 6, 8]). Numerical method is strongly devoloped for solving this problem [1, 2, 6, 10, 11]. Specially, in [11] E. I. Parmuzin and V. P. Shutyaev proved that the numerical solution of the discrete problem converges to that of the continous problem. However, they did not mention the pertubation of the observation data. In this paper, we consider the case where the input data is pertubed and we use the Galerkin approximation to prove the convergence of the solution of this problem.

Let X, H be Hilbert spaces with $X \subset H$. The inner product of H is denoted by $(\cdot, \cdot)$. Suppose that X is dense in H and H is identified with its

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dual space. Denote by $X^*$and H^* the dual spaces of X and H respectively. Then $X \subset H \subset X^*$. Denote by $L_2(0,T;H), L_2(0,T;X)$ and $L_2(0,T;X^*)$ the spaces of measurable functions $f(t),\ t \in [0,T], T < \infty$, with the corresponding values in H, X, X^*. Set $Y = L_2(0,T;X)$ and $Y^* = L_2(0,T;X^*)$.

We introduce

$$W(0,T) = \big\{ g \mid g \in Y,\ \frac{dg}{dt} \in Y^* \big\}.$$

Then $W(0,T)$ is a Hilbert space with the norm

$$\| g \|_{W(0,T)} = \left(\int_0^T \| g \|_X^2 \, dt + \int_0^T \| \frac{dg}{dt} \|_{X^*}^2 dt \right)^{\frac{1}{2}} \leq \infty.$$

Suppose that $A : Y \longrightarrow Y$ is a linear or nonlinear operator, $u \in H, f \in Y^*$. Consider the problem of finding $\varphi \in W(0,T)$ which satifies the following evolution model conditions

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + A(\varphi) = f, \quad t \in (0,T), \\ \varphi\big|_{t=0} = u. \end{cases} \tag{1}$$

Denote by Y_{obs} the observational space. Then Y_{obs} is a Hilbert space. Suppose that the process (1) can be observed by observational operator $C : Y \longrightarrow Y_{obs}$, with C is a linear bounded operator.

We consider the data assimilation problem, i.e we reconstruct the initial condition $u \in H$ satisfying (1) by using the measurement of the solution $\varphi(x,t)$ at the final time instant $C\varphi(x,t) = \varphi(x,T) = \varphi_T$.

To emphasize the dependence of solution φ of problem (1) on the initial condition u, we write $\varphi = \varphi(u)$. Let $u_0 \in X$ be a prediction of u. We introduce the functional

$$J_\alpha(u) = \frac{\alpha}{2} \| u - u_0 \|_X^2 + \frac{1}{2} \| C\varphi - \varphi_T \|_{Y_{obs}}^2$$

where $\alpha = \text{const} \geq 0$.

In oder to solve the data assimilation problem, we use the variational method, i.e. finding u and φ such that

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + A(\varphi) = f, \quad t \in (0,T) \\ \varphi\big|_{t=0} = u \\ J_\alpha(u) = \inf\limits_v J_\alpha(v) \end{cases} \tag{2}$$

In the next section, we consider problem (1) in case the operator A from Y to Y^* is linear and satisfies following condition

$$\Big(A(t)\nu, \nu \Big) \geq \bar{a} \| \nu \|_X, \quad \bar{a} > 0, \quad \forall \nu \in X, \quad t \in [0,T], \tag{3}$$

The functional $J_0(u)$ is induced by replacing $\alpha = 0$ in functional $J_\alpha(u)$ of the data assimilation problem (2). The observation in this case is pertubed, i.e $\|\varphi_T^\epsilon - \varphi_T\|_{Y_{obs}} \leq \epsilon$, for a given $\epsilon > 0$.
Therefore data assimilation problem (2) becomes: finding functions u and φ satisfying

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + A(t)\varphi = f, \quad t \in (0,T) \\ \varphi|_{t=0} = u \\ J_0(u) = \inf\limits_{v \in H} J_0(v) \end{cases} \tag{4}$$

where

$$J_0(u) = \| C\varphi(u) - \varphi_T^\epsilon \|_{Y_{obs}}^2 .$$

2 The proof of the convergence by using Galerkin method

In Hilbert space H, we choose an orthogonal basis system $\{\psi^n(x)\}$ and we approximate $\varphi(x,t)$ and u in the following form

$$\varphi^N(x,t) = \sum_{n=1}^{N} \varphi_n(t)\psi^n(x), \quad u^N(x) = \sum_{n=1}^{N} u_n \psi^n(x).$$

Therefore, Problem (1) becomes

$$\begin{cases} \sum_{n=1}^N \dfrac{\partial\{\varphi_n(t)\psi^n(x)\}}{\partial t} + \sum_{n=1}^N A(t)\varphi_n(t)\psi^n(x) = f, \quad t \in (0,T) \\ \sum_{n=1}^N \varphi_n(0)\psi^n(x) = \sum_{n=1}^N u_n \psi^n(x) \end{cases}$$

or

$$\begin{cases} \sum_{n=1}^N \dfrac{\partial \varphi_n(t)}{\partial t}\psi^n(x) + \sum_{n=1}^N A(t)\varphi_n(t)\psi^n(x) = f, \quad t \in (0,T) \\ \sum_{n=1}^N \varphi_n(0)\psi^n(x) = \sum_{n=1}^N u_n \psi^n(x). \end{cases}$$

