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ON CHARACTERIZATIONS OF CONVEX VECTOR FUNSTIONS AND OPTIMIZATION

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Abstract

In this paper, we present characterizations of convex vector functions via generalized monotonicity of their directional derivatives and differentials. By applying these results to vector optimization, we have established some necessary/sufficient conditions for optimality of vector optimization problems, especially the Kuhn-Tucker condition for constrained problems. The results obtained in this paper generalize some corresponding well-known results of W. Fenchel [8], O.L. Mangasarian [9] and R.T. Rockafellar [7] in the scalar case.

1 Introduction

Since convex analysis is almost complete then recently people have taken their attention to generalized convexity. The notion of convex vector functions has been studied by many authors (see, [2-6, 10]) because this plays an important role in vector optimization. In this paper, we establish some characterizations of convex vector functions via generalized monotonicity of their directional derivatives and their differentials and then we use these results to establish some necessary/sufficient conditions for optimality of vector optimization problems. The paper is structured as follows. In the next section, we present some preliminaries concerning concepts of directional derivatives, cone order, convex vector functions and monotonicity. Section 3 is devoted to studying characterizations

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of convex vector functions via generalized monotonicity of their directional derivative and differentials. In section 4 we consider the structure of the set of minimum points of convex vector functions. Especially, necessary and sufficient conditions of minimum points of vector functions are established. Section 5 establishes sufficient conditions of Kuhn-Tucker type for vector problems with constraints. The results obtained in this paper generalize some well-known results of W. Fenchel [8], O.L. Mangasarian [9] and R.T. Rockafellar [7] in the scalar case.

2 Preliminaries

Let $D \subset R^n$ be a nonempty set and let $x \in D$. We say that a direction $v \in R^n$ is a feasible direction of D at x if there exists $t_0 > 0$ such that $x + tv \in D, \forall t \in [0, t_0]$. The set of feasible directions of D at x is denoted by $T_D(x)$. When D is convex, it is easy to see that $T_D(x)$ is a convex cone. If $x \in riD$ then $T_D(x)$ is the subspace of R^n which paralells the affine hull of D. Especially, if $x \in intD$ then $T_D(x) = R^n$.

Let $f : D \to R^m, x \in D, v \in T_D(x)$. The directional derivative of f at x in the direction v, denoted by $f'(x; v)$, is defined as the following limit (if such exists)

$$f'(x; v) = \lim_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Lemma 2.1. *Let f be a vector function from a set $D \subset R^n$ to R^m. Let $x \in D$ and $v \in T_D(x)$. Assume that $f'(x; v)$ exists. Then*

i) $f'(x; \lambda v) = \lambda f'(x; v),\ \forall \lambda \geq 0$.

ii) $(\xi f)'(x; v) = \xi f'(x; v),\ \forall \xi \in L(R^m, R)$, (where, $L(R^m, R)$ denotes the space of linear functionals on R^m).

Proof. i) Let $\lambda > 0$. We have

$$\lim_{t \downarrow 0} \frac{f(x + t\lambda v) - f(x)}{t} = \lim_{t \downarrow 0} \lambda \frac{f(x + t\lambda v) - f(x)}{t\lambda} = \lambda \lim_{t \downarrow 0} \frac{f(x + t\lambda v) - f(x)}{t\lambda}.$$

Hence, $f'(x; \lambda v)$ exists and

$$f'(x; \lambda v) = \lambda f'(x; v).$$

If $\lambda = 0$ then the equality is obvious.

ii) We have,

$$\begin{aligned}(\xi f)'(x;v) &= \lim_{t\downarrow 0}\frac{\xi f(x+tv)-\xi f(x)}{t}.\\ &= \lim_{t\downarrow 0}\xi\big[\frac{f(x+tv)-f(x)}{t}\big].\\ &= \xi\big[\lim_{t\downarrow 0}\frac{f(x+tv)-f(x)}{t}\big].\\ &= \xi f'(x;v).\end{aligned}$$

The proof is complete. □

The following lemma is a basic result of vector analysis which will be needed in the sequel.

Lemma 2.2. *Let f be a vector function from a set $D \subset R^n$ with $intD \neq \emptyset$ to R^m and let $x \in intD$. If f is differentiable at x then*
i) $f'(x;v) = Df(x)(v),\ \forall v \in R^n$,
ii) $D(\xi f)(x) = \xi Df(x),\ \forall \xi \in L(R^m, R)$,
(where, $Df(x)$ denotes the differential of f at x).

Let $f : D \subset R^n \to R^m$, $x \in D, v \in T_D(x)$. The generalized directional derivative of f at x in the direction v (proposed by C. Cusano, M. Fini, D. Torre in [10]), denoted by ${f'}_D(x;v)$, is defined as the following set

$${f'}_D(x;v) := \big\{l = \lim_{k\to\infty}\frac{f(x+t_kv)-f(x)}{t_k}, t_k \downarrow 0\big\}.$$

From definitions we have immediately

Lemma 2.3. *If $f'(x;v)$ exists then ${f'}_D(x;v) = f'(x;v)$.*

We recall that a nonempty set $C \subset R^m$ is said to be a cone if

$$tc \in C, \forall c \in C, t \geq 0.$$

Put $lC := C \cap (-C)$. A cone C is said to be pointed if $lC = \{0\}$. The polar cone of a cone C is defined as the set

$$C' := \{\xi \in L(R^m, R)\ :\ \xi(c) \geq 0, \forall c \in C\}.$$

A cone $C \subset R^m$ specifies on R^m an order defined by

$$x \preceq y \Leftrightarrow y - x \in C.$$

When $x \preceq y$ and not $y \preceq x$ then we write $x \prec y$. If $intC \neq \emptyset$ then $x \ll y$ means $y - x \in intC$. A nonempty set $A \subset R^m$ is said to be lower bounded if there exist $a \in R^m$ such that

$$a \preceq x,\ \forall x \in A.$$

The following lemma will be needed in the next section.

