

A NOTE ON WEAK LIFTING MODULES

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Abstract

It is proved that the notions weak lifting and weak \oplus -supplemented are same notions.

1 Introduction

Throughout the paper all rings have identity and all modules are unitary. In [3] and [4], authors defined weak lifting modules: A module M is said to be a *weak lifting*, if for each semisimple submodule N of M , there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $M_2 \cap N \ll M_2$. In both papers, they have posed many results but some of these results are actually redundant.

In [3], it is defined that a module M is called *weak \oplus -supplemented*, if for each semisimple submodule N of M , there exists a direct summand K of M such that $M = N + K$ and $N \cap K \ll K$.

Authors considered these two definitions in different manners, however these notions are the same notions.

Lemma 1.1. *Let N and L be two submodules of M so that $M = N + L$, also let N be semisimple, then $M = X \oplus L$ for some semisimple submodule X of N .*

Proof Obviously, $N \cap L$ is a direct summand of N , because N is semisimple. So, $N = (N \cap L) \oplus A$ for some submodule A of N . Then $M = [(N \cap L) \oplus A] + L = A + L$ but $A \cap L = (N \cap L) \cap A = 0$. So, $M = A \oplus L$. \square

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Theorem 1.2. *For an R -module M , the following statements are equivalent:*

1. *Every semisimple submodule of M has a supplement.*
2. *M is weak \oplus -supplemented.*
3. *M is weak lifting.*

Proof (1 \Rightarrow 2) Let N be a semisimple submodule of M . By assumption, N has a supplement L in M such that $M = N + L$ and $N \cap L \ll L$. By Lemma 1.1, L is a direct summand of M .

(2 \Rightarrow 3) Let M be weak- \oplus -supplemented module and N be a semisimple submodule of M . Then N has a direct summand supplement L in M , i.e., $M = N + L = L \oplus L'$ and $N \cap L \ll L$ for some submodule L' of M . By Lemma 1.1, $M = X \oplus L$ where $X \leq N$ and we already have $L \cap N \ll L$. Hence M is weak lifting.

(3 \Rightarrow 1) Let N be a semisimple submodule of M . By assumption, there exists a decomposition K of M such that $M = K \oplus K'$, $K \leq N$ and $N \cap K' \ll K'$ for some submodule K' of M . Then $M = N + K'$. \square

The following result is proved in ([3], Lemma 2.4) by help of an extra property, namely (D_3) property, however (D_3) property is not necessary. We shall give the definition of (D_3) property for completeness:

A module M is said to have (D_3) property, if whenever M_1 and M_2 are direct summands in M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M .

Proposition 1.3. *Any direct summand of a weak lifting module is weak lifting.*

Proof Let M be a weak lifting module and N be a direct summand of M so that $M = N \oplus K$ for some submodule K of M . Let S be a semisimple submodule of N . Since $S \subseteq M$, then $M = S + T$ and $S \cap T \ll T$ for some submodule T of M . By the modular law, $N = S + (N \cap T)$. Then by Lemma 1.1, $M = S' \oplus T$ for some $S' \subseteq S$. By the modular law again, we get $N = S' \oplus (N \cap T)$, that is, $N \cap T$ is a direct summand of N . We claim that $T = (N \cap T) \oplus K$: Since $S' \subseteq N$ and $M = N \oplus K = S' \oplus T$, then K can be considered as a submodule of T , then $(N \cap T) \oplus K \subseteq T$. Conversely, if $t \in T \subseteq M = N \oplus K$, then $t = n + k$ for some $n \in N$ and $k \in K$, since $n = t - k \in T$, then $n \in N \cap T$. Hence $N \cap (S \cap T) \subseteq S \cap T \ll T = (N \cap T) \oplus K$. Since $S \cap T \subseteq N \cap T$, then by ([5], 19.3.(5)), $S \cap (N \cap T) \ll N \cap T$. \square

In ([3], Theorem 2.7), authors prove that finite direct sum of w - \oplus -supplemented modules is w - \oplus -supplemented. And by Theorem 1.2, we can say that finite direct sum of weak lifting modules are weak lifting. By considering with

Proposition 1.3, in [4], Proposition 2.1, Proposition 2.4 and Theorem 2.2 are unnecessary.

In [3], Proposition 2.3, Lemma 2.4, Theorem 2.8 are redundant and (D_3) property is not necessary in Proposition 2.6.

The following Lemma shows that, we should be careful about speaking direct summandness of a semisimple submodule.

Lemma 1.4. *Let M be a module with $\text{Rad}M \trianglelefteq M$. Then there is no nonzero semisimple direct summand of M .*

Proof Let N be a semisimple direct summand of M , then $M = N \oplus K$ for some submodule K of M . Then by ([5], 21.6) $\text{Rad}M = \text{Rad}K$ and since $\text{Rad}M \trianglelefteq M$, then $K \trianglelefteq M$ by ([1], Proposition 5.16). If $N \neq 0$, $K = M$, impossible. Therefore $N = 0$. \square

As a result of above Lemma, we may say, for instance, over a domain R , divisible R -modules have no nonzero semisimple direct summands.

Therefore, in [4], Proposition 1.7, Theorem 1.8, should have been written more carefully.

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