East-West J. of Mathematics: Vol. 13, No 1(2011) pp. 61-66

SOME REMARKS ON R_e-MULTIPLICATION MODULES OVER FIRST STRONGLY GRADED RINGS

Rashid Abu-Dawwas* and **Mashhoor Refai**†

**Department of Mathematics*
Yarmouk University, Irbid, Jordan
e-mail: rrashid@yu.edu.jo

†*President Office*
Princess Sumaya University for Technology, Amman, Jordan
e-mail: m.refai@psut.edu.jo

Abstract

Let G be a group, R be a first strongly G - graded ring and M be a G - graded R - module such that M_e is a nonzero R_e - torsion free R_e - multiplication module. In this paper, we introduce several results concerning the R_e - modules $\{M_g : g \in \text{supp}(R, G)\}$.

Introduction

Let R be a commutative ring with unity 1 and M be a left R - module. If N is an R - submodule of M, then the ideal $\{r \in R : rM \subseteq N\}$ of R will be denoted by $(N :_R M)$. A left R - module M is called multiplication R - module if for every R - submodule N of M, $N = IM$ for some ideal I of R. In this case, we can take $I = (N :_R M)$. Let P be a maximal ideal of R. Then we define $T_P(M) = \{m \in M : (1-p)m = 0 \text{ for some } p \in P\}$. Clearly, $T_P(M)$ is an R - submodule of M. We say that M is P - cyclic if there exist $p \in P$ and $m \in M$ such that $(1-p)M \subseteq Rm$. Moreover, M is multiplication R - module if and only if $M = T_P(M)$ or M is P - cyclic for every maximal ideal P of R. For more details, one can look in $[1, 2, 3]$. A commutative ring R with unity is a G - graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. We

---

Key words: Graded rings, Graded modules, Multiplication modules.
2000 AMS Mathematics Subject Classification: 13A02

denote this by (R,G) and we consider $\text{supp}(R,G) = \{g \in G : R_g \neq 0\}$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, R_e is a subring of R and $1 \in R_e$. Let M be a left R - module. Then M is a G - graded R - module (in short, M is gr - R - module) if there exist additive subgroups M_g of M indexed by the elements $g \in G$ such that $M = \bigoplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. We denote this by (M,G) and we consider $\text{supp}(M,G) = \{g \in G : M_g \neq 0\}$. The elements of M_g are called homogeneous of degree g. If $x \in M$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in M_g. Clearly, M_g is R_e - submodule of M for all $g \in G$. For more details, one can look in [4, 5].

1 R_e-Multiplication Modules over First Strongly Graded Rings

In this section, we introduce several results concerning the R_e - modules $\{M_g : g \in \text{supp}(R,G)\}$ under the assumption that (R,G) is first strong and M_e is a nonzero R_e - torsion free R_e - multiplication module.

Definition 1.1. [5] (R,G) is called first strong if $1 \in R_g R_{g^{-1}}$ for all $g \in$supp(R,G).

Proposition 1.2. [5] *Let R be a G - graded ring. Then (R,G) is first strong if and only if supp(R,G) is a subgroup of G and $R_g R_h = R_{gh}$ for all $g, h \in$supp(R,G).*

Definition 1.3. [5] *(M,G) is called first strong if supp(R,G) is a subgroup of G and $R_g M_h = M_{gh}$ for all $g \in$supp(R,G), $h \in G$.*

Proposition 1.4. [5] *Let R be a G - graded ring. Then (R,G) is first strong if and only if every gr - R - module is first strongly graded.*

Now, we introduce our results.

