East-West J. of Mathematics: Vol. 13, No 1(2011) pp. 51-56

POSET PROPERTIES OF INFINITE ANTIMATROIDS

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Abstract

This paper extends the notion of antimatroid from finite case to infinite one, followed by dealing with poset properties of infinite antimatroids. All the results here imply that poset theory will be a key source for studying on infinite antimatroids.

1 Introduction and Preliminaries

With the development of finite matroids, infinite matroids are getting advantages step by step. Conversely, infinite matroids accelerate the improvement of finite matroids, and further, matroid theory. As the "opposite" to matroids, antimatroids have been developed fruitfully though its corresponding results are not as many as that of finite matroids. Generally, people think a mathematical antimatroid-like construction as an infinite matroid if it is defined on an infinite set and generalizes the definition of the finite antimatroids. On the other hand, [3,4] provides two equivalent ways to define a finite antimatroid: one is from set theory and another is from convex geometry theory. [6] generalizes the definition of finite antimatroid from convex geometry theory. [5] discusses some characterizations of an infinite antimatroid using the definition as [6]. According to my knowledge, it has not been found the extension of finite antimatroid from set theory. Here, this paper will first extend the definition of finite antimatroid from set theory to infinite case. This is a new way to define an infinite antimatroid. Afterwards, it will discuss with some corresponding *poset properties* of an infinite antimatroid, that is, completely properties and

This research is supported by Science and Technology Item of Baoding (11ZG005)
Key words: infinite antimatroid; poset; feasible set; lattice.
2000 AMS Mathematics Subject Classification: 05B35; 06A06

invariants determined by the abstract poset for an infinite antimatroid $(E,\mathcal{F})$ and not requiring explicit knowledge of the set system $\mathcal{F}$. For two infinite antimatroid defined in [6] and this paper, the relations between them will be studied in the future.

First of all, it starts by reviewing and presenting the knowledge what we need later on. In the following, we will work over a ground–possibly infinite–set E. The set of all subsets of E will be denoted by 2^E. A *set system* over E is a non-empty family contained in 2^E. Let $\mathcal{F} \subseteq 2^E$ be a set system. We say that a *basis of a subset* $A \subseteq E$ of $(E,\mathcal{F})$ is a(inclusion-wise) maximal element in $\mathcal{F}$ of A; a *loop* is an element of E that is contained in no basis; $\mathcal{F}$ is *normal* if it does not contain loops. In a poset P, "a covers b" will be in notation $b \prec a$.

Definition 1 [2,p.63&2,p.17] A *Boolean lattice* is a complemented distributive lattice.

Lemma 1 [2,p.62] In a bounded distributive lattice: an element can have only one complement; besides, if a has a complement, then it also has a relative complement in any interval containing it.

All the other knowledge about poset and lattice are referred to [1,2].

The following is to define an infinite antimatroid by set theory and the other two definitions needed in later on.

Definition 2 (1) Let $\mathcal{F} \subseteq 2^E$ be a set system with $\emptyset \in \mathcal{F}$. An *infinite antimatroid* is a pair $(E,\mathcal{F})$ with $\mathcal{F}$(called *feasible sets*) satisfying the following conditions.
(I) $\mathcal{F}$ is normal;
(II) $\mathcal{F}$ is closed under union, i.e. $X_\alpha \in \mathcal{F}$ $(\alpha \in \mathcal{A})$ implies $\bigcup\limits_{\alpha\in\mathcal{A}} X_\alpha \in \mathcal{F}$;
(III) For $X, Y \in \mathcal{F}, Y \subset X$, there is $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{F}$.

(2) A lattice L with $1 \in L$(i.e. L has a maximum element 1) is said to be *join-distributive* if for every $x \in L \setminus \{1\}$, the interval $[x, j(x)]$ is Boolean, where $j(x)$ is the join of all elements in L that cover x.

(3) A lattice L is a *pre-semimodular* if $x \wedge y \prec x, y$ implies $x, y \prec x \vee y$.

Remark 1 (1) By [3,p.77, Theorem 1.1& 4,p.271, Proposition 8.2.7], one has that the notion of infinite antimatroid presented in definition 2 is the extension of the concept of *finite antimatroid* (cf.[3,p.22& 4,p.291,definition 8.2.6]).

Definition 2(2) is clearly the extension of *join-distributive for finite case* (cf.[4,p.323&3,p.8]).

(2) We see that a pre-semimodular lattice L is not always a *semimodular*

lattice(cf.[2,p.225]), but if L has finite length, then by [2,p.226, Theorem 2], L is semimodular. In other words, the definition of pre-semimodular conforms with the laws.

(3) For an infinite antiamtroid $(E, \mathcal{F})$, (I) and (II) in definition 2 together follows that $(E, \mathcal{F})$ has a unique basis E and A has a unique basis for every $A \subseteq E$.

Furthermore, the feasible sets of $(E, \mathcal{F})$ ordered by inclusion form a lattice, with lattice operations: $X \vee Y = X \cup Y$, and $X \wedge Y$ is the unique basis of $X \cap Y$.

Lemma 2 Every join-distributive lattice is pre-semimