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HEREDITARY TORSION THEORIES AND CONNECTEDNESS, DISCONNECTEDNESS OF TOPOLOGICAL SPACES

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Abstract

Using topological congruences, a Hoehnke radical for topological spaces can be defined as in universal algebra. For most of the well-known classes of algebras, an ideal-hereditary Hoehnke radical (= hereditary torsion theory) always determines a corresponding pair of Kurosh-Amitsur radical and semisimple classes. Here it is shown that an ideal-hereditary Hoehnke radical of topological spaces need not determine a corresponding pair of Kurosh-Amitsur radical and semisimple classes (= connectednesses and disconnectednesses). In fact, it is shown that there are exactly five hereditary torsion theories of topological spaces of which two are not Kurosh-Amitsur radicals.

1 Introduction

The work presented here is for topological spaces, but the context is much broader using universal algebraic tools. The overall theme is general radical theory which has its roots in three main distinct areas in mathematics: the theory of radical and semisimple classes of algebraic structures like rings and groups, the theory of torsion and torsionfree classes of modules and more generally in abelian categories and the connectednesses and disconnectednesses of topological spaces and graphs. There are many similarities between these theories, but also some significant and interesting differences. Here we will present

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a peculiar property of the general radical theory of topological spaces which is not to be found in any of the well-known algebraic structures. In order to understand and appreciate this exceptional behaviour, some background material will be required.

The origins of radical theory goes back to the early twentieth century with the work of Wedderburn on finite dimensional algebras. This was extended to ring theory with Köthe's nilradical, the Jacobson radical and subsequently many other radicals. These, together with some developments in group theory led to the axiomization of the radical concept by Kurosh and independently Amitsur for rings and groups and omega-groups in the early fifties. In the sixties, Hoehnke [8] used congruences to define a radical for universal algebras; now known as a Hoehnke radical. The motivating reason for this radical is that the semisimple algebras have a subdirect representation. In an environment where there are distinguished substructures (i.e., where a congruence is completely determined by one of the congruence classes, as for example in Ω-groups), it was then shown under which conditions a Hoehnke radical will become a Kurosh-Amitsur radical; see for example Mlitz [11]. A second main stream that contributed to the development and existence of general radical theory has its origins in the torsion theory of modules. Torsion theories of modules were defined in terms of equivalence classes of injective modules, but when Dickson [4] generalized torsion theories to abelain categories, this approach was abandoned. It turns out that the torsion and torsion-free classes correspond to the radical and semisimple classes of general radical theory respectively. It is then interesting and rather pleasing that the hereditary torsion theories of topological spaces have a connection with injective topological spaces (= indiscrete spaces) as will be seen below. The third contribution to the general radical theory is to be found in the category of topological spaces. Connectednesses and disconnectednesses of topological spaces were defined by Preuß [15] as classes of spaces on which certain maps are constant. Then Arhangel'skiĭ and Wiegandt [1] showed that these classes are just the Kurosh-Amitsur radical and semisimple classes of rings and groups by replacing the algebraic notions involved in the characterization with their categorical equivalent topologial versions. This was followed by a theory of connectednesses and disconnectednesses for graphs, S-acts and more generally for abstract relational structures.

With such similar radical theories in so many divergent branches of mathematics, the need arose for a common language to describe them all. Category theory proved to be a suitable tool; initially only catering for the radical theories from an algebraic environment, for example by Šul'geĭfer [16], Suliński [17] and Holcombe and Walker [9]. Such categories exclude the connectednesses and disconnectednesses of topological spaces and graphs. A unified treatment for these two cases were given by Fried and Wiegandt [5,6] by considering graphs and topological spaces as abstract relational stuctures. But this approach excluded the algebraic cases. In [3] and [18] less stringent conditions were imposed on a general category to make it suitable to describe the radical theory of the classical algebraic stuctures as well as those of topological spaces and graphs. Subsequently, the most comprehensive theory covering all known radical theories was given by Márki, Mlitz and Wiegandt [10] in a general categorical setting but with an universal algebraic flavor.

Quite recently, it has been shown that the connectednesses and disconnectednesses of both topological spaces and graphs (i.e the Kurosh-Amitsur radical and semisimple classes) can be obtained from Hoehnke radicals as has been done for universal algebras using congruences. In [2] and [19] it was shown how to define congruences on graphs and topological spaces. These congruences then lead in a natural way to appropriate versions of the algebraic isomorphism theorems and subdirect products. As in universal algebra, one can then define a Hoehnke radical for graphs and topological spaces. Necessary and sufficient conditions to ensure that the Hoehnke radical becomes a Kurosh-Amitsur radical have been determined.

The fact that connectednesses and disconnectednesses of topological spaces and graphs can be defined as Hoehnke radicals, has now brought many questions to the fore. Here we address one of them. In the classical torsion theory, torsionfree classes (= semisimple classes) are always hereditary and a hereditary torsion theory means that the associated torsion class (= radical class) is hereditary. For associative rings, the semisimple classes are also always hereditary, so also here a hereditary radical would mean the associated radical class is hereditary. But for more general classes, e.g. not necessarily associative rings or near-rings, semisimple classes need not be hereditary. Thus, for a radical in general, it is now customary to call it ideal-hereditary if both its semisimple class and its radical class are hereditary. The relationships between Hoehnke radicals, Kurosh-Amitsur radicals, connectednesses and disconnectednesses and torsion theories with or without additional properties (like hereditariness) have been investigated and clarified for most concrete categories. For associative rings and similar types of algebraic categories, in fact for Ω-groups in general, it is well-known that any ideal-hereditary Hoehnke radical is a Kurosh-Amitsur radical. In terms of torsion theories, this statement says that any hereditary torsion theory is Kurush-Amitsur. The question which will be addressed here is whether every ideal-hereditary Hoehnke radical of topological spaces is a Kurosh-Amistsur radical (i.e., whether every hereditary torsion theory of topological spaces gives rise to a corresponding pair of connectedness and disconnectedness). Wiegandt [20] has shown that for $\mathcal{S}$-acts this is not the case: a hereditary torsion theory need not determine a Kurosh-Amitsur radical. It will be shown here that this is also the case for topological spaces.

The notion of a congruences on a topological space is still fairly new and we will recall the definition and first properties of such congruences as well as the defintion of a Hoehnke radical for topological spaces and its connection to connectednesses and disconnectednesses from [19]. In fact, what was shown in [19] is that the notions and methods of algebra can be used to describe the connectednesses and disconnectednesses of topology. In other words, the connectedness-disconnectedness theory coincides with the Kurosh-Amitsur radical theory in all respects. We will then give the necessary build up to the problem that will be addressed here. In the last section, an example is given to show that an ideal-hereditary Hoehnke radical of topological spaces (= hereditary torsion theory) need not determine a connectedness - disconnectedness pair (i.e. need not be a Kurosh-Amistsur radical). In other words, the category of topological spaces is an example in which there exist injective objects and also a torsionfree class with hereditary Hoehnke radical which is not Kurosh-Amitsur. This example also shows that a complete and idempotent Hoehnke radical need not be a Kurosh-Amitsur radical. It is thus in order to look at the salient properties of such radicals. It will be seen that these are mostly simmilar to those of the more restrictive Kurosh-Amitsur radicals. In the last result, all the ideal-hereditary Hoehnke radicals are determined and it will be seen that of these five, three are Kurosh-Amitsur radicals.

2 Congruences on a topological space

The topological spaces $(X, \mathcal{T})$ under discussion, will always have $X \neq \emptyset$. The one-element space will be denoted by T and we identify all one-element spaces with this *trivial space* T. A subset of a topological space will always be regarded as a topological space with respect to the relative topology, unless explicitly mentioned otherwise.

Definition 2.1. *[19] Let $(X, \mathcal{T})$ be a topological space. A congruence ρ on X is a pair $\rho = (\sim, \mathbb{T})$ where*

(C1) $\sim$ is an equivalence relation on X;

(C2) $\mathbb{T}$ is a topology on X, called the congruence topology, with $\mathbb{T} \subseteq \mathcal{T}$; and

(C3) For all $x \in X$, $x \in U \in \mathbb{T}$ implies the equivalence class $[x] \subseteq U$.

A congruence $\rho = (\sim, \mathbb{T})$ is a strong congruence on X if $\mathbb{T} = \{U \subseteq X | U$ is open in X and $x \in U$ implies $[x] \subseteq U\}$.

Examples. Some examples of congruences on $(X, \mathcal{T})$ are:

(1) $\iota_X = (\backsimeq, \mathcal{T})$; the *identity congruence* on X where $a \backsimeq b$ is equality $a = b$. Note that if $\mathbb{T}$ is any topology on X with $\mathbb{T} \subseteq \mathcal{T}$, then $(\backsimeq, \mathbb{T})$ is a congruence on X.