We note that $u^N \in H_N$, where H_N is the linear space generated by $\{\psi^n\}_{n=1,N}$ and the functional $J_0(u)$ becomes

$$J_0(u^N) = \| \sum_{n=1}^{N} \varphi_n(T) C\psi^n(x) - \varphi_T^\epsilon \|_H^2 .$$

Suppose that, for the approximation method in [6], we can find u_*^N such that

$$J_N^* \leq J_0(u_*^N) \leq J_N^* + \epsilon_N, \tag{5}$$

where $J_N^* = \inf\limits_{u^N \in X_N} J_0(u^N)$ and $\epsilon_N \longrightarrow 0$ when $N \longrightarrow \infty$.

The following theorem is the main result of this paper.

Main theorem. *Suppose that u_*^N satisfies the condition (5) and u^* is a solution of (4). Then u_*^N weakly converges to u^* in X and $\varphi^N(u_*^N)$ weakly converges to φ^* in $L_2(0,T;H)$, where φ^* is the solution of the following problem*

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + A(t)\varphi = f, \quad t \in (0,T) \\ \varphi|_{t=0} = u^*. \end{cases}$$

Before proving this theorem, we need the following lemma.

Lemma. *Let $u \in H$ and*

$$u^N = \sum_{n=1}^{N} u_n \psi^n \to u \quad \textit{in } H.$$

Then

$$|J_0(u) - J_0(u^N)| \longrightarrow 0 \quad \textit{when} \quad N \longrightarrow \infty.$$

Proof. We have

$$\begin{aligned} J_0(u) - J_0(u^N) &= \| C\varphi(u) - \varphi_T^\epsilon \|_X^2 - \| C\varphi^N(u^N) - \varphi_T^\epsilon \|_X^2 \\ &= \| C\varphi(u) - C\varphi^N(u^N) + C\varphi^N(u^N) - \varphi_T^\epsilon \|_X^2 \\ &\quad - \| C\varphi^N(u^N) - \varphi_T^\epsilon \|_X^2 \\ &= \| C(\varphi(u) - \varphi^N(u^N)) \|_X^2 \\ &\quad + 2 < C(\varphi(u) - \varphi^N(u^N)), C\varphi^N(u^N) - \varphi_T^\epsilon > \\ &\le \|C\|^2 \|\varphi(u) - \varphi^N(u^N)\|_Y^2 \\ &\quad + 2\|C\| \|\varphi(u) - \varphi^N(u^N)\|_Y (\|C\| \|\varphi^N(u^N)\|_Y \\ &\quad + \|\varphi^N(u^N)\|_Y + \|\varphi_T^\epsilon\|). \end{aligned}$$

Hence $\varphi^N(u^N)$ converges to $\varphi(u)$ in Y and $\|C\|, \|\varphi^N(u^N)\|_Y, \|\varphi_T^\epsilon\|$ are bounded. Therefore

$$|J_0(u) - J_0(u^N)| \longrightarrow 0 \quad \text{when} \quad N \longrightarrow \infty$$

and the lemma follows. □

Proof of Main theorem. For a given $\delta > 0$, there exists $u^\delta \in H$ such that

$$J_0^* \le J_0(u^\delta) \le J_0^* + \frac{\delta}{2}.$$

Hence $J^* = \inf J_0(u)$. Moreover, by the above lemma, there exists a positive integer N^* such that

$$|J_0(u^\delta) - J_0(u^N)| < \frac{\delta}{2}$$

for all $N \geq N^*$. The above inequality implies $-\frac{\delta}{2} < J_0(u^\delta) - J_0(u^N) < \frac{\delta}{2}$. Therefore $J_0(u^N) < J_0(u^\delta) + \frac{\delta}{2}$.

So, we have

$$J_N^* \leq J_0(u^N) < J_0(u^\delta) + \frac{\delta}{2} \leq J_0^* + \delta.$$

Hence $\limsup J_N^* \leq J_0^*$. Since, $J_0^* \leq J_N^*$, we have $\liminf J_N^* \geq J_0^*$. Therefore $\lim J_N^*$ exists and

$$\lim J_N^* = J_0^*. \tag{6}$$

It implies that

$$0 \leq J_0(u_*^N) - J_0^* \leq |J_0(u_*^N) - J_N^*| + |J_N^* - J_0^*|.$$

So, by (5), $|J_0(u_*^N) - J_N^*|$ approaches to 0. By (6), $|J_N^* - J^*|$ approaches to 0. Therefore, u_*^N is a minimum sequense of functional $J_0(u)$. Hence, $J_0(u)$ is weak lower semi-continuous. Hence u_*^N weakly converges to u^*. Therefore, $\varphi(u_*^N)$ weakly converges to φ^* in Y, and the proof of Main theorem is completed. □

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