Lemma 2.4. *Assume that the ordered cone $C \subset R^m$ is closed and convex. Let $c \in R^m$. Then*

i) $c \in C \Leftrightarrow \xi(c) \geq 0,\ \forall \xi \in C' \setminus \{0\}$.

ii) Supposing that $intC \neq \emptyset$. Then

$$c \in intC \Leftrightarrow \xi(c) > 0,\ \forall \xi \in C' \setminus \{0\}.$$

Proof. i) $\underline{\Rightarrow}$: It is immediate from the definition of C'.

$\underline{\Leftarrow}$: Suppose to the contrary that, $c \notin C$. By the strong separation theorem, there exists $\xi \in L(R^m, R)$ such that

$$\xi(c) < \xi(x),\ \forall x \in C.$$

If there exists $x_0 \in C$ such that $\xi(x_0) < 0$ then $\xi(c) < \xi(tx_0) = t\xi(x_0)$, for every $t > 0$. Take $t \to +\infty$, we meet a contradiction. Hence, $\xi(x) \geq 0$, for all $x \in C$. Therefore, $\xi \in C'$ and $\xi(c) < \xi(0) = 0$ which contradicts the hypothesis. Thus, $c \in C$.

ii) $\underline{\Rightarrow}$: There exists $r > 0$ such that $B(c, r) \subset C$. Suppose to the contrary that there exists $\xi \in C' \setminus \{0\}$ such that $\xi(c) = 0$. Since $\xi(x) \geq 0$ for every $x \in B(c, r)$, it implies $\xi(x) = 0$, for every $x \in B(c, r)$. Hence, $\xi = 0$ which contradicts the assumption above.

$\underline{\Leftarrow}$: Suppose in the contrary that, $c \notin intC$. By the separation theorem, there exists $\xi \in L(R^m, R) \setminus \{0\}$ such that

$$\xi(c) < \xi(x),\ \forall x \in intC.$$

If there exists $x_0 \in intC$ such that $\xi(x_0) < 0$ then $\xi(c) < \xi(tx_0) = t\xi(x_0)$, for every $t > 0$. Take $t \to +\infty$, we meet a contradiction. Hence, $\xi(x) \geq 0$, for every $x \in intC$. Thus, $\xi \in C' \setminus \{0\}$ and $\xi(c) \leq \xi(0) = 0$ which contradicts the hypothesis. The proof is complete. □

Lemma 2.5. *Let $C \subset R^m$ be a convex cone which is not the whole space. If $intC \neq \emptyset$ then $\{0\} \cup intC$ is a pointed cone.*

Proof. Suppose in the contrary that $\{0\} \cup intC$ is not pointed. Then there exist $c \in intC \cap (-intC)$. Since $intC$ is convex then $0 = \frac{1}{2}c + \frac{1}{2}(-c) \in intC$. This implies $C = R^m$ which contradicts assumptions. The proof is complete. □

A vector function f from a nonempty convex set $D \subset R^n$ to R^m is said to be convex with respect to C if for every $x, y \in D,\ \lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \preceq \lambda f(x) + (1-\lambda) f(y),$$

i.e.,

$$\lambda f(x) + (1-\lambda)f(y) \in f(\lambda x + (1-\lambda)y) + C.$$

Supposing that $intC \neq \emptyset$. f is said to be strictly convex with respect to C if for every $x, y \in D, x \neq y,\ \lambda \in (0,1)$, one has

$$f(\lambda x + (1-\lambda)y) \ll \lambda f(x) + (1-\lambda)f(y),$$

i.e.,

$$\lambda f(x) + (1-\lambda)f(y) \in f(\lambda x + (1-\lambda)y) + intC.$$

The lemma below shows the relation between scalar convex functions and vector convex functions which will be needed in the next section.

Lemma 2.6. ([Lemma 21, 2]) *Assume that the ordered cone $C \subset R^m$ is closed and convex. Let f be a vector function from a nonempty and convex set $D \subset R^n$ to R^m. Then,*

i) f is convex with respect to C if and only if ξf is convex, for every $\xi \in C' \setminus \{0\}$.

ii) Supposing that $intC \neq \emptyset$. f is strictly convex if only if ξf is strictly convex, for every $\xi \in C' \setminus \{0\}$.

Lemma 2.7. *Assume that the ordered cone $C \subset R^m$ is convex. Let $D \subset R^n$ be a nonempty convex set and $f : D \subset R^n \to R^m$ be a convex vector function with respect to C. Let $x \in D$ and $v \in T_D(x) \setminus \{0\}$. Then,*

i) The function $g(t) := \frac{f(x+tv)-f(x)}{t}$, $t > 0$, is increasing with respect to C , i.e.,

$$0 < t < t' \Rightarrow g(t) \preceq g(t').$$

ii) Supposing that $intC \neq \emptyset$ and f is strictly convex. Then the function $g(t)$, $t > 0$, is strictly increasing with respect to C, i.e.,

$$0 < t < t' \Rightarrow g(t) \ll g(t').$$

Proof. Let $0 < t < t'$. We have, $x + tv = \lambda x + (1-\lambda)(x + t'v)$, where, $\lambda = 1 - \frac{t}{t'} \in (0,1)$.

i) Since f is convex then $f(x+tv) \preceq \lambda f(x) + (1-\lambda)f(x+t'v)$. Since $\lambda = 1 - \frac{t}{t'}$ this implies $\frac{f(x+tv)-f(x)}{t} \preceq \frac{f(x+t'v)-f(x)}{t'}$.

ii) Since f is strictly convex then $f(x+tv) \ll \lambda f(x) + (1-\lambda)f(x+t'v)$. Since $\lambda = 1 - \frac{t}{t'}$ this implies $\frac{f(x+tv)-f(x)}{t} \ll \frac{f(x+t'v)-f(x)}{t'}$. The proof is complete. □

Lemma 2.8. ([Proposition 4.3, 6]) *Assume that the ordered cone $C \subset R^m$ is closed, convex and pointed. Let $D \subset R^n$ be a nonempty convex set and $f : D \subset R^n \to R^m$ be convex. Let $x \in D$ and $v \in T_D(x)$. If the set $\{\frac{f(x+tv)-f(x)}{t} | t > 0\}$ is bounded below then $f'(x;v)$ exists.*