(2) $\upsilon_X = (\leftrightsquigarrow, \mathcal{I}_X)$ where $\leftrightsquigarrow$ is the universal equivalence relation on X, i.e., $a \leftrightsquigarrow b$ for all $a, b \in X$ and $\mathcal{I}_X$ is the indiscrete topology on X.

(3) If $\sim$ is any equivalence on X, then $(\sim, \mathcal{I}_X)$ is a congruence on X.

(4) Let $f : X \to Y$ be a continuous mapping. The *kernel of* f, denoted by *ker* $f = (\sim_f, \mathbb{T}_f)$, is the congruence on X defined by $a \sim_f b \Leftrightarrow f(a) = f(b)$ for $a, b \in X$ and $\mathbb{T}_f = \{U \subseteq X \mid U = f^{-1}(V)$ for some $V \subseteq Y$ open$\}$.

(5) Let $f : X \to Y$ be a continuous mapping. The *strong kernel of* f is the congruence *sker* $f = (\sim_f, \mathbb{T}_{sf})$ on X defined by $a \sim_f b \Leftrightarrow f(a) = f(b)$ for $a, b \in X$ and $\mathbb{T}_{sf} = \{U \subseteq X \mid U$ is open in X and $U = f^{-1}(f(U))\}$. It can be verified that *sker* f is actually a strong congruence on X. For a quotient map f, *ker* f = *sker* f. In particular, ι_X =*ker* 1_X = *sker* 1_X is a strong congruence.

Quotients determined by congruences. Every congruence $\rho = (\sim, \mathbb{T})$ on X determines a topological space $(X/\rho, \mathcal{T}/\rho)$ with $X/\rho = \{[x] \mid x \in X\}$ and topology $\mathcal{T}/\rho = \{\pi_\rho(U) \mid U \in \mathbb{T}\}$ where $\pi_\rho : X \to X/\rho$ is defined by $\pi_\rho(x) = [x]$. This is a surjective continuous map with kernel ρ. This space X/ρ is called the *weak quotient space determined by* ρ and π_ρ is the *weak quotient map* (or just the canonical map). In general X/ρ need not be a quotient space with π_ρ a quotient map. Using condition $(C3)$ and the surjectivity of the map π_ρ, it can be shown that $\mathcal{T}/\rho = \{W \subseteq X/\rho \mid \pi_\rho^{-1}(W) \in \mathbb{T}\}$. Note that $(X/\iota_X, \mathcal{T}/\iota_X) \cong (X, \mathcal{T})$ and $(X/\upsilon_X, \mathcal{T}/\upsilon_X) \cong T$ (the trivial space). In fact, $X/\rho \cong X \Leftrightarrow \rho = \iota_X$, $X/\rho \cong T \Leftrightarrow \rho = \upsilon_X$ and $\iota_X = \upsilon_X \Leftrightarrow X \cong T$.

Ordering of congruences. For two congruences $\rho = (\sim_\rho, \mathbb{T}_\rho)$ and $\gamma = (\sim_\gamma, \mathbb{T}_\gamma)$ on X, we say ρ *is contained in* γ, written as $\rho \sqsubseteq \gamma$, provided $\sim_\rho \subseteq \sim_\gamma$ and $\mathbb{T}_\gamma \subseteq \mathbb{T}_\rho$. We need an extension of this ordering: Let $f : X \to Y$ be a continuous map with $\rho = (\sim, \mathbb{T})$ a congruence on X. By $f(\rho)$ we mean the pair $f(\rho) = (f(\sim), f(\mathbb{T}))$ where $f(\sim) = \{(f(a), f(b)) \mid a, b \in X$ with $a \sim b\}$ and $f(\mathbb{T}) = \{f(U) \mid U \in \mathbb{T}\}$. This need not be a congruence on Y, but we will compare it with congruences on Y. If $\gamma = (\sim_\gamma, \mathbb{T}_\gamma)$ is a congruence on Y, then we write $f(\rho) \sqsubseteq \gamma$ if:

(i) $f(\sim) \subseteq \sim_\gamma$ i.e., $a \sim b$ implies $f(a) \sim_\gamma f(b)$; and

(ii) $f^{-1}(\mathbb{T}_\gamma) \subseteq \mathbb{T}$; i.e., for every $W \in \mathbb{T}_\gamma$ we have $f^{-1}(W) \in \mathbb{T}$.

In such a case, we say $f(\rho)$ is *contained in* γ. The use of $\sqsubseteq$ here is in harmony with the earlier use if we consider the identity map $1_X : X \to X$ and compare $\rho = 1_X(\rho)$ and γ. It can easily be verified that for any congruence ρ on X, $\iota_X \sqsubseteq \rho \sqsubseteq \upsilon_X$.

Note that $Con(X) := \{\theta \mid \theta$ is a congruenece on $X\}$ is a complete lattice where $\bigcup_{i \in I} \theta_i = (\sim_\cup, \mathbb{T}_\cup)$ and $\bigcap_{i \in I} \theta_i = (\sim_\cap, \mathbb{T}_\cap)$ for congruences $\theta_i = (\sim_i, \mathbb{T}_i)$ on X are defined as follows:

For $a, b \in X$,

$a \sim_\cup b \Leftrightarrow$ there are $i_1, i_2, ..., i_n \in I$ and $a_{i_1}, a_{i_2}, ..., a_{i_n} \in X, n \geq 2$, such that

$a = a_{i_1} \sim_{i_1} a_{i_2} \sim_{i_2} a_{i_3} \sim_{i_3} ... \sim_{i_{n-2}} a_{i_{n-1}} \sim_{i_{n-1}} a_{i_n} = b$; and

$a \sim_\cap b \Leftrightarrow a \sim_i b$ for all $i \in I$.

The congruence topologies are defined by $\mathbb{T}_\cup := \bigcap_{i \in I} \mathbb{T}_i$ and $\mathbb{T}_\cap$ is the topology on X with basis $\mathcal{B} := \{B \subseteq X \mid B$ is a finite intersection $B = \bigcap_{j=1}^{n} U_j$ where $U_j \in \mathbb{T}_{i_j}$ for some $i_j \in I, j = 1, 2, 3, ..., n\}$.

Extension of a congruence. Let $(S, \mathcal{T}_S)$ be a subspace of $(X, \mathcal{T})$ with $\theta = (\sim_S, \mathbb{T}_S)$ a congruence on S. We define an extension of θ to a congruence

$\overline{\theta} = (\sim_X, \mathbb{T}_X)$ on X which will retain the congruence classes of S. Let $a, b \in X$:

For both a and b in S, $a \sim_X b \Leftrightarrow a \sim b$; and

for both a and b not in S, $a \sim_X b \Leftrightarrow a = b$.

Let $\mathbb{T}_X = \{U \subseteq X \mid U \text{ is open}, U \cap S \in \mathbb{T}_S \text{ and } x \in U \text{ implies } [x]_X \subseteq U\}$. Then $\overline{\theta} = (\sim_X, \mathbb{T}_X)$ is a congruence on X and for any $a \in X$,

$$[a]_X = \begin{cases} [a]_S \text{ if } a \in S \\ \{a\} \text{ if } a \notin S. \end{cases}$$

This congruence $\overline{\theta}$ is called the *extension of* θ *to* X. It can be verified that if θ is a strong congruence on S, then so is its extension $\overline{\theta}$ on X.

The homeomorphism (isomorphism) theorems and two useful corollaries are given next. We use $\cong$ to denote homeomorphic spaces.

First Homeomorphism Theorem: Let $f : (X, \mathcal{T}) \to (Y, \mathcal{F})$ be a continuous map with $\rho = \ker f$. Then $(X/\rho, \mathcal{T}/\rho) \cong f(X)$.

Second Homeomorphism Theorem: Let $(X, \mathcal{T})$ be a topological space with $\rho = (\sim, \mathbb{T})$ a congruence on X and $\pi_\rho : X \to X/\rho$ the canonical map $\pi_\rho(a) = [a]$. Let S be a non-empty subset of X. Then ρ induces a congruence on the subspace S, denoted by $S \cap \rho = (\sim_S, \mathbb{T}_S)$, by restricting ρ to S in a natural way: For all $a, b \in S$, $a \sim_S b \Leftrightarrow a \sim b$ and $\mathbb{T}_S = \{U \cap S \mid U \in \mathbb{T}\}$. This congruence $S \cap \rho$ is called the *restriction of the congruence* ρ *to* S. The subspace of X/ρ determined by $\pi_\rho(S)$ will be denoted by $(S + \rho)/\rho$. Then the weak quotient space of S determined by the congruence $S \cap \rho$ on S is homeomorphic to the subspace $\{[a] \mid a \in S\}$ of $(X/\rho, \mathcal{T}/\rho)$; i.e., $S/S \cap \rho \cong (S + \rho)/\rho$.