Proposition 1.5. *Let R be a first strongly G - graded ring and M be a gr - R - module. Then M_e is R_e - multiplication if and only if M_g is R_e - multiplication for all $g \in$ supp(R,G).*

Proof Suppose M_e is R_e - multiplication. Let $g \in$ supp (R,G) and let N be an R_e - submodule of M_g. Then $R_{g^{-1}}N$ is an R_e - submodule of M_e. Since M_e is R_e - multiplication, $R_{g^{-1}}N = IM_e$ for some ideal I of R_e and then $N = R_e N = R_g R_{g^{-1}} N = R_g I M_e = I R_g M_e = I M_g$. Hence M_g is R_e-multiplication. The converse is obvious. □

Lemma 1.6. *Let R be a first strongly G - graded ring and M be a gr - R - module such that M_e is a nonzero R_e - torsion free R_e - multiplication module. If A is a proper ideal of R_e, then $AM_g \neq M_g$ for all $g \in$ supp(R,G).*

Proof Let $g \in \text{supp}(R,G)$. Suppose $AM_g = M_g$. Then for $m \in M_e$, $R_e m = IM_e$ for some ideal I of R_e since M_e is R_e - multiplication. Now, $R_e m = IM_e = IR_{g^{-1}}M_g = IR_{g^{-1}}AM_g = IAR_{g^{-1}}M_g = IAM_e = AIM_e = AR_e m = R_e Am = Am$. So, $m = am$ for some $a \in A$ and then $(1-a)m = 0$ and since M_e is R_e - torsion free, $1 = a \in A$ a contradiction. Hence, $AM_g \neq M_g$. $\square$

Proposition 1.7. *Let R be a first strongly G - graded ring and M be a gr - R - module such that M_e is a nonzero R_e - torsion free R_e - multiplication module. If A and B are two ideals of R_e and $g \in supp(R,G)$ such that $AM_g \subseteq BM_g$, then $A \subseteq B$.*

Proof Let $a \in A$ and let $X = \{r \in R_e : ra \in B\}$. Then X is an ideal of R_e. It is enough to prove that $X = R_e$. Suppose $X \neq R_e$. Then there exists a maximal ideal P of R_e such that $X \subseteq P$. By Lemma 1.6, $PM_e \neq M_e$ and then there exists $m \in M_e$ such that $m \notin PM_e$. Since M_e is R_e - multiplication, $R_e m = IM_e$ for some ideal I of R_e. If $I \subseteq P$, then $m \in R_e m = IM_e \subseteq PM_e$ a contradiction. So, $I \nsubseteq P$ and then $I + P = R_e$ and hence $1 = \alpha + p$ for some $\alpha \in I$, $p \in P$. Now, $(1-p)am \in (1-p)AM_e = (1-p)AR_{g^{-1}}M_g = (1-p)R_{g^{-1}}AM_g \subseteq (1-p)R_{g^{-1}}BM_g = (1-p)BR_{g^{-1}}M_g = (1-p)BM_e = B(1-p)M_e = B\alpha M_e \subseteq BIM_e = BR_e m = R_e Bm = Bm$. So, $(1-p)am = bm$ for some $b \in B$ and then $((1-p)a - b)m = 0$. Since M_e is R_e - torsion free, $(1-p)a = b \in B$ and then $1 - p \in X \subseteq P$ a contradiction. Hence, $X = R_e$ and then $a \in B$. $\square$

Proposition 1.8. *Let R be a first strongly G - graded ring and M be a gr - R - module such that M_e is a nonzero R_e - torsion free R_e - multiplication module. If R_e is Noetherian, then M_g is R_e - Noetherian for all $g \in supp(R,G)$.*

Proof Let $g \in \text{supp}(R,G)$ and let $N_1 \subseteq N_2 \subseteq \cdots$ be a chain of R_e - submodules of M_g. By Proposition 1.5, M_g is R_e - multiplication and then $N_k = I_k M_e$ for some ideal I_k of R_e, $k = 1, 2, \cdots$. So, $I_1 M_e \subseteq I_2 M_e \subseteq \cdots$ and then by Proposition 1.7, $I_1 \subseteq I_2 \subseteq \cdots$. Since R_e is Noetherian, there exists $m \in \mathbb{N}$ such that $I_n = I_m$ for all $n \geq m$ and then $I_n M_e = I_m M_e$ for all $n \geq m$ and hence $N_n = N_m$ for all $n \geq m$. Therefore, M_g is R_e-Noetherian. $\square$

Proposition 1.9. *Let R be a first strongly G - graded ring and M be a gr - R - module such that M_e is a nonzero R_e - torsion free R_e - multiplication module. If M_g is R_e - Noetherian for some $g \in supp(R,G)$, then R_e is Noetherian.*

