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# A NOTE ON WEAK LIFTING MODULES

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**Abstract**

It is proved that the notions weak lifting and weak $\oplus$-supplemented are same notions.

## 1 Introduction

Throughout the paper all rings have identity and all modules are unitary. In [3] and [4], authors defined weak lifting modules: A module $M$ is said to be a *weak lifting*, if for each semisimple submodule $N$ of $M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $M_2 \cap N \ll M_2$. In both papers, they have posed many results but some of these results are actually redundant.

In [3], it is defined that a module $M$ is called *weak $\oplus$-supplemented*, if for each semisimple submodule $N$ of $M$, there exists a direct summand $K$ of $M$ such that $M = N + K$ and $N \cap K \ll K$.

Authors considered these two definitions in different manners, however these notions are the same notions.

**Lemma 1.1.** *Let $N$ and $L$ be two submodules of $M$ so that $M = N + L$, also let $N$ be semisimple, then $M = X \oplus L$ for some semisimple submodule $X$ of $N$.*

**Proof** Obviously, $N \cap L$ is a direct summand of $N$, because $N$ is semisimple. So, $N = (N \cap L) \oplus A$ for some submodule $A$ of $N$. Then $M = [(N \cap L) \oplus A] + L = A + L$ but $A \cap L = (N \cap L) \cap A = 0$. So, $M = A \oplus L$. □

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**Theorem 1.2.** *For an $R$-module $M$, the following statements are equivalent:*

1. *Every semisimple submodule of $M$ has a supplement.*
2. *$M$ is weak $\oplus$-supplemented.*
3. *$M$ is weak lifting.*

**Proof** ($1 \Rightarrow 2$) Let $N$ be a semisimple submodule of $M$. By assumption, $N$ has a supplement $L$ in $M$ such that $M = N + L$ and $N \cap L \ll L$. By Lemma 1.1, $L$ is a direct summand of $M$.

($2 \Rightarrow 3$) Let $M$ be weak-$\oplus$-supplemented module and $N$ be a semisimple submodule of $M$. Then $N$ has a direct summand supplement $L$ in $M$, i.e., $M = N + L = L \oplus L'$ and $N \cap L \ll L$ for some submodule $L'$ of $M$. By Lemma 1.1, $M = X \oplus L$ where $X \leq N$ and we already have $L \cap N << L$. Hence $M$ is weak lifting.

($3 \Rightarrow 1$) Let $N$ be a semisimple submodule of $M$. By assumption, there exists a decomposition $K$ of $M$ such that $M = K \oplus K'$, $K \leq N$ and $N \cap K' \ll K'$ for some submodule $K'$ of $M$. Then $M = N + K'$. □

The following result is proved in ([3], Lemma 2.4) by help of an extra property, namely $(D_3)$ property, however $(D_3)$ property is not necessary. We shall give the definition of $(D_3)$ property for completeness:

A module $M$ is said to have $(D_3)$ property, if whenever $M_1$ and $M_2$ are direct summands in $M$ with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of $M$.

**Proposition 1.3.** *Any direct summand of a weak lifting module is weak lifting.*

**Proof** Let $M$ be a weak lifting module and $N$ be a direct summand of $M$ so that $M = N \oplus K$ for some submodule $K$ of $M$. Let $S$ be a semisimple submodule of $N$. Since $S \subseteq M$, then $M = S + T$ and $S \cap T \ll T$ for some submodule $T$ of $M$. By the modular law, $N = S + (N \cap T)$. Then by Lemma 1.1, $M = S' \oplus T$ for some $S' \subseteq S$. By the modular law again, we get $N = S' \oplus (N \cap T)$, that is, $N \cap T$ is a direct summand of $N$. We claim that $T = (N \cap T) \oplus K$: Since $S' \subseteq N$ and $M = N \oplus K = S' \oplus T$, then $K$ can be considered as a submodule of $T$, then $(N \cap T) \oplus K \subseteq T$. Conversely, if $t \in T \subseteq M = N \oplus K$, then $t = n + k$ for some $n \in N$ and $k \in K$, since $n = t - k \in T$, then $n \in N \cap T$. Hence $N \cap (S \cap T) \subseteq S \cap T \ll T = (N \cap T) \oplus K$. Since $S \cap T \subseteq N \cap T$, then by ([5], 19.3.(5)), $S \cap (N \cap T) \ll N \cap T$. □

In ([3], Theorem 2.7), authors prove that finite direct sum of $w$-$\oplus$-supplemented modules is $w$-$\oplus$ - supplemented. And by Theorem 1.2, we can say that finite direct sum of weak lifting modules are weak lifting. By considering with

Proposition 1.3, in [4], Proposition 2.1, Proposition 2.4 and Theorem 2.2 are unnecessary.

In [3], Proposition 2.3, Lemma 2.4, Theorem 2.8 are redundant and $(D_3)$ property is not necessary in Proposition 2.6.

The following Lemma shows that, we should be careful about speaking direct summandness of a semisimple submodule.

**Lemma 1.4.** *Let $M$ be a module with $RadM \trianglelefteq M$. Then there is no nonzero semisimple direct summand of $M$.*

**Proof** Let $N$ be a semisimple direct summand of $M$, then $M = N \oplus K$ for some submodule $K$ of $M$. Then by ([5], 21.6) $RadM = RadK$ and since $RadM \trianglelefteq M$, then $K \trianglelefteq M$ by ([1], Proposition 5.16). If $N \neq 0$, $K = M$, impossible. Therefore $N = 0$. □

As a result of above Lemma, we may say, for instance, over a domain $R$, divisible R-modules have no nonzero semisimple direct summands.

Therefore, in [4], Proposition 1.7, Theorem 1.8, should have been written more carefully.

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