Third Homeomorphism Theorem: Let $(X, \mathcal{T})$ be a topological space; let $\alpha = (\sim_\alpha, \mathbb{T}_\alpha)$ and $\beta = (\sim_\beta, \mathbb{T}_\beta)$ be two congruences on X with $\alpha \sqsubseteq \beta$. The *quotient of* β *by* α, written as $\beta/\alpha = (\sim_{\beta/\alpha}, \mathbb{T}_{\beta/\alpha})$, is the congruence on the weak quotient space $(X/\alpha, \mathcal{T}/\alpha)$ defined by:

- For $[a]_\alpha, [b]_\alpha \in X/\alpha$, $[a]_\alpha \sim_{\beta/\alpha} [b]_\alpha \Leftrightarrow a \sim_\beta b$; and
- $\mathbb{T}_{\beta/\alpha} = \{W \subseteq X/\alpha \mid W = \pi_\alpha(U) \text{ for some } U \in \mathbb{T}_\beta\}$ where $\pi_\alpha : X \to X/\alpha$ is the canonical map.

Then $(X/\alpha)/(\beta/\alpha)$ is homeomorphic to X/β.

Corollaries:

(1) Let $(X, \mathcal{T})$ be a topological space with $\alpha = (\sim_\alpha, \mathbb{T}_\alpha)$ a congruence on X. Then $\gamma = (\sim, \mathbb{T})$ is a congruence on the weak quotient space $(X/\alpha, \mathcal{T}/\alpha)$ if and only if $\gamma = \beta/\alpha$ for some congruence $\beta = (\sim_\beta, \mathbb{T}_\beta)$ on X with $\alpha \sqsubseteq \beta$.

(2) Let θ be a congruence on X. Then there is a one-to-one correspondence between the set of all congruences α on X for which $\theta \sqsubseteq \alpha$ and the set of all congruences on X/θ given by $\alpha \to \alpha/\theta$. This correspondence preserves containment, joins and intersections.

Subdirect products. Let $\prod_{i \in I} X_i$ denote the product of the topological spaces $X_i, i \in I$ with $p_j : \prod_{i \in I} X_i \to X_j$ the j-th projection. A subspace Y of $\prod_{i \in I} X_i$ is said to be a *subdirect product* of the spaces $X_i, i \in I$, if $p_i(Y) = X_i$ for all

$i \in I$. The characterization of such products in terms of congruences is given by the topological version of the well-known algebraic result:

Theorem 2.2. *[19] A topological space X is a subdirect product of spaces $X_i, i \in I$, if and only if for every $i \in I$ there is a congruence θ_i on X such that $X/\theta_i \cong X_i$ and $\bigcap_{i \in I} \theta_i = \iota_X$.*

We will need the next two results from elementary topology. Both are well-known and are easy consequences of the Embedding Lemma, but they are reformulated here in terms of subdirect products and we give their proofs using congruences. Recall that the *Sierpiński space* is the two point space with exactly one proper open subset.

Proposition 2.3. *Any non-trivial T_0-space is a subdirect product of copies of the Sierpiński space.*

Proof. Let $(X, \mathcal{T})$ be a T_0-space with at least two elements. Let $\mathcal{B}$ be a basis for $\mathcal{T}$. For every $B \in \mathcal{B}$, let $\gamma_B = (\sim_B, \mathbb{T}_B)$ be the congruence on X where $\sim_B$ is the equivalence relation on X with the two equivalence classes B and $X - B$ and the congruence topology is given by $\mathbb{T}_B = \{\emptyset, B, X\}$. Then the weak quotient space $X_B := (X/\gamma_B, \mathcal{T}/\gamma_B)$ is a two-element space with one proper open set; hence homeomorphic to the Sierpiński space. We show $\bigcap_{B \in \mathcal{B}} \gamma_B = \iota_X$. Suppose $\bigcap_{B \in \mathcal{B}} \gamma_B = (\sim, \mathbb{T})$. For $x, y \in X$ with $x \sim y$, the definition of the intersection of congruences gives $x \sim_B y$ for all $B \in \mathcal{B}$. Since X is a T_0-space, this is only possible if $x = y$. The intersection congruence topology $\mathbb{T}$ is by definition the topology on X which has as basis finite intersections $\bigcap_{i=1}^{n} U_i$ where each U_i is from one of the congruence topologies $\mathbb{T}_{B_i}$ for some $B_i \in \mathcal{B}$. Thus $\mathbb{T} = \mathcal{T}$ and so $\bigcap_{B \in \mathcal{B}} \gamma_B = (\sim, \mathbb{T}) = (\stackrel{\circ}{=}, \mathcal{T}) = \iota_X$. By Theorem 2.2 the result follows. □

Proposition 2.4. *Any non-trivial space X is a subdirect product of copies of the Sierpiński space and the two-element indiscrete space.*

Proof. Let $(X, \mathcal{T})$ be a space with at least two elements. Let $\mathcal{B}$ be a basis for $\mathcal{T}$. As in the previous proof, for all $B \in \mathcal{B}$, let $\gamma_B = (\sim_B, \mathbb{T}_B)$ be the congruence on X where $\sim_B$ is the equivalence relation on X with the two equivalence classes B and $X - B$ and the congruence topology is given by $\mathbb{T}_B = \{\emptyset, B, X\}$. Furthermore, for any $a \in X$, let $\gamma_a = (\sim_a, \mathbb{T}_a)$ be the congruence on X where $\sim_a$ is the equivalence relation on X with the two equivalence classes $\{a\}$ and $X - \{a\}$ and the congruence topology is given by $\mathbb{T}_a = \{\emptyset, X\}$. Then $X_B := (X/\gamma_B, \mathcal{T}/\gamma_B)$ is homeomorphic to the Sierpiński space and $X_a := (X/\gamma_a, \mathcal{T}/\gamma_a)$ is homeomorphic to the indiscrete space on a two-element set.

Again it can be verified that $(\bigcap_{B\in\mathcal{B}} \gamma_B) \cap (\bigcap_{a\in X} \gamma_a) = \iota_X$ and by Theorem 2.2 it follows that X is a subdirect product of copies of the Sierpiński space and the two-element indiscrete space. □

3 Hoehnke radicals, connectednesses and disconnectednesses

From [19] we now recall the definition and results on Hoehnke radicals and their relationship to connectednesses and disconnectednesses. All considerations will be in a *universal class* $\mathcal{W}$ of topological spaces. This means $\mathcal{W}$ is a non-empty class of spaces which is *hereditary* (if Y is a subspace of $X \in \mathcal{W}$, then $Y \in \mathcal{W}$) and *closed under continuous images* (if $f : X \to Y$ is a surjective continuous map with $X \in \mathcal{W}$, then also $Y \in \mathcal{W}$). Clearly then, we must have $T \in \mathcal{W}$. When we discuss examples, we will take the universal class to be the class of all topological spaces. Any subclass $\mathcal{M}$ of $\mathcal{W}$ will always be assumed to be *abstract*; i.e. $\mathcal{M}$ contains the trivial space T and all homeomorphic copies of spaces in $\mathcal{M}$ (this assumption will mostly not be mentioned explicitly). If Y is a subspace of X it will be denoted by $Y \prec X$ and a surjective continuous mapping from X to Y will be denoted by $X \longrightarrow\!\!\!\!\triangleright\, Y$. We need to recall many relevant definitions, constructions and results. These, and much more, can be found in [1]. At times, Kurosh-Amitsur will be written as KA.

A class $\mathcal{R} \subseteq \mathcal{W}$ is a *KA-radical class* (= *connectedness*) if it satisfies:

$X \in \mathcal{R} \Leftrightarrow$ for every $X \longrightarrow\!\!\!\!\triangleright\, Y, Y \neq T$, there is $S \prec Y, S \neq T$, with $S \in \mathcal{R}$.

A class $\mathcal{S} \subseteq \mathcal{W}$ is a *KA-semisimple class* (= *disconnectedness*) if it satisfies:

$X \in \mathcal{S} \Leftrightarrow$ for every $S \prec X, S \neq T$, there is $S \longrightarrow\!\!\!\!\triangleright\, Y, Y \neq T$, with $Y \in \mathcal{S}$.

The two operators $\mathcal{U}$ and $\mathcal{D}$ on a class $\mathcal{M} \subseteq \mathcal{W}$, called the *upper radical operator* and *semisimple operator* respectively, are given by:

$\mathcal{UM} = \{X \in W \mid X \longrightarrow\!\!\!\!\triangleright\, Y \neq T \text{ implies } Y \notin \mathcal{M}\}$ and

$\mathcal{DM} = \{X \in W \mid T \neq S \prec X \text{ implies } S \notin \mathcal{M}\}$.

Note that $\mathcal{M} \cap \mathcal{UM} = \{T\} = \mathcal{M} \cap \mathcal{DM}$. The class $\mathcal{UM}$ is always closed under continuous images and $\mathcal{DM}$ is always hereditary. If $\mathcal{M}$ is hereditary, then $\mathcal{UM}$ is a KA-radical class and if $\mathcal{M}$ is closed under continuous images, then $\mathcal{DM}$ is a KA-semisimple class. It can be shown that any KA-radical class $\mathcal{R}$ is closed under continuous images; hence $\mathcal{DR}$ is a KA-semisimple class; called the *semisimple class corresponding to* $\mathcal{R}$. Likewise, it can be shown that any KA-semisimple class $\mathcal{S}$ is hereditary, hence $\mathcal{US}$ is a KA-radical class called the *radical class corresponding to* $\mathcal{S}$.