Corollary 2.9. *Assume that the ordered cone $C \subset R^m$ is closed, convex and pointed. Let $D \subset R^n$ be a nonempty convex set and $f : D \subset R^n \to R^m$ be convex. Let $x \in riD$ (where, riD denotes the relative interior of D) and $v \in T_D(x)$. Then $f'(x;v)$ exists.*

Proof. Since $x \in riD$ then $T_D(x)$ is a subspace. Hence, $-v \in T_D(x)$. Then there exist $t_1, t_2 > 0$ such that $x + t_1 v, x - t_2 v \in D$. Let $0 < t < t_1$. One has, $x = \lambda(x + tv) + (1 - \lambda)(x - t_2 v)$, where, $\lambda = \frac{t_2}{t+t_2} \in (0, 1)$. It implies, $f(x) \preceq \lambda f(x + tv) + (1 - \lambda)f(x - t_2 v)$. Take $\lambda = \frac{t_2}{t+t_2}$ into account, we have

$$\frac{f(x) - f(x - t_2 v)}{t_2} \preceq \frac{f(x + tv) - f(x)}{t}, \forall t \in (0, t_1).$$

Then $f'(x;v)$ exists by Lemma 2.8. The proof is complete. □

Denote by $L(R^n, R^m)$ the space of linear maps from R^n to R^m.

Definition 2.10. Let $D \subset R^n$ be a nonempty set. A map $F : D \to L(R^n, R^m)$ is said to be monotone with respect to C if

$$(F(x) - F(y))(x - y) \succeq 0,\ \forall x, y \in D,$$

i.e.,

$$(F(x) - F(y))(x - y) \in C,\ \forall x, y \in D.$$

Supposing that $intC \neq \emptyset$. F is said to be strictly monotone if

$$(F(x) - F(y))(x - y) \gg 0,\ \forall x, y \in D, x \neq y,$$

i.e.,

$$(F(x) - F(y))(x - y) \in intC,\ \forall x, y \in D,\ x \neq y.$$

We recall a well-known result of Fenchel [8] and Mangasarian [9] which will be needed in the next section.

Theorem 2.11. *Let $D \subset R^n$ be a nonempty, open and convex set and let $f : D \to R$ be a differentiable function on D. Then*

i) f is convex if and only if Df is a monotone map on D, i.e.,

$$(Df(y) - Df(x))(y - x) \geq 0,\ \forall x, y \in D.$$

ii) f is strictly convex if and only if Df is strictly monotone, i.e.,

$$((Df(y) - Df(x))(y - x) > 0,\ \forall x, y \in D, x \neq y.$$

3 Characterizations of convex vector functions

In this section, we assume that R^m is ordered by a closed and convex cone C. Let $g : (a, b) \to R$ and let $x \in (a, b)$. We recall that the right derivative of g at x is defined as the following limit (if such exists)

$$g'_+(x) = \lim_{y \downarrow x} \frac{g(y) - g(x)}{y - x}.$$

Lemma 3.1. *Let $D \subset R^n$ be a nonempty convex set and let $f : D \to R$. Assume that for every $x \in D, v \in T_D(x)$, $f'(x; v)$ exists finitely. Then,*

i) f is convex if only if

$$f(y) - f(x) \geq f'(x; y - x),\ \forall x, y \in D.$$

ii) f is strictly convex if and only if

$$f(y) - f(x) > f'(x; y - x),\ \forall x, y \in D, x \neq y.$$

Proof. i) $\underline{\Rightarrow}$: If f is convex then, for every $x, y \in D$, the function $\varphi(t) = \frac{f(x+t(y-x))-f(x)}{t}$ is increasing on (0.1]¿ This implies

$$f(y) - f(x) \geq \lim_{t \downarrow 0} \frac{f(x + t(y - x)) - f(x)}{t} = f'(x; y - x).$$

$\underline{\Leftarrow}$: Suppose in the contrary that f is not convex. Then there exist $x_0, y_0 \in D, \lambda_0 \in (0, 1)$ such that

$$f(x_0 + \lambda_0(y_0 - x_0)) > f(x_0) + \lambda_0(f(y_0) - f(x_0)). \tag{1}$$

Put $g(t) := f(x_0 + t(y_0 - x_0)) - f(x_0) - t(f(y_0) - f(x_0)), t \in [0, 1]$. Since $f'(x; v)$ exists and finite for every $x \in D, v \in T_D(x)$ then $g(t)$ is continuous on $[0, 1]$. By (1), $g(\lambda_0) > 0$. Since $g(0) = g(1) = 0$ then $g(t)$ attains maximun at some $t_0 \in (0, 1)$. Clearly, $g(t_0) > 0$. Put $t_1 = sup\{t \in [0, t_0) | g(t) = 0\}$. Since g is continuous then $g(t_1) = 0$. Hence, $g(t) > 0$ for every $t \in (t_1, t_0)$. Then there exist $\lambda \in (t_1, t_0)$ such that $g'_+(\lambda) \geq 0$. We have,

$$\begin{aligned}
&0 > g(1) - g(\lambda) = \lambda(f(y_0) - f(x_0)) + f(x_0) - f(x_0 + \lambda(y_0 - x_0)) \\
&= \lambda(f(y_0) - f(x_0)) + f(x_0) - f(y_0) + f(y_0) - f(x_0 + \lambda(y_0 - x_0)) \\
&\geq \lambda(f(y_0) - f(x_0)) + f(x_0) - f(y_0) + (1 - \lambda)f'((x_0 + \lambda(y_0 - x_0); (y_0 - x_0)) \\
&= (1 - \lambda)[f'((x_0 + \lambda(y_0 - x_0); (y_0 - x_0)) - (f(y_0) - f(x_0))] \\
&= (1 - \lambda)g'_+(\lambda) \geq 0.
\end{aligned}$$

This is a contradiction. Thus, f is convex.

ii) $\underline{\Rightarrow}$: If f is strictly convex then for every $x, y \in D, x \neq y$, the function $\varphi(t) = \frac{f(x+t(y-x))-f(x)}{t}$ is strictly increasing on $(0.1]$. This implies,

$$f(y) - f(x) = \varphi(1) > \varphi(\frac{1}{2}) = \frac{f(x + \frac{1}{2}(y-x)) - f(x)}{\frac{1}{2}}$$
$$\geq \lim_{t \downarrow 0} \frac{f(x + t(y-x)) - f(x)}{t} = f'(x; y-x).$$