Proof Suppose $g \in \text{supp}(R,G)$ such that M_g is R_e - Noetherian. Let $I_1 \subseteq I_2 \subseteq \cdots$ be a chain of ideals of R_e. Then $I_1 M_g \subseteq I_2 M_g \subseteq \cdots$ is a chain of

R_e - submodules of M_g. Since M_g is R_e - Noetherian, there exists $m \in \mathbb{N}$ such that $I_n M_g = I_m M_g$ for all $n \geq m$. By Proposition 1.7, $I_n = I_m$ for all $n \geq m$. Hence, R_e is Noetherian. □

Proposition 1.10. *Let R be a first strongly G - graded ring and M be a gr - R - module such that M_e is a nonzero R_e - torsion free R_e - multiplication module. Then R_e is Noetherian if and only if M_g is R_e - Noetherian for all $g \in$ supp(R, G).*

Proposition 1.11. *Let R be a first strongly G - graded ring and M be a gr - R - module such that M_e is a nonzero R_e - torsion free R_e - multiplication module. Then*
$(IM_g :_{R_e} M_g) = I$ for every ideal I of R_e and $g \in$ supp(R, G).

Proof Let I be an ideal of R_e and $g \in \text{supp}(R, G)$. Clearly, $I \subseteq (IM_g :_{R_e} M_g)$. Let $r \in (IM_g :_{R_e} M_g)$. Then $rM_g \subseteq IM_g$ and then $\langle r \rangle$ is an ideal of R_e with $\langle r \rangle M_g \subseteq IM_g$. By Proposition 1.7, $\langle r \rangle \subseteq I$ and then $r \in I$, i.e., $(IM_g :_{R_e} M_g) \subseteq I$ and hence
$(IM_g :_{R_e} M_g) = I$. □

A nonzero R - module M is called faithful if $rM \neq 0$ for all nonzero $r \in R$.

Proposition 1.12. *Let R be a first strongly G - graded ring and M be a gr - R - module such that M_e is a nonzero R_e - torsion free R_e - multiplication module. Suppose for $g \in$ supp(R, G), M_g is R_e - faithful. Then M_g is finitely generated.*

Proof Suppose M_g is not finitely generated. Then by [1, Theorem 3.1], there exists a maximal ideal A of R_e such that $M_g = AM_g$. By Proposition 1.5, M_g is R_e - multiplication and then by [1, Theorem 2.5], there exists a maximal R_e - submodule N of M_g such that $N = BM_g$ for some maximal ideal B of R_e. Now, $N = BM_g = BAM_g = ABM_g = AN \subseteq AM_g$. By Proposition 1.7, $B = AB \subseteq A$. Since A and B are maximal, $A = B$ and then $M_g = AM_g = BM_g = N$ a contradiction. Hence, M_g is finitely generated. □

For $g \in G$, define $Z(M_g) = \{r \in R_e : rm = 0 \text{ for some nonzero } m \in M_g\}$.

Proposition 1.13. *Let R be a G - graded ring and M be a gr - R - module such that M_e is R_e - faithful multiplication module. Suppose for $g \in$ supp(R, G), M_g is R_e - faithful. Then $Z(M_g) = Z(R_e)$.*

Proof Let $r \in Z(R_e)$. Then there exists a nonzero $s \in R_e$ such that $rs = 0$. Since M_g is R_e - faithful, $sM_g \neq 0$ and $r(sM_g) = 0$ and then $r \in Z(M_g)$, i.e., $Z(R_e) \subseteq Z(M_g)$. Let $t \in Z(M_g)$. Then there exists a nonzero $m \in M_g$ such that $tm = 0$. Now, $R_{g^{-1}}m$ is an R_e - submodule of M_e and M_e is R_e - multiplication, then $R_{g^{-1}}m = IM_e$ for some ideal I of R_e. But $tIM_e = tR_{g^{-1}}m = R_{g^{-1}}tm = 0$, so $tI = 0$ since M_e is R_e - faithful. Since $IM_e \neq 0$, we have $I \neq 0$ and then $t \in Z(R_e)$, i.e., $Z(M_g) \subseteq Z(R_e)$. Hence $Z(M_g) = Z(R_e)$. $\square$

An R - submodule N of M is called prime R - submodule if whenever $r \in R$ and $m \in M$ such that $rm \in N$, then either $rM \subseteq N$ or $m \in N$.