Definition 3.1. *A mapping ρ which assigns to each $X \in \mathcal{W}$ a congruence* $\rho(X) = \rho_X =$
$(\sim_{\rho_X}, \mathbb{T}_{\rho_X})$ *on X, is called a Hoehnke radical on $\mathcal{W}$, if it satisfies the following two conditions:*

(H1) For any continuous map $f : X \to Y$, $f(\rho(X)) \sqsubseteq \rho(f(X))$; and

(H2) For all $X \in \mathcal{W}$, $\rho(X/\rho_X) = \iota_{X/\rho_X}$.

The class $\mathcal{S}_\rho = \{X \in \mathcal{W} \mid \rho(X) = \iota_X\}$ is called the associated semisimple class and $\mathcal{R}_\rho = \{X \in \mathcal{W} \mid \rho(X) = \upsilon_X\}$ the associated radical class.

Of course, if $f : X \to Y$ is a surjective continuous map, condition (H1) can be written as $f(\rho_X) \sqsubseteq \rho_Y$. In general, $\mathcal{S}_\rho$ need not be KA-semisimple and $\mathcal{R}_\rho$ need not be KA-radical. We do have $\mathcal{S}_\rho \cap \mathcal{R}_\rho = \{T\}$, $\mathcal{R}_\rho$ is always closed under continuous images and $\mathcal{R}_\rho = \mathcal{U}\mathcal{S}_\rho$. The next result shows that a Hoehnke radical is very general.

Theorem 3.2. *(a) Let ρ be a Hoehnke radical on $\mathcal{W}$. Then, for every $X \in \mathcal{W}$, $\rho(X) = \cap\{\theta \mid \theta$ is a congruence on X for which $X/\theta \in \mathcal{S}_\rho\}$ and $\mathcal{S}_\rho$ is closed under subdirect products.*

(b) Conversely, let $\mathcal{M} \subseteq \mathcal{W}$ be any abstract class. Then ρ defined by $\rho(X) = \cap\{\theta \mid \theta$ is a congruence on X for which $X/\theta \in M\}$ for all $X \in \mathcal{W}$ is a Hoehnke radical on $\mathcal{W}$ and $\mathcal{S}_\rho = \overline{\mathcal{M}}$; the subdirect closure of $\mathcal{M}$.

Any disconnectedness (or connectedness) can be obtained from a Hoehnke radical. For this we need two conditions. The Hoehnke radical ρ on $\mathcal{W}$ is said to be *complete* if it satisfies the following condition: whenever $X \in \mathcal{W}$ and θ is a strong congruence on X with $[a]_\theta \in \mathcal{R}_\rho$ for all $a \in X$, then $\theta \sqsubseteq \rho_X$. The radical ρ is *idempotent* if for all $X \in \mathcal{W}$ and all $a \in X$, the equivalence class $[a]_{\rho_X} \in \mathcal{R}_\rho$. Then:

Theorem 3.3. *(a) Let ρ be a Hoehnke radical on $\mathcal{W}$. Suppose ρ is complete, idempotent and for all $X \in \mathcal{W}$, the congruence ρ_X is strong. Then $\mathcal{S}_\rho$ is a disconnectedness and $\mathcal{R}_\rho = \mathcal{U}\mathcal{S}_\rho$ is a connectedness.*

(b) Let $\mathcal{S} \subseteq \mathcal{W}$ be a disconnectedness. Then there is a Hoehnke radical ρ on $\mathcal{W}$ which is complete, idempotent, for all $X \in \mathcal{W}$ the congruence ρ_X is strong, $\mathcal{S}_\rho = \mathcal{S}$ and $\mathcal{R}_\rho = \mathcal{U}\mathcal{S}$.

It is thus possible to define connectednesses and disconnectednesses for topological spaces in terms of congruences. We therefore call a Hoehnke radical ρ on $\mathcal{W}$ which is complete, idempotent and for which ρ_X is a strong congruence for all X, a *Kurosh-Amitsur radical*.

4 Ideal-hereditary Hoehnke radicals

Our terminolgy here is largely motivated by that in use for associative ring theory or similar classes of algebras. For a thorough overview of the radical theory of associative rings, Gardner and Wiegandt [7] can be consulted. In particular, to appreciate the similarities between the algebraic and the topological with the use of congruences in what will be presented below, Mlitz and Veldsman [12] can consulted. As background, recall, that a class of rings $\mathcal{M}$ is hereditary if $I \lhd A \in \mathcal{M}$ implies $I \in \mathcal{M}$ ($I \lhd A$ means I is an ideal of A). Let ρ be a Kurosh-Amisur radical of associative rings with corresponding radical class $\mathcal{R}$ and semisimple class $\mathcal{S}$. Then:

(1) $\mathcal{R}$ is hereditary iff $\rho(A) \cap I \subseteq \rho(I)$ for all rings A and ideals I of A.

(2) $\mathcal{S}$ is hereditary iff $\rho(I) \subseteq \rho(A) \cap I$ for all rings A and ideals I of A.

(3) ρ is ideal-hereditary iff $\rho(I) = \rho(A) \cap I$ for all rings A and ideals I of A.

If ρ is only a Hoehnke radical, then:

(4) $\rho(A) \cap I \subseteq \rho(I)$ for all rings A and ideals I of A implies ρ is idempotent (i.e., $\rho(\rho(A)) = \rho(A)$ for all rings A) and $\mathcal{R}_\rho$ is hereditary.

(5) $\rho(I) \subseteq \rho(A) \cap I$ for all rings A and ideals I of A implies ρ is complete (i.e., $\rho(I) = I \lhd A$ implies $I \subseteq \rho(A)$) and $\mathcal{S}_\rho$ is hereditary.

Since we know that an idempotent and complete Hoehnke radical of rings is a Kurosh-Amitsur radical, we thus know that an ideal-hereditary Hoenke radical of associative rings coincides with a Kurosh-Amitsur radical and both the radical and semisimple classes are hereditary.

Definition 4.1. *Let ρ be a Hoehnke radical of topological spaces. Then ρ is called:*

(1) r-hereditary if for every space X and subspace Y, $\rho_X \cap Y \sqsubseteq \rho_Y$.

(2) s-hereditary if for every space X and subspace Y, $\rho_Y \sqsubseteq \rho_X \cap Y$.

(3) Ideal-hereditary if it is both r-hereditary and s-hereditary.

(4) A hereditary torsion theory if it is an ideal-hereditary Hoehnke radical.

We will need:

Proposition 4.2. *Let ρ be a Hoehnke radical of topological spaces. If ρ is r-hereditary, then ρ is idempotent and $\mathcal{R}_\rho$ is hereditary.*

Proof. Let $x \in X$, X any space. For the idempotence of ρ, we need to show that $Y := [x]_{\rho X} \in \mathcal{R}_\rho$; i.e., $\rho_Y = \upsilon_Y$. Let $a, b \in Y$. Then $a \sim_{\rho X} b$ and hence also $a \sim_{\rho_X \cap Y} b$. From $\rho_X \cap Y \sqsubseteq \rho_Y$ we then get $a \sim_{\rho Y} b$. Thus the two equivalences $\sim_{\rho Y}$ and $\leftrightsquigarrow_Y$ coincide. Next we show $\mathbb{T}_{\rho_Y} = \mathcal{I}_Y$. Let $\emptyset \neq U \in \mathbb{T}_{\rho_Y}$, say $u \in U$. From $\rho_X \cap Y \sqsubseteq \rho_Y$ we get $\emptyset \neq W \in \mathbb{T}_{\rho_X}$ such that $U = Y \cap W$. For any $t \in Y$, it follows that $t \sim_{\rho_X} u \in U \subseteq W \in \mathbb{T}_{\rho_X}$ and so $t \in [u]_{\rho_X} \subseteq W$. Thus $U = Y$ and $\mathbb{T}_{\rho_Y} = \mathcal{I}_Y$. To show $\mathcal{R}_\rho$ is hereditary, let Y be a subspace of $X \in \mathcal{R}_\rho$. Then $\upsilon_Y = \upsilon_X \cap Y = \rho_X \cap Y \sqsubseteq \rho_Y$. This gives $\upsilon_Y = \rho_Y$ and so $Y \in \mathcal{R}_\rho$. □

Proposition 4.3. *Let ρ be a Hoehnke radical of topological spaces determined by the class $\mathcal{M}$. Then conditions (a), (b) and (c) below are equivalent:*

(a) ρ is s-hereditary.