$\underline{\Leftarrow}$: Suppose in the contrary that f is not strictly convex. Then there exists $x_0, y_0 \in D, x_0 \neq y_0, \lambda_0 \in (0,1)$ such that

$$f(x_0 + \lambda_0(y_0 - x_0)) \geq f(x_0) + \lambda_0(f(y_0) - f(x_0)).$$

If $f(x_0 + \lambda_0(y_0 - x_0)) > f(x_0) + \lambda_0(f(y_0) - f(x_0))$ then by a proof similar i) we get a contradiction. Now consider the case $f(x_0 + \lambda_0(y_0 - x_0)) = f(x_0) + \lambda_0(f(y_0) - f(x_0))$. Put $g(t) = f(x_0 + t(y_0 - x_0)) - f(x_0) - t(f(y_0) - f(x_0)), t \in [0,1]$. Then, $g(t)$ is continuous on $[0,1]$ and $g(\lambda_0) = g(0) = g(1) = 0$. Put $\alpha_1 = min\{g(t) | t \in [0, \lambda_0]\}$, $\alpha_2 = min\{g(t) | t \in [\lambda_0, 1]\}$. If $\alpha_1 = 0$ then

$$0 = g(\lambda_0) - g(0) = f(x_0 + \lambda_0(y_0 - x_0)) - f(x_0) - \lambda_0(f(y_0) - f(x_0))$$
$$> \lambda_0[f'(x_0; y_0 - x_0) - (f(y_0) - f(x_0)] = \lambda_0 {g'}_{+}(0) = 0.$$

We meet a contradiction. If $\alpha_2 = 0$ we also get a contradiction similarly. Hence, one has $\alpha_i < 0, i = 1, 2$. By continuity of g, there exist $[t_1, t_2] \subset [0, \lambda_0]$ such that $g(t_2) = 0, g(t) < 0, t \in [t_1, t_2], min\{g(t) | t \in [t_1, t_2]\} > \alpha_2$. Then there exist $s_1 \in [t_1, t_2]$ such that ${g'}_{+}(s_1) \geq 0$. By the continuity of g, there exists $s_2 \in [\lambda_0, 1]$ such that $min\{g(t) | t \in [\lambda_0, 1]\} = g(s_2)$. Then,

$$\begin{aligned}
0 &> g(s_2) - g(s_1) \\
&= f(x_0 + s_2(y_0 - x_0)) - f(x_0 + s_1(y_0 - x_0)) - (s_2 - s_1)(f(y_0) - f(x_0)) \\
&\geq (s_2 - s_1)[f'((x_0 + s_1(y_0 - x_0); (y_0 - x_0)) - (f(y_0) - f(x_0))] \\
&= (s_2 - s_1){g'}_{+}(s_1) \geq 0.
\end{aligned}$$

This is a contradiction. Thus, f is strictly convex. □

Theorem 3.2. *Let $D \subset R^n$ be a nonempty convex set and let $f : D \to R^m$. Assume that for every $x \in D, v \in T_D(x)$, $f'(x; v)$ exists. Then,*

i) f is convex if and only if

$$f(y) - f(x) \in f'(x; y-x) + C, \ \forall x, y \in D.$$

ii) Supposing that $intC \neq \emptyset$. f is strictly convex if and only if

$$f(y) - f(x) \in f'(x; y-x) + intC, \ \forall x, y \in D, x \neq y.$$

Proof. From hypothesises and Lemma 2.1, $(\xi f)'(x;v)$ exists for every $\xi \in L(R^m, R), x \in D, v \in T_D(x)$.

i) From Lemma 2.6, Lemma 3.1, Lemma 2.1, Lemma 2.4, we have

$$\begin{aligned}
f \text{ is convex} &\Leftrightarrow \xi f \text{ is convex}, \ \forall \xi \in C' \setminus \{0\}. \\
&\Leftrightarrow \xi f(y) - \xi f(x) \geq (\xi f)'(x; y-x), \ \forall x, y \in D, \xi \in C' \setminus \{0\}. \\
&\Leftrightarrow \xi\big[f(y) - f(x) - f'(x; y-x)\big] \geq 0, \ \forall x, y \in D, \xi \in C' \setminus \{0\}. \\
&\Leftrightarrow f(y) - f(x) \succeq f'(x; y-x), \ \forall x, y \in D.
\end{aligned}$$

ii) From Lemma 2.6, Lemma 3.1, Lemma 2.1, Lemma 2.4, we have

$$\begin{aligned}
&f \text{ is strictly convex} \Leftrightarrow \xi f \text{ is strictly convex}, \ \forall \xi \in C' \setminus \{0\}. \\
&\Leftrightarrow \xi f(y) - \xi f(x) > (\xi f)'(x; y-x), \ \forall x, y \in D, x \neq y, \xi \in C' \setminus \{0\}. \\
&\Leftrightarrow \xi\big[f(y) - f(x) - f'(x; y-x)\big] > 0, \ \forall x, y \in D, x \neq y, \xi \in C' \setminus \{0\}. \\
&\Leftrightarrow f(y) - f(x) \gg f'(x; y-x), \ \forall x, y \in D, x \neq y.
\end{aligned}$$

The proof is complete. □

When f is differentiable, we get the following corollary which generalizes the corresponding well-known result in convex analysis.

Corollary 3.3. *Let $D \subset R^n$ be a nonempty, open and convex set and let $f : D \to R^m$ be differentiable on D. Then,*

i) f is convex if and only if

$$f(y) - f(x) \in Df(x)(y-x) + C, \ \forall x, y \in D.$$

ii) Supposing that $intC \neq \emptyset$. f is strictly convex if and only if

$$(f(y) - f(x) \in Df(x)(y-x) + intC, \ \forall x, y \in D, x \neq y).$$

Proof. From Lemma 2.2, for every $x \in D, v \in R^n$, $f'(x;v) = Df(x)(v)$ then by applying Theorem 3.2, we obtain immediately the corollary.

□

A result of C. Cusano, M. Fini and D. Torre in [10] is a very speacial case of the above theorem.