Proposition 1.14. *Let R be a first strongly G - graded ring and M be a gr - R - module such that M_e is a nonzero R_e - torsion free R_e - multiplication module. If P is a prime ideal of R_e, then PM_g is a prime R_e - submodule of M_g for all $g \in supp(R, G)$.*

Proof Firstly, we prove that PM_e is a prime R_e - submodule of M_e. Let $a \in R_e$ and $x \in M_e$ such that $ax \in PM_e$. If $a \in P$, then $aM_e \subseteq PM_e$ and then it is done. Suppose $a \notin P$. let $X = \{r \in R_e : rx \in PM_e\}$. Then X is an ideal of R_e. Suppose $X \neq R_e$. Then there exists a maximal ideal Q of R_e such that $X \subseteq Q$. Since M_e is R_e - multiplication, either $T_Q(M_e) = M_e$ or M_e is Q - cyclic. If $T_Q(M_e) = M_e$, then there exists $q \in Q$ with $(1-q)x = 0$ and then $1-q \in X \subseteq Q$ a contradiction. So, M_e is Q - cyclic that is there exist $m \in M_e$ and $q \in Q$ with $(1-q)M_e \subseteq R_e m$. Hence
$(1-q)PM_e = P(1-q)M_e \subseteq PR_e m = R_e Pm \subseteq Pm$. Now, $(1-q)x = sm$ for some $s \in R_e$ and $(1-q)ax = pm$ for some $p \in P$. Thus $asm = a(1-q)x = (1-q)ax = pm$ and then $(as-p)m = 0$. Since M_e is R_e torsion free, $as = p$. Then $(1-q)as = (1-q)p \in P$. Since P is prime, either $(1-q) \in P$ or $as \in P$. But $P \subseteq X \subseteq Q$, so $(1-q) \notin P$ and then $as \in P$. Since P is prime and $p \notin P$, $s \in P$. Thus $(1-q)x = sm \in Pm \subseteq PM_e$ and then $(1-q) \in X \subseteq Q$ a contradiction. Hence $X = R_e$ and then $x \in PM_e$. So, PM_e is a prime R_e - submodule of M_e. Let $g \in \text{supp}(R, G)$, $r \in R_e$ and $m \in M_g$ such that $rm \in PM_g$. Then $rR_{g^{-1}}m = R_{g^{-1}}rm \in R_{g^{-1}}PM_g = PR_{g^{-1}}M_g = PM_e$. Since PM_e is prime, either $rM_e \subseteq PM_e$ or $R_{g^{-1}}m \in PM_e$. If $rM_e \subseteq PM_e$, then
$rM_g = rR_gM_e = R_g rM_e \subseteq R_g PM_e = PR_gM_e = PM_g$. If $R_{g^{-1}}m \in PM_e$, then
$m \in R_e m = R_g R_{g^{-1}}m \subseteq R_g PM_e = PR_gM_e = PM_g$. Hence PM_g is a prime R_e - submodule of M_g. $\square$

References

[1] Z. Abd El-Bast and P. F. Smith, *Multiplication Modules,* Comm. Algebra, **16** (4) (1988), 755 – 779.

[2] Z. Abd El-Bast and P. F. Smith, *Multiplication Modules and Theorems of Mori and Mott,* Comm. Algebra, **16** (4) (1988), 781 – 796.

[3] D. S. Lee and H. B. Lee, *Some Remarks on Faithful Multiplication Modules,* Journal of the Chungcheong Mathematical Society, **6** (1993), 131 – 137.

[4] C. Nastasescu and F. Van Oystaeyen, "Graded Ring Theory", Mathematical Library 28, North Holland, Amesterdam, 1982.

[5] M. Refai, *Various types of strongly graded rings,* Abhath Al-Yarmouk Journal (Pure Sciences and Engineering Series), **4** (2) (1995), 9 – 19.