(b) $\mathcal{S}_\rho$ is hereditary; and

(c) If Y is a subspace of X with $X \in \mathcal{M}$, then $Y \in \overline{\mathcal{M}} = \mathcal{S}_\rho$.
Moreover, if ρ is s-hereditary, then it is complete.

Proof. (a) $\Rightarrow$ (b): Suppose ρ is s-hereditary and let Y be a subspace of $X \in \mathcal{S}_\rho$. Then $\rho_Y \sqsubseteq \rho_X \cap Y = \iota_X \cap Y = \iota_Y$. Thus $\rho_Y = \iota_Y$ and hence $Y \in \mathcal{S}_\rho$.

(b) $\Rightarrow$ (c) is trivial since $\mathcal{M} \subseteq \overline{\mathcal{M}} = \mathcal{S}_\rho$.

(c) $\Rightarrow$ (a): Suppose (c) holds and let Y be a subspace of X. Let θ be any congruence on X with $X/\theta \in \mathcal{M}$. By the Second Homeomorphism Theorem and the assumption (c), we have $Y/(\theta \cap Y) \cong (Y + \theta)/\theta$ which is a subspace of $X/\theta \in \mathcal{M}$ and hence $Y/(\theta \cap Y) \in \overline{\mathcal{M}} = \mathcal{S}_\rho$. Thus $\rho_Y \sqsubseteq \theta \cap Y$ and $\rho_Y \sqsubseteq \rho_X \cap Y$ follows.

Lastly, suppose ρ is s-hereditary and let θ be a strong congruence on X with $[x]_\theta \in \mathcal{R}_\rho$ for all $x \in X$. We need to show $\theta \sqsubseteq \rho_X$. Firstly, let $a, b \in X$ with $a \sim_\theta b$. Then $[b]_\theta \in \mathcal{R}_\rho$ and $\nu_{[b]_\theta} = \rho_{[b]_\theta} \sqsubseteq \rho_X \cap [b]_\theta$ by the two assumptions. Now $a, b \in [b]_\theta$ and so $a \leftrightsquigarrow_{[b]_\theta} b$ which by the previous equality gives $a \sim_{\rho_{[b]_\theta}} b$ and by the inclusion we get $a \sim_{\rho_X} b$. Thus $\sim_\theta \subseteq \sim_{\rho_X}$. Secondly, let $U \in \mathbb{T}_{\rho_X}$. Then U is open in X and for any $u \in U$, the first part gives $[u]_\theta \subseteq [u]_{\rho_X}$. Thus $\bigcup_{u \in U} [u]_\theta \subseteq \bigcup_{u \in U} [u]_{\rho_X} = U \subseteq \bigcup_{u \in U} [u]_\theta$. Hence $U = \bigcup_{u \in U} [u]_\theta$ and since θ is a strong congruence, $U \in \mathbb{T}_\theta$. The required inclusion $\theta \sqsubseteq \rho_X$ follows. □

Corollary 4.4. *Let ρ be an ideal-hereditary Hoehnke radical (= hereditary torsion theory) in a universal class of topological spaces. Then ρ is idempotent, complete and both the radical class $\mathcal{R}_\rho$ and the semisimple class $\mathcal{S}_\rho$ are hereditary.*

For all the well-known classes of algebras, any ρ as above (ideal-hereditary Hoehnke radical) will be a Kurosh-Amitsur radical. For topological spaces, this need not be the case as will be seen in the next section.

5 Hereditary torsion theories.

In this section, the universal class $\mathcal{W}$ is the class of all topological spaces and an example will be given to show that a hereditary torsion theory of topological spaces need not be a Kurosh-Amitsur radical. Because this example also shows that a complete idempotent Hoehnke radical need not be Kurosh-Amitsur, we will look at the salient properties of such radicals. It will be seen that they agree to a large extend with those of the Kurosh-Amitsur radicals and one can mostly use the arguments from Preuß [13] and [14] to justify their validity. Necessary and sufficient conditions are given to ensure that an ideal-hereditary Hoehnke radical of topological spaces is a Kurosh-Amitsur radical. In conclusion we determine all the ideal-hereditary Hoehnke radicals; in particular also showing which ones are Kurosh-Amitsur radicals.

To start, we firstly consider the possible congruences on a topological space. For any set X, we will use $\mathcal{I}_X$ and $\mathcal{D}_X$ to denote the indiscrete and discrete toplogies on X respectively (on a two-element set we will rather use $\mathcal{I}_2$ and $\mathcal{D}_2$). Let $(X, \mathcal{T})$ be a topological space with $\gamma = (\sim, \mathbb{T}_\gamma)$ a congruence on X. This means $\mathbb{T}_\gamma$ is any topology on X with $\mathcal{I}_X \subseteq \mathbb{T}_\gamma \subseteq \mathbb{T}_{s(\sim)} \subseteq \mathcal{T}$ where $\mathbb{T}_{s(\sim)}$ denotes the strong congruence topology on X with respect to $\sim$. When $\sim$ is the equality relation $\Bumpeq$ on X (also known as the diagonal), then $\mathbb{T}_{s(\Bumpeq)} = \mathcal{T}$ and $\mathbb{T}_\gamma$ can be any topology on X with $\mathcal{I}_X \subseteq \mathbb{T}_\gamma \subseteq \mathcal{T}$. In this case $[x] = \{x\}$ for all $x \in X$ and the weak quotient space $(X/\gamma, \mathcal{T}/\gamma)$ is identified with $(X, \mathbb{T}_\gamma)$ since $\mathcal{T}/\gamma = \{\pi(U) \mid U \in \mathbb{T}_\gamma\} = \mathbb{T}_\gamma$. When $\sim$ is the universal relation $\leftrightsquigarrow$ on X (i.e. $a \leftrightsquigarrow b$ for all $a, b \in X$), then $\mathcal{I}_X = \mathbb{T}_\gamma = \mathbb{T}_{s(\leftrightsquigarrow)}$ and the only congruence on X with respect to the universal relation is $\gamma = (\leftrightsquigarrow, \mathbb{T}_\gamma) = (\leftrightsquigarrow, \mathcal{I}_X) = \upsilon_X$. Any congruence $\gamma = (\sim, \mathbb{T}_\gamma)$ on $(X, \mathcal{T})$ gives the following chain of congruences:

$$\iota_X = (\Bumpeq, \mathcal{T}) \sqsubseteq (\sim, \mathbb{T}_{s(\sim)}) \sqsubseteq \gamma = (\sim, \mathbb{T}_\gamma) \sqsubseteq (\sim, \mathcal{I}_X) \sqsubseteq (\leftrightsquigarrow, \mathcal{I}_X) = \upsilon_X.$$

We use I_2 to denote the two-element indiscrete space, S_2 for the Sierpiński space (with topology $\mathcal{S}_2$) and D_2 for the two-element discrete space. On I_2 there are only two congruences ι_{I_2} and υ_{I_2}, on S_2 there are three namely $\iota_{S_2}, (\Bumpeq, \mathcal{I}_2)$ and υ_{S_2} and D_2 has four congruences $\iota_{D_2}, (\Bumpeq, \mathcal{S}_2), (\Bumpeq, \mathcal{I}_2)$ and υ_{D_2}. Recall, an object Q in a category is called *injective* if for any given morphism $g : C \to Q$ and monomorphism $f : C \to B$ there exists a morphism $h : B \to Q$ such that $h \circ f = g$. It is known (in any case easy to prove) that a topological space is injective in the category of all topological spaces precisely when it is an indiscrete space.

Example 5.1. *An ideal-hereditary Hoehnke radical which is not Kurosh-Amitsur. For each $(X, \mathcal{T})$, let $\rho_X = (\sim_X, \mathbb{T}_X)$ where $\sim_X$ is $\Bumpeq$ and $\mathbb{T}_X = \mathcal{I}_X$ where $\mathcal{I}_X$ is the indiscrete topology on X. Then ρ_X is a congruence on X and in general it need not be a strong congruence. Let $(X/\rho_X, \mathcal{T}/\rho_X)$ be the weak quotient space determined by the congruence ρ_X with $\pi : X \longrightarrow\!\!\triangleright X/\rho_X$ the canonical map $\pi(x) = [x] = \{x\}$. We identify $\{x\}$ and x and hence $(X/\rho_X, \mathcal{T}/\rho_X) = (X, \mathcal{I}_X)$. It is straightforward to check that ρ is a Hoehnke radical which is complete and idempotent. It is actually ideal-hereditary: Let Y be a subspace of a space X. We show that $\rho_X \cap Y = \rho_Y$. For any $a, b \in Y$, $a \sim_{\rho_X \cap Y} b$ iff $a \sim_{\rho_X} b$ iff $a = b$ iff $a \sim_{\rho_Y} b$ and so the two equivalences $\sim_{\rho_X \cap Y}$ and $\sim_{\rho_Y}$ coincide. Moreover, $\mathbb{T}_{\rho_X \cap Y} = \{V \cap Y \mid V \in \mathcal{T}_X = \mathcal{I}_X\} = \{\emptyset, Y\} = \mathcal{I}_Y = \mathbb{T}_Y$. Thus $\rho_X \cap Y = \rho_Y$. This means ρ is an ideal-hereditary Hoehnke radical (= hereditary torsion theory) with a hereditary radical class $\mathcal{R}_\rho = \{T\}$ and a hereditary semisimple class $\mathcal{S}_\rho = \{X \mid X$ is an indiscrete space (= injective space)$\}$; but ρ is not a Kurosh-Amitsur radical (ρ_X need not be a strong congruence). In any case, it is known that the class of indiscrete spaces is not a disconnectedness (it is herditary and productive, but not upwards-closed: the function $f : S_2 \to I_2$ with*

$f(0) = 0$ and $f(1) = 1$ is surjective and continuous with $f^{-1}(0)$ and $f^{-1}(1)$ indiscrete subspaces but S_2 is not indiscrete).