Corollary 3.4. ([Theorem 4.1, 10]) *Assume that the ordered cone C is closed, convex and pointed. If $f : R^n \to R^m$ is convex then*

$$f(y) - f(x) \in {f'}_D(x; y-x) + C, \forall x, y \in R^n.$$

Proof. For every $x \in R^n, v \in T_{R^n}(x) = R^n,\ f'(x;v)$ exists by Corollary 2.9 . Hence by Theorem 3.2 i) and Lemma 2.3, we have

$$f(y) - f(x) \succeq f'(x; y-x) = f'_D(x; y-x), \forall x, y \in R^n.$$

The proof is complete. $\square$

The following theorem expands well-known results of Fenchel[8] and Mangasarian[9] to the vector case.

Theorem 3.5. *Let $D \subset R^n$ be a nonempty, open and convex set and let $f : D \to R^m$ be a differentiable function on D. Then*

i) f is convex if and only if Df is a monotone map on D, i.e.,

$$(Df(y) - Df(x))(y - x) \in C,\ \forall x, y \in D.$$

ii) Supposing that $intC \neq \emptyset$. f is strictly convex if and only if Df is strictly monotone, i.e.,

$$((Df(y) - Df(x))(y - x) \in intC,\ \forall x, y \in D, x \neq y.$$

Proof. i) We have

f is convex
$\Leftrightarrow \xi f$ is convex, $\forall \xi \in C' \setminus \{0\}$, (by Lemma 2.6)
$\Leftrightarrow (D(\xi f)(y) - D(\xi f)(x))(y - x) \geq 0, \forall x, y \in D, \xi \in C' \setminus \{0\}$, (Theorem 2.11)
$\Leftrightarrow \xi[(Df(y) - Df(x))(y - x)] \geq 0,\ \forall x, y \in D,\ \xi \in C' \setminus \{0\}$, (by Lemma 2.2)
$\Leftrightarrow (Df(y) - Df(x))(y - x) \succeq 0,\ \forall x, y \in D$, (by Lemma 2.4).

i) We have

f is strictly convex
$\Leftrightarrow \xi f$ is strictly convex, $\forall \xi \in C' \setminus \{0\}$, (by Lemma 2.6)
$\Leftrightarrow (D(\xi f)(y) - D(\xi f)(x))(y - x) > 0, \forall x, y \in D, \xi \in C' \setminus \{0\}$, (Theorem 2.11)
$\Leftrightarrow \xi[(Df(y) - Df(x))(y - x)] > 0,\ \forall x, y \in D,\ \xi \in C' \setminus \{0\}$, (by Lemma 2.2)
$\Leftrightarrow (Df(y) - Df(x))(y - x) \gg 0,\ \forall x, y \in D$, (by Lemma 2.4).

The proof is complete. $\square$

4 Optimality without constraint

In this section, the ordered cone $C \subset R^m$ is assumed convex.

Definition 4.1. ([Definition 2.1, 1]) Let $A \subset R^m$ be nonempty and $x \in A$. We say that

i) x is an ideal efficient point of A with respect to C if $x \preceq a$, for every $a \in A$. The set of ideal efficient points of A is denoted by $IMin(A|C)$

ii) x is a Pareto efficient point of A with respect to C if for any $a \in A$, $a \preceq x$ implies $x \preceq a$. The set of Pareto efficient points of A is denoted by $Min(A|C)$

iii) x is a properly efficient point of A with respect to C if there exists a cone $K \subsetneq R^m$ such that $C \setminus lC \subset intK$ and $x \in Min(A|K)$. The set of properly efficient points of A is denoted by $PrMin(A|C)$

iv) supposing that $intC \neq \emptyset$, x is a weakly efficient point of A with respect to C if $x \in Min(A|intC \cup \{0\})$. The set of weakly efficient points of A is denoted by $WMin(A|C)$.

When there is no afraid of confusion, we omit 'with respect to C' and $'|C$' in the definition above. It is immediately from definitions that if the ordered cone C is pointed and $IMinA \neq \emptyset$ then $IMinA$ is a singleton.

Lemma 4.2. *Assume that the ordered cone $C \subset R^m$ is convex and pointed. Let $A \subset R^m$ be nonempty and let $x_* \in A$. Then*

i) $x_* \in MinA$ if and only if $a \notin x_* - (C \setminus \{0\})$, for every $a \in A$.

ii) Supposing that $intC \neq \emptyset$. Then, $x_* \in WMinA$ if and only if $a \notin x_* - intC$, for every $a \in A$.

Proof. It is immediately from definitions. □

Definition 4.3. Let $D \subset R^n$ be nonempty and let $f : D \to R^m, x_* \in D$. We say that

i) x_* is a local ideal (resp., Pareto, properly, weakly) minimum point of f if there exists a neighbourhood V of x_* such that $f(x_*) \in IMin(f(D \cap V)$ (resp., $f(x_*) \in Min(f(D \cap V), f(x_*) \in PrMin(f(D \cap V), f(x_*) \in WMin(f(D \cap V))$.

ii) x_* is a global ideal (resp., Pareto, properly, weakly) minimum point of f (on D) if $f(x_*) \in IMinf(D)$ (resp., $f(x_*) \in Minf(D), f(x_*) \in PrMinf(D)$, $f(x_*) \in WMinf(D))$.

Proposition 4.4. *Assume that the ordered cone $C \subset R^m$ is convex and pointed. Let $D \subset R^n$ be a nonempty and convex set. Let $f : D \subset R^n \to R^m$ be convex and let $x_* \in D$. If x_* is a local $G\_minimum$ point of f then x_* is also a global $G\_minimum$ point of f, where, $G \in \{Pareto,\ ideal,\ properly\}$. Supposing that $intC \neq \emptyset$ then the statement above is also valid for the case $G = weakly$.*

Proof. From assumptions, there exists a neighbourhood V of x_* such that $f(x_*) \in G\_Min(f(D) \cap V)$. Let $x \in D$ be arbitrary. then there exists $t \in (0, 1)$

such that $x_* + t(x - x_*) \in D \cap V$. Since f is convex then

$$(1-t)f(x_*) + tf(x) \in f(x_* + t(x - x_*) + C.$$

Firstly, consider the case $G = Pareto$. By Lemma 4.2, one has, $f(x_* + t(x - x_*) \notin f(x_*) - (C \setminus \{0\})$. Hence, $(1-t)f(x_*) + tf(x) \notin f(x_*) - (C \setminus \{0\})$. This implies $f(x) \notin f(x_*) - (C \setminus \{0\})$, i.e., $f(x_*) \in Minf(D)$. Thus, x_* is a global Pareto minimum point of f on D.