Example 5.2. *A non-trivial ideal-herditary Kurosh-Amitsur radical.*
The class of indiscrete spaces should not be discarded as uninteresting from a radical theory point of view. It is actually a hereditary connectedness. This is not a new result; what is new here is to describe it as a Hoehnke radical using congruences. For any space $(X, \mathcal{T})$, let $\alpha_X = (\sim_{\alpha_X}, \mathbb{T}_{\alpha_X})$ be the congruence with $x \sim_{\alpha_X} y$ iff for any $U \subseteq X$ open, $x \in U \Leftrightarrow y \in U$ and let $\mathbb{T}_{\alpha_X} = \mathcal{T}$. It can be checked that α_X is a strong congruence on X and α is a complete and idempotent Hoehnke radical. It is also ideal-hereditary: Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. For $a, b \in Y$, $a \sim_{\alpha_X \cap Y} b$ iff $a \sim_{\alpha_X} b$ iff for any $U \subseteq X$ open, $a \in U \Leftrightarrow b \in U$ iff for any $V \subseteq Y$ open, $a \in V \Leftrightarrow b \in V$ iff $a \sim_{\alpha_Y} b$. In addition, $\mathbb{T}_{\alpha_X \cap Y} = \{V \cap Y \mid V \in \mathbb{T}_{\alpha_X} = \mathcal{T}\} = \mathcal{T}_Y = \mathbb{T}_{\alpha_Y}$ and so $\alpha_X \cap Y = \alpha_Y$. In summary we have: α is an ideal-hereditary Kurosh-Amitsur with $\mathcal{R}_\alpha = \{X \mid X$ is an indiscrete space$\}$ a hereditary connectedness and $\mathcal{S}_\alpha = \{X \mid X$ is a T_0-space$\}$ a hereditary disconnectedness. All disconnectednesses of topological spaces are hereditary, but apart from the class of all indiscrete spaces, there are only two other hereditary connectednesses, namely the two trivial ones $\{T\}$ and $\{X \mid X$ is any space$\}$ (cf. [1]).

For a Hoehnke radical ρ, we will write $\rho(X) = \rho_X = (\sim_{\rho_X}, \mathbb{T}_{\rho_X})$ or just $(\sim_X, \mathbb{T}_X)$ for all spaces X. The topology on X is mostly not given explicitly, but when necessary and nothing else is mentioned, it will be $\mathcal{T}$. The set X with open sets the congruence topology $\mathbb{T}_{\rho_X}$ will be denoted by X_ρ, i.e., X_ρ is the topological space $X_\rho = (X, \mathbb{T}_{\rho_X})$.

Proposition 5.3. *Let ρ be a complete and idempotent Hoehnke radical of topological spaces with corresponding semisimple class $\mathcal{S}_\rho$ and radical class $\mathcal{R}_\rho$. Then:*
(1) For every space X, there is a surjective continuous map $q : X \to X_S$ with $X_S \in \mathcal{S}_\rho$ and if $f : X \to Y$ is any surjective continuous map with $Y \in \mathcal{S}_\rho$, then there is a continuous mapping $g : X_S \to Y$ such that $g \circ q = f$. For every $a \in X_S$, $q^{-1}(a) \in \mathcal{R}_\rho$ and if B is a subspace of X with $B \in \mathcal{R}_\rho$, then there is an $a \in X_S$ such that $B \subseteq q^{-1}(a)$.
(2) $\mathcal{R}_\rho = U\mathcal{S}_\rho, \mathcal{R}_\rho \cap \mathcal{S}_\rho = \{T\}, \mathcal{S}_\rho \subseteq \mathcal{D}\mathcal{R}_\rho$ and $X \in \mathcal{D}\mathcal{R}_\rho \Rightarrow X_\rho \in \mathcal{S}_\rho$.
(3) $\mathcal{S}_\rho$ is closed under subdirect products and is weakly hereditary (i.e., for any subspace Y of $X, X \in \mathcal{S}_\rho \Rightarrow Y_\rho \in \mathcal{S}_\rho$).
(4) If $\mathcal{S}_\rho$ consists of T_0-spaces with at least one space that is not T_1, then $S_2 \in \mathcal{S}_\rho$ and $\mathcal{R}_\rho$ is the class of all indiscrete spaces.
(5) If $\mathcal{S}_\rho$ contains a non-T_0-space, then $I_2 \in \mathcal{S}_\rho$ and $\mathcal{R}_\rho = \{T\}$.
(6) $\mathcal{R}_\rho$ is closed under continuous images and products.

Proof. (1) By the definition of a Hoehnke radical and Theorem 3.2, the map q is just the weak quotient map $X \to X/\rho_X = X_S \in \mathcal{S}_\rho$ and the idempotence

gives $q^{-1}(a) = [a]_{\rho_X} \in \mathcal{R}_\rho$. For the last property, let B be a subspace of X with $B \in \mathcal{R}_\rho$. Then $\rho_B = \nu_B$ (which is a strong congruence on B) and we can extend ρ_B to a strong congruence $\gamma = (\sim_\gamma, \mathbb{T}_\gamma)$ on X with $[b]_{\gamma_B} = [b]_{\rho_B} = B \in \mathcal{R}_\rho$ for all $b \in B$ and for $x \in X - B$, $[x]_{\gamma_X} = \{x\} \in \mathcal{R}_\rho$. By assumption, ρ is complete and so $\gamma \sqsubseteq \rho_X$. This means, for any $b \in B$, $B = [b]_{\gamma_B} \subseteq [b]_{\rho_X} = q^{-1}(q(b))$.
(2) We only check the last implication: Let $(X, \mathcal{T}) \in \mathcal{DR}_\rho$. By the idempotence $[x]_{\rho_X} \in \mathcal{R}_\rho$ and so $[x]_{\rho_X} = \{x\}$ for all $x \in X$. Thus $\rho_X = (\sim_{\rho_X}, \mathbb{T}_{\rho_X}) = (\Bumpeq, \mathbb{T}_{\rho_X})$ and hence $X_\rho = (X, \mathbb{T}_{\rho_X}) = (X/\rho_X, \mathbb{T}/\rho_X) \in \mathcal{S}_\rho$.
(3) The first statement follows from Theorem 3.2. For the second, let $X \in \mathcal{S}_\rho$ with Y a subspace of X. For any $y \in Y$, $[y]_{\rho_Y} \in \mathcal{R}_\rho$ by the idempotence and then by (1) above, $[y]_{\rho_Y} \subseteq [y]\rho_X = \{y\}$; the last equality follows since $X \in \mathcal{S}_\rho$. Thus $\rho_Y = (\Bumpeq, \mathbb{T}_{\rho_Y})$ and so $Y_\rho = (Y, \mathbb{T}_{\rho_Y}) = (Y/\rho_Y, \mathbb{T}_{\rho_Y}) \in \mathcal{S}_\rho$.
(4) Suppose $\mathcal{S}_\rho$ consists of T_0-spaces with $Z \in \mathcal{S}_\rho$, Z is not a T_1 space. This means there are distinct x and y in Z and $U \subseteq Z$ open such that $x \in U$, $y \notin U$ and whenever $y \in V$ for $V \subseteq Z$ open, then also $x \in V$. Then the subspace $Y = \{x, y\}$ of Z is just the Sierpiński space S_2. Since $\mathcal{S}_\rho$ is weakly hereditary and $Z \in \mathcal{S}_\rho$, we have $Y\rho = (Y, \mathbb{T}_{\rho_Y}) \in \mathcal{S}_\rho$ where $\rho_Y = (\sim_{\rho_Y}, \mathbb{T}_{\rho_Y}) = (\Bumpeq, \mathbb{T}_{\rho_Y})$. The topology $\mathbb{T}_{\rho_Y}$ on Y must be the Sierpiński topology since $(Y, \mathbb{T}_{\rho_Y}) \in \mathcal{S}_\rho$ eliminates the indiscrete topology as a possibility. Thus $S_2 \in \mathcal{S}_\rho$. Next we show $\mathcal{R}_\rho = \{X \mid X \text{ indiscrete}\}$: Let $X \in \mathcal{R}_\rho$. If X contains a proper open subset W, then there is a surjective continuous map $f : X \to S_2$ which gives the absurdity $S_2 \in \mathcal{R}_\rho \cap \mathcal{S}_\rho = \{T\}$. Hence X must be indiscrete. Conversely, if X is any indiscrete space and $\rho_x \neq \upsilon_X$, then $(X/\rho_X, \mathcal{T}/\rho_X)$ is a non-trivial indiscrete T_0-space in $\mathcal{S}_\rho$; clearly not possible. Thus $\rho_x = \upsilon_X$ and $X \in \mathcal{R}_\rho$.
(5) Suppose now $W \in \mathcal{S}_\rho$ is not a T_0-space. Then there are distinct a and b in W and no open set in W that contains the one but not the other. This means the subspace $R = \{a.b\}$ of W is just the indiscrete space I_2 which by (3) above must be in $\mathcal{S}_\rho$. Since $\mathcal{R}_\rho$ is closed under continuous images and I_2 is a continuous image of any non-trivial space, $\mathcal{R}_\rho = \{T\}$ follows.
(6) We already know $\mathcal{R}_\rho$ is closed under continuous images for any Hoehnke radical. We show $\mathcal{R}_\rho$ is closed under products in several steps. (i) $\mathcal{R}_\rho$ is closed under finite products: Let $X, Y \in \mathcal{R}_\rho$ and choose $(a, b) \in X \times Y$. Let $[(a, b)]$ denote the equivalence class of (a, b) in $X \times Y$ with respect to $\rho_{X \times Y} = (\sim, \mathbb{T})$. Then $X \times Y \in \mathcal{R}_\rho$ will follow from the idempotence if we can show $[(a, b)] = X \times Y$. Let $(x, y) \in X \times Y$. Now $(a, b) \in \{a\} \times Y \cong Y \in \mathcal{R}_\rho$ and so $\{a\} \times Y \subseteq [(a, b)]$ by (1) above. Likewise, from $(a, b) \in X \times \{b\} \cong X \in \mathcal{R}_\rho$ we have $X \times \{b\} \subseteq [(a, b)]$. Thus $(x, b) \in \{x\} \times Y \cong Y \in \mathcal{R}_\rho$ and $(x, b) \in X \times \{b\} \subseteq [(a, b)]$ give $\{x\} \times Y \subseteq [(a, b)]$. Hence $(x, y) \in [(a, b)]$. We conclude that $X \times Y = [(a, b)] \in \mathcal{R}_\rho$ and a simple induction extends this conclusion to any finite product. (ii) $\mathcal{R}_\rho$ is closed under arbitrary products: For this we distinguish three cases: (a) Every space in $\mathcal{S}_\rho$ is a T_0-space with at least one space that is not a T_1-space; (b) $\mathcal{S}_\rho$ contains at least one space that is not a T_0-space; and (c) all the spaces in $\mathcal{S}_\rho$ are T_1-spaces. For cases (a) and (b),