For the cases $G = ideal, weakly$, the proofs are completely similar.

Finally, consider the case $G = properly$. Then there exists a cone $K \subsetneq R^m$ such that $C \setminus \{0\} \subset intK$ and $f(x_*) \in Min(f(D \cap V)|K)$. If $f(x) \in f(x_*) - K$ then $(1-t)f(x_*) + tf(x) \in f(x_*) - K$. Then, $f(x_* + t(x - x_*) \in f(x_*) - K$. This implies $f(x_*) \in f(x_* + t(x - x_*) - K$ since $f(x_*) \in Min(f(D \cap V|K))$. Hence, $f(x_*) \in (1-t)f(x_*) + tf(x) - K$, i.e., $f(x_*) \in f(x) - K$. Thus, $f(x_*) \in Min(f(D)|K)$. This means that x_* is a global properly minimum point of f. □

Proposition 4.5. *Assume that the ordered cone C is convex and pointed. Let $D \subset R^n$ be a nonempty and convex set. Let $f : D \subset R^n \to R^m$ be convex. Then the set of ideal minimum points of f is a convex set.*

Proof. Let x_*, y_* be ideal minimum points of f, $\lambda \in (0,1)$. Since C is pointed then, $f(x_*) = f(y_*) = IMinf(D)$. By convexity of f, one has, $f(\lambda x_* + (1-\lambda)y_*) \preceq \lambda f(x_*) + (1-\lambda)f(y_*) = f(x_*)$. This implies $f(\lambda x_* + (1-\lambda)y_*) = f(x_*) = IMinf(D)$. Hence, $\lambda x_* + (1-\lambda)y_*$ is an ideal minimum point of f. □

Remark 4.6. In general, the set of Pareto minimum points of a convex vector function is not convex. For instant, consider R^2 with the order generated by the nonnegative orthant cone. Let $D = \{(x,y) | (x-1)^2 + (y-1)^2 \leqslant 1,\ y \leqslant -x+1\}$ and let f be the identify map on D. Then f is convex, $(0,1), (1,0)$ are Pareto minimum points of f . However, $z = \frac{1}{2}(0,1) + \frac{1}{2}(1,0)$ is not a Pareto minimum point of f.

Proposition 4.7. *Assume that the ordered cone C is convex and pointed with $intC \neq \emptyset$. Let $D \subset R^n$ be a nonempty and convex set and let $f : D \subset R^n \to R^m$ be strictly convex. If f has an ideal minimum point then this point is unique.*

Proof. Suppose in the contrary that f has two ideal minimum points x_*, y_*. Then $f(\frac{1}{2}x_* + \frac{1}{2}y_*) \ll \frac{1}{2}f(x_*) + \frac{1}{2}f(y_*) = IMinf(D)$. We get a contradiction. □

Remark 4.8. We note that the set of Pareto-minimum points of a strictly convex vector function in general is not a singleton. For instant, let's consider R^2 with the order generated by the cone $C := \{(x,y) \in R^2 | y - x \geq 0, y + x \geq 0\}$, $f : R \to R^2$ defined by $f(x) = (x, x^2)$. Then f is strictly convex. However, the set of Pareto minimum points of f is the interval $[-\frac{1}{2}, \frac{1}{2}]$, obviously not a singleton.

Theorem 4.9. *Assume that the ordered cone C is convex, closed and pointed. Let $D \subset R^n$ be convex and nonempty and let $f : D \to R^m, x_* \in D$. Assume that $f'(x_*; v)$ exists, for every $v \in T_D(x_*)$. We have*

i) If x_* is a local G_minimum point of f, then

$$0 \in G\_Min\{f'(x_*; v) | v \in T_D(x_*)\},$$

where, $G \in \{Ideal, properly\}$. In addition, assume that f is convex, then the converse statement is also true.

ii) Supposing that $intC \neq \emptyset$. If x_* is a local weakly minimum point of f, then

$$0 \in WMin\{f'(x_*; v) | v \in T_D(x_*)\}.$$

In addition, assume that f is convex, then the converse statement is also true.

iii) Supposing that f is convex. If $0 \in Min\{f'(x_*; v) | v \in T_D(x_*)\}$, then x_* is a Pareto-minimum point of f.

Proof. i) *G=ideal.* Let $v \in T_D(x_*)$ be arbitrary. For $t > 0$ small enough, one has, $\frac{f(x_*+tv)-f(x_*)}{t} \succeq 0$. By taking $t \downarrow 0$, since C is closed, we have $f'(x_*; v) \succeq 0$. Hence, $0 \in IMin\{f'(x_*; v) | v \in T_D(x_*)\}$.

Conversely, assume that f is convex. From Theorem 3.2, for every $x \in D$, one has

$$f(x) - f(x_*) \succeq f'(x_*, x - x_*) \succeq 0.$$

This means that f attains ideal minimum at x_*.

G=properly. There exists a neighbourhood V of x_* and there exists a convex cone $K \subsetneq R^m$ such that $C \setminus \{0\} \subset intK$ and $f(x_*) \in Min(f(D \cap V|K)$. Let $v \in T_D(x_*)$ be arbitrary. By Lemma 2.6, $\{0\} \cup intK$ is a pointed cone. From Lemma 4.2, for $t > 0$ small enough, one has $f(x_* + tv) \notin f(x_*) - intK$. Hence, $\frac{f(x_*+tv)-f(x_*)}{t} \notin -intK$. Take $t \downarrow 0$, one has, $f'(x_*; v) \notin -intK$. Therefore,

$$0 \in PrMin(\{f'(x_*; v) | v \in T_D(x_*)\} | C).$$

Conversely, there exists a convex cone $K \subsetneq R^m$ such that $C \setminus \{0\} \subset intK$ and $0 \in Min(\{f'(x_*; v) | v \in T_D(x_*)\} | K)$. Since $\{0\} \cup intK$ is a pointed cone then,

$$f'(x_*; v) \notin -intK, \ \forall v \in T_D(x_*). \tag{2}$$

From asumptions and Theorem 3.2, for every $x \in D$, one has

$$f(x) - f(x_*) \in f'(x_*, x - x_*) + C. \tag{3}$$

By (2) and (3), we have $f(x) - f(x_*) \notin -intK$. This implies, $f(x_*) \in Min(f(D)|\{0\} \cup intK)$, i.e., x_* is a properly minimum point of f.

ii) Let $v \in T_D(x_*)$ be arbitrary. By Lemma 4.2, for $t > 0$ small enough, one has $f(x_* + tv) \notin f(x_*) - intC$. Hence, $\frac{f(x_*+tv)-f(x_*)}{t} \notin -intC$. Take $t \downarrow 0$, we have $f'(x_*; v) \notin -intC$, i.e. , $0 \in WMin\{f'(x_*; v) | v \in T_D(x_*)\}$.

Conversely, assume that f is convex. By assumptions and Theorem 3.2, for every $x \in D$, one has

$$f(x) - f(x_*) \succeq f'(x_*, x - x_*) \notin -intC.$$

This implies $f(x) - f(x_*) \notin -intC$. Hence, $f(x_*) \in WMinf(D)$.

iii) By assumptions and Theorem 3.2, for every $x \in D$, one has

$$f(x) - f(x_*) \succeq f'(x_*, x - x_*) \notin -(C \setminus \{0\}).$$

Hence $f(x) - f(x_*) \notin -(C \setminus \{0\})$,i.e., $f(x_*) \in Minf(D)$. The proof is complete. $\square$

Remark 4.10. We note that the converse statement of Theorem 4.9 iii) is not true in general. For instant, let $f : R \to R^2$ defined by

$$f(x) = \begin{cases} (x, x^2), & x < 0 \\ (x, 0), & x \geq 0. \end{cases}$$

R^2 is ordered by the nonnegative orthant cone. Then f is convex, 0 is a Pareto-minimum point of f. However, $f'(0; -1) = (-1, 0) \prec (0, 0)$. Hence $(0, 0) \notin Min\{f'(0; v) | v \in T_D(0)\}$.

Definition 4.11. Let $D \subset R^n$ be nonempty and $x_* \in D$. Let $A \subseteq R^n$ be an open set which contains D and let $f : A \to R^m$ be differentiable at x_*. We say that x_* is a Pareto (resp., ideal, properly, weakly) stable point of f on D if

$$0 \in Min\{Df(x_*)(v) | v \in T_D(x_*)\},$$

(resp.,$0 \in IMin\{Df(x_*)(v) | v \in T_D(x_*)\}, 0 \in PrMin\{Df(x_*)(v) | v \in T_D(x_*)\}$, $0 \in WMin\{Df(x_*)(v) | v \in T_D(x_*)\}$.)

The following corollary generalizes well-known results in the scalar case on optimal conditions for differentiable functions.

Corollary 4.12. *Assume that the ordered cone $C \subset R^m$ is convex, closed and pointed. Let $D \subset R^n$ be nonempty and convex. Let $x_* \in D$, $A \subset R^n$ be an open set which contains D. Let $f : A \to R^m$ be differentiable at x_*. Then,*

i) If x_* is a local G_minimum of f on D then, x_* is a G_stable point of f on D, where $G \in \{ideal, properly\}$. In addition, assume that f is convex on D, then the converse statement is also true.

ii) Supposing that $intC \neq \emptyset$. If x_* is a local weakly minimum of f on D then, x_* is a weakly stable point of f on D. In addition, assume that f is convex on D, then the converse statement is also true.

iii) Supposing that f is convex. If x_* is a Pareto stable point of f on D then, x_* is a Pareto minimum point of f on D.

Proof. It is immediately from Theorem 4.9 and from the equality $f'(x_*, v) = Df(x_*)(v)$, for every $v \in T_D(x_*)$. $\square$

5 Optimality with constraints

Let $f, g, g_1, ..., g_q, h_1, ..., h_p$ be functions from R^n to R^m. Assume that R^m is ordered by a convex, closed and pointed cone C . Let us consider the constrained vector optimization problem

$$\text{Min} f(x)$$
$$\begin{cases} h_i(x) = 0, i = 1, ..., p \\ g_j(x) \in -C, j = 1, ..., q. \end{cases} \qquad (P_1)$$

When $intC \neq \emptyset$, we also consider the problem

$$\text{Min} f(x)$$
$$\begin{cases} h_i(x) = 0, i = 1, ..., p \\ g(x) \notin intC. \end{cases} \qquad (P_2)$$

Denote by D_k the set of feasible solutions of $(P_k), k = 1, 2$, i.e.,

$$D_1 = \{x \in R^n | h_i(x) = 0, i = 1, ..., p;\ g_j(x) \in -C, j = 1, ..., q\}$$
$$D_2 = \{x \in R^n | h_i(x) = 0, i = 1, ..., p;\ g(x) \notin intC\}.$$

Definition 5.1. Let $\bar{x} \in D_k, k = 1, 2$. We say that $\bar{x}$ is an ideal (resp., weakly) optimal solution of (P_k) if $\bar{x}$ is an ideal minimum (resp., weakly minimum) point of f on D_k.

Lemma 5.2. *If $g_j, j = 1, ..., q$, are convex and $h_i, i = 1, ..., p$, are affine then D_1 is convex.*

Proof. Let $x, y \in D_1, \lambda \in [0, 1]$. Then, $g_j(x), g_j(y) \preceq 0$ and $h_i(x) = h_i(y) = 0$. Since C is convex then $\lambda g_j(x)+(1-\lambda)g_j(y) \preceq 0$. By convexity of $g_j, j = 1, ..., q$, then $g_j(\lambda x + (1 - \lambda)y) \preceq 0$. Since $h_i, i = 1, ..., p$, are affine then $h_i(\lambda x + (1 - \lambda)y) = 0$. Hence, $\lambda x + (1 - \lambda)y \in D_1$. Thus, D_1 is convex.