the result follows trivially since by (4) and (5) above, $\mathcal{R}_\rho$ is either the class of all indiscrete spaces or the class of trivial spaces; both closed under arbitrary products. Suppose thus $\mathcal{S}_\rho$ consists of T_1-spaces and let $X_i \in \mathcal{R}_\rho$ for all $i \in I$, I some index set. Let $X = \prod_{i\in I} X_i$ with $\pi : X \to X/\rho_X$ the weak quotient map. Since $X/\rho_X \in \mathcal{S}_\rho$, it then follows that $[a] = \pi^{-1}(\pi(a))$ is closed in X for all $a \in X$ where $[a] = [a]_{\rho_X} \in \mathcal{R}_\rho$. To complete the proof, it will thus suffice to show that for $a = (a_i) \in X$, $X = \overline{[a]}$, the closure of $[a]$. Let $x = (x_i) \in X$ and suppose $x \in U = \prod_{i\in I} U_i$, U a base element. This means U_i is open in X_i for all $i \in I$ and there is a finite subset J of I such that $U_i = X_i$ for all $i \in I - J$. For each $i \in I$, let $H_i = \begin{cases} X_i \text{ if } i \in J \\ \{a_i\} \text{ if } i \in I - J \end{cases}$. By the first part, $H := \prod_{i\in I} H_i \in \mathcal{R}_\rho$ and from $a \in H$, we have $H \subseteq [a]$. For all $j \in J$, choose $u_j \in U_j$ and let $w = (w_i) \in X$ be defined by $w_i = \begin{cases} u_i \text{ if } i \in J \\ a_i \text{ if } i \in I - J \end{cases}$. Then $w \in U \cap H \subseteq U \cap [a]$ and we are done. □

Finally, all the ideal-hereditary Hoehnke radicals ρ of topological spaces are determined. There are exactly five such radicals of which three are Kurosh-Amitsur radicals, i.e. for such ρ, the classes $\mathcal{R}_\rho$ and $\mathcal{S}_\rho$ form a corresponding pair of connectednesses and disconnectednesses.

Theorem 5.4. *Let ρ be an ideal-herediary Hoehnke radical of topological spaces. Then ρ is one of the following five radicals:*
(a) $\rho_X = \upsilon_X$ for all X. This is a Kurosh-Amitsur radical with $\mathcal{R}_\rho$ the class of all spaces and $\mathcal{S}_\rho = \{T\}$.
(b) $\rho_X = (\sim_X, \mathbb{T}_X)$ is the congruence with $x \sim_X y$ iff for any $U \subseteq X$ open, $x \in U \Leftrightarrow y \in U$ and $\mathbb{T}_X = \mathcal{T}$. This is a Kurosh-Amitsur radical with $\mathcal{R}_\rho = \{X \mid X$ is an indiscrete space$\}$ and $\mathcal{S}_\rho = \{X \mid X$ is a T_0-space$\}$.
(c) $\rho_X = \iota_X$ for all X. This is a Kurosh-Amitsur radical with $\mathcal{R}_\rho = \{T\}$ and $\mathcal{S}_\rho$ is the class of all spaces.
(d) $\rho_X = (\Bumpeq, \mathcal{I}_X)$ for all X which is not a Kurosh-Amitsur radical. Here $\mathcal{R}_\rho = \{T\}$ and $\mathcal{S}_\rho = \{X \mid X$ is an indiscrete space$\}$.
(e) $\rho_X = (\Bumpeq, \mathbb{T}_X)$ where $\mathbb{T}_X = \begin{cases} \mathcal{T} \text{ if } (X, \mathcal{T}) \text{ is a } T_1\text{-space} \\ \mathcal{I}_X \text{ otherwise} \end{cases}$. This is not Kurosh-Amitsur, $\mathcal{R}_\rho = \{T\}$ and $\mathcal{S}_\rho = \{X \mid X$ is an indiscrete space or a T_1-space$\}$.

Proof. We know that ρ is complete, idempotent and both $\mathcal{R}_\rho$ and $\mathcal{S}_\rho$ are hereditary. Since the two-point indiscrete space has only two congruences, we consider the two cases (1) $I_2 \in \mathcal{R}_\rho$ when $\rho_{I_2} = \upsilon_{I_2}$ and (2), $I_2 \in \mathcal{S}_\rho$ when $\rho_{I_2} = \iota_{I_2}$.
(1) Suppose $I_2 \in \mathcal{R}_\rho$ and then distinguish the two subcases (1.1) $S_2 \in \mathcal{R}_\rho$ and (1.2) $S_2 \notin \mathcal{R}_\rho$. For (1.1) we have $I_2, S_2 \in \mathcal{R}_\rho$. By Proposition 2.4, any space is a

subdirect product of copies of I_2 and S_2. Since $\mathcal{R}_\rho$ is closed under products and is hereditary, we can conclude that in this case $\mathcal{R}_\rho$ is the class of all spaces and then $\mathcal{S}_\rho = \{T\}$ by Proposition 5.3(2); hence ρ is the Kurosh-Amitsur radical with $\rho_X = \upsilon_X$ for all X giving (a). For (1.2) we have $S_2 \notin \mathcal{R}_\rho$ and hence $S_2 \in \mathcal{S}_\rho$. Indeed, S_2 has only three congruences and and we know $\rho_{S_2} \neq \upsilon_{S_2}$. If $\rho_{S_2} = (\eqcirc, \mathcal{I}_2)$, then the weak quotient $S_2/\rho_{S_2} \cong I_2 \in \mathcal{R}_\rho \cap \mathcal{S}_\rho = \{T\}$; a contradiction. Hence $\rho_{S_2} = \iota_{S_2}$. Thus we have $I_2 \in \mathcal{R}_\rho$ and $S_2 \in \mathcal{S}_\rho$. Note firstly that any space X in $\mathcal{S}_\rho$ is T_0. Indeed, let x and y be two distinct elements from X. The subspace $Y = \{x, y\}$ cannot be I_2 since $\mathcal{S}_\rho$ is hereditary and $I_2 \in \mathcal{R}_\rho$. Thus Y must be S_2 or D_2. In either case, there is at least one open set in X that contains one point of Y but not the other. Thus X is a T_0-space. Since the non-T_1-space S_2 is in $\mathcal{S}_\rho$, Proposition 5.3 (4) yields $\mathcal{R}_\rho = \{X \mid X$ is an indiscrete space$\}$. Moreover, by Proposition 2.3 any T_0-space is a subdirect product of copies of $S_2 \in \mathcal{S}_\rho$ and since $\mathcal{S}_\rho$ is closed under subdirect products (cf. Theorem 3.2), $\mathcal{S}_\rho = \{X \mid X$ is a T_0-space$\}$ holds. In Example 5.2 above we have seen that this is also a Kurosh-Amitsur radical giving (b).