□

Assume that g, g_j, h_i are differentiable at $\bar{x} \in R^n$. Put

$$A(\bar{x}) := \{j | g_j(\bar{x}) = 0\}$$
$$S_1(\bar{x}) := \{v \in R^n | Dh_i(\bar{x})(v) = 0, i = 1, ..., p; Dg_j(\bar{x})(v) \in -C, j \in A(\bar{x})\}$$
$$S_2(\bar{x}) := \begin{cases} \{v \in R^n | Dh_i(\bar{x})(v) = 0, i = 1, ..., p\}, \text{ if } g(\bar{x}) \neq 0, \\ \{v \in R^n | Dh_i(\bar{x})(v) = 0, i = 1, ..., p; Dg(\bar{x})(v) \notin intC\}, \text{if } g(\bar{x}) = 0. \end{cases}$$

Lemma 5.3. *i) If $g_j, h_i,\ j = 1, ..., q, i = 1, ...p$, are differentiable at $\bar{x} \in D_1$ then $T_{D_1}(\bar{x}) \subseteq S_1(\bar{x})$.*

ii) If $g, h_i,\ i = 1, ..., p$, are differentiable at $\bar{x} \in D_2$ then $T_{D_2}(\bar{x}) \subseteq S_2(\bar{x})$.

Proof. i) Let $v \in T_{D_1}(\bar{x})$ be arbitrary. For $t > 0$ small enough, one has $h_i(\bar{x} + tv) = 0, g_j(\bar{x} + tv) \preceq 0, i = \bar{1,p}, j \in A(\bar{x})$. This together closedness of C imply

$$Dh_i(\bar{x})(v) = h'_i(\bar{x}; v) = \lim_{t \downarrow 0} \frac{h_i(\bar{x} + tv) - h_i(\bar{x})}{t} = 0, i = \bar{1,p}$$

$$Dg_j(\bar{x})(v) = g'_j(\bar{x}; v) = \lim_{t \downarrow 0} \frac{g_j(\bar{x} + tv) - g_j(\bar{x})}{t} \preceq 0, j \in A(\bar{x}).$$

Hence, $v \in S_1(\bar{x})$.

ii) It is completely similar the proof above. □

Theorem 5.4. *Assume that f is convex, D_1 is convex and $f, g_j, h_i, j = 1, ..., q, i = 1, ..., p$, are differentiable at $\bar{x} \in D_1$. If there exist numbers $\bar{\lambda}_1, ..., \bar{\lambda}_q \geq 0, \bar{\mu}_1, ..., \bar{\mu}_p \in R$ such that*

$$(i)\ Df(\bar{x}) + \sum_{j=1}^{q} \bar{\lambda}_j Dg_j(\bar{x}) + \sum_{i=1}^{p} \bar{\mu}_i Dh_i(\bar{x}) = 0$$

$$(ii)\ \bar{\lambda}_j g_j(\bar{x}) = 0, \forall j = 1, ..., q$$

then $\bar{x}$ is an ideal optimal solution of (P_1).

Proof. Let $v \in S_1(\bar{x})$ be arbitrary. From (i) and (ii), one has

$$Df(\bar{x})(v) = -\sum_{j=1}^{q} \bar{\lambda}_j Dg_j(\bar{x})(v) + \sum_{i=1}^{p} \bar{\mu}_i Dh_i(\bar{x})(v) - \sum_{j \in A(\bar{x})} \bar{\lambda}_j Dg_j(\bar{x})(v) \succeq 0.$$

By Lemma 5.3, $Df(\bar{x})(v) \succeq 0$, for every $v \in T_{D_1}(\bar{x})$. Hence, $\bar{x}$ is an ideal stable point of f on D_1. Then by Corollary 4.12, $\bar{x}$ is an ideal minimum point of f on D_1. The theorem is proved. □

Theorem 5.5. *Let f be convex, D_2 be convex and let$f, g, h_i, i = 1, ..., p$, be differentiable at $\bar{x} \in D_2$. Assume that there exist numbers $\bar{\lambda} \geq 0, \bar{\mu}_1, ..., \bar{\mu}_p \in R$ such that*

$$(i)\ Df(\bar{x}) + \bar{\lambda} Dg(\bar{x}) + \sum_{i=1}^{p} \bar{\mu}_i Dh_i(\bar{x}) = 0$$

$$(ii)\ \bar{\lambda} g(\bar{x}) = 0.$$

Then,

a) If $g(\bar{x}) \neq 0$ then $\bar{x}$ is an ideal optimal solution of (P_2).

b) If $g(\bar{x}) = 0$ then $\bar{x}$ is a weakly optimal solution of (P_2). **Proof.** a) From

(ii) one has $\bar{\lambda} = 0$. Then (i) implies

$$Df(\bar{x}) - \sum_{i=1}^{p} \bar{\mu}_i Dh_i(\bar{x}).$$

Let $v \in S_2(\bar{x})$ be arbitrary. One has $Df(\bar{x})(v) - \sum_{i=1}^{p} \bar{\mu}_i Dh_i(\bar{x})(v) = 0$. By Lemma 5.3, one has $Df(\bar{x})(v) = 0$, for every $v \in T_{D_2}(\bar{x})$. Hence, $\bar{x}$ is an ideal stable point of f on D_2. By Corollary 4.12, $\bar{x}$ is an ideal optimal solution of (P_2).

b) Let $v \in S_2(\bar{x})$ be arbitrary. From (ii), we have, $Df(\bar{x})(v) = -\bar{\lambda} Dg(\bar{x})(v) - \sum_{i=1}^{p} \bar{\mu}_i Dh_i(\bar{x})(v) \not\ll 0$. By Lemma 5.3, one has $Df(\bar{x})(v) \not\ll 0$, for every $v \in T_{D_2}(\bar{x})$, i.e., $0 \in WMin\{Df(\bar{x})(v) | v \in T_{D_2}(\bar{x})\}$. By Corollary 4.12, $\bar{x}$ is a weakly optimal solution of (P_2). The proof is complete.

□

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