(2) Suppose $I_2 \in \mathcal{S}_\rho$. Then $\mathcal{R}_\rho = \{T\}$ for if $X \in \mathcal{R}_\rho$ and X has more than one element, there is a continuous mapping from X onto I_2. Proposition 5.3(6) then gives $I_2 \in \mathcal{R}_\rho \cap \mathcal{S}_\rho = \{T\}$; a contradiction. The idempotency of ρ means $[x]_{\rho X} \in \mathcal{R}_\rho = \{T\}$; hence $[x]_{\rho X} = \{x\}$ for all $x \in X$ and for all spaces X. Thus $\rho_X = (\eqcirc, \mathbb{T}_X)$ for all X. If we identify $\{x\}$ with x, the weak quotient of X by ρ_X is $(X, \mathbb{T}_X)$ and $(X, \mathbb{T}_X) \in \mathcal{S}_\rho$. We thus have $(X, \mathcal{T}) \in \mathcal{S}_\rho \Leftrightarrow \mathbb{T}_X = \mathcal{T}$. Since $I_X \subseteq \mathbb{T}_X \subseteq \mathcal{T}$, it follows that every indiscrete space is in $\mathcal{S}_\rho$. We now proceed by looking at several subcases. Firstly, either (2.1) $S_2 \in \mathcal{S}_\rho$ or (2.2), $S_2 \notin \mathcal{S}_\rho$. For (2.1) we have both I_2 and S_2 in $\mathcal{S}_\rho$. By Proposition 2.4 and the fact that $\mathcal{S}_\rho$ is closed under subdirect products we get all topological spaces in $\mathcal{S}_\rho$ and thus (c). For (2.2) we have $S_2 \notin \mathcal{S}_\rho$ and since $\mathcal{S}_\rho$ is hereditary, $\mathcal{S}_\rho$ is contained in the class of all topological spaces that do not have S_2 as a subspace, i.e., $\mathcal{S}_\rho \subseteq \{(X, \mathcal{T}) \mid a \in U \in \mathcal{T} \Rightarrow \overline{\{a\}} \subseteq U\}$. Moreover, for a space $(X, \mathcal{T})$ and subspace $(Y, \mathcal{T}_Y)$, we know that $(Y, (\mathbb{T}_X)_Y)$ is a subspace of $(X, \mathbb{T}_X)$. As usual, $\mathcal{T}_Y$ and $(\mathbb{T}_X)_Y$ denote the relative topologies on Y with respect to $\mathcal{T}$ and $\mathbb{T}_X$ respectively. By the ideal-heredity of ρ, we have $\rho_{(Y,\mathcal{T}_Y)} = \rho_{(X,\mathcal{T})} \cap (Y, \mathcal{T}_Y)$ which means $\mathbb{T}_Y = \{Y \cap W \mid W \in \mathbb{T}_X\} = (\mathbb{T}_X)_Y$. This brings us to the last two subcases to consider, namely (2.2.1) $D_2 \notin \mathcal{S}_\rho$ and (2.2.2) $D_2 \in \mathcal{S}_\rho$.

(2.2.1) Suppose $D_2 \notin \mathcal{S}_\rho$. Then $\mathcal{S}_\rho = \{X \mid X$ is indiscrete$\}$. Indeed, suppose $(X, \mathcal{T}) \in \mathcal{S}_\rho$ and $\emptyset \neq U \subsetneq X$ is open. Choose $p \in U$ and $q \in X - U$. Then $Y = (\{p, q\}, \mathcal{T}_Y)$ is a subspace of $(X, \mathcal{T}) \in \mathcal{S}_\rho$ which gives $Y \in \mathcal{S}_\rho$. Clearly $Y \neq I_2$; but neither can it be S_2 or D_2. Consequently X can only be an indiscrete space. Since $(X, \mathbb{T}_X) \in \mathcal{S}_\rho$ we get $\rho_X = (\eqcirc, \mathcal{I}_X)$ for all X which gives (d).

(2.2.2) Suppose $D_2 \in \mathcal{S}_\rho$. For any space $(X, \mathcal{T})$, let $\mathcal{B}_X$ be the class of all subsets of X that are both open and closed. Let $\overline{\mathcal{B}_X}$ be the topology generated by $\mathcal{B}_X$. We claim $\mathcal{I}_X \subseteq \mathcal{B}_X \subseteq \overline{\mathcal{B}_X} \subseteq \mathbb{T}_X \subseteq \mathcal{T} \subseteq \mathcal{D}_X$. Only $\mathcal{B}_X \subseteq \mathbb{T}_X$ needs verification, the other inclusions being obvious. Let U be a proper subset of X that is both open and closed. Define a mapping $f : X \to D_2$ by $f(U) = \{0\}$ and $f(X-U) = \{1\}$. Then f is a surjective continuous map and by (H1) we get $f^{-1}(\mathcal{D}_2) \subseteq \mathbb{T}_X$. In particular, $U = f^{-1}(\{0\}) \in \mathbb{T}_X$. Thus every zero-dimensional space (spaces with a basis of subsets that are both open and closed) is in $\mathcal{S}_\rho$; in particular all discrete spaces are in $\mathcal{S}_\rho$. We can now show $\mathcal{S}_\rho = \{X \mid X$ is an indiscrete space or a T_1-space$\}$. Indeed, let $(X, \mathcal{T})$ be from $\mathcal{S}_\rho$ and suppose it has proper open subsets. Then it must be a T_1space: Let $a \in X$ and suppose $b \in \overline{\{a\}}$, $b \neq a$. Now $\overline{\{a\}} \neq X$, for if $\overline{\{a\}} = X$ and U is any proper open subset of X, we have $a \in U$ and from the above $\overline{\{a\}} \subseteq U$ which is not possible. Choose thus $c \in X - \overline{\{a\}}$ and consider the subspace $Y = (\{b, c\}, \mathcal{T}_Y)$ of $X \in \mathcal{S}_\rho$. Clearly $Y \neq I_2$; hence $Y = D_2$ and there is $W \in \mathcal{T}$ with $\{b\} = Y \cap W$. But then $\{a\} \cap W \neq \emptyset$ which contradicts $Y \cap W = \{b\}$. Thus $\{a\}$ must be closed in X and $(X, \mathcal{T})$ is T_1. Conversely, suppose $(X, \mathcal{T})$ is a T_1-space. To show $X \in \mathcal{S}_\rho$ we need $\mathbb{T}_X = \mathcal{T}$. Let U be a proper open subset of X and let $a \in U$. Take $(Y, \mathcal{T}_Y)$ to be the subspace of $(X, \mathcal{T})$ with $Y = (X - U) \cup \{a\}$. Now $\{a\} = Y \cap U = Y \cap \{a\}$ and so $\{a\}$ is open and closed in Y. But then $\{a\} \in \mathbb{T}_Y = \{Y \cap W \mid W \in \mathbb{T}_X\}$; i.e., $\{a\} = Y \cap W_a$ for some $W_a \in \mathbb{T}_X$. Thus $a \in W_a \subseteq U$ and the equality $\mathbb{T}_X = \mathcal{T}$ follows. For any space $(X, \mathcal{T})$, it is clear that if $(X, \mathbb{T}_X)$ is a T_1-space, then so is $(X, \mathcal{T})$. But here the converse also holds: Suppose $(X, \mathcal{T})$ is a T_1-space. Then $(X, \mathcal{T}) \in \mathcal{S}_\rho$ and from $\rho_X = \iota_X$ we get $(X, \mathbb{T}_X) = (X, \mathcal{T})$ which is T_1. So, if $(X, \mathcal{T})$ is not a T_1-space, then neither is $(X, \mathbb{T}_X)$ which forces $\mathbb{T}_X = \mathcal{I}_X$ since $(X, \mathbb{T}_X) \in \mathcal{S}_\rho$. We may thus conclude that for any $(X, \mathcal{T})$,

$$\mathcal{T}_X = \begin{cases} \mathcal{T} \text{ if } (X, \mathcal{T}) \text{ is a } T_1\text{-space} \\ \mathcal{I}_X \text{ otherwise} \end{cases}$$

which gives (e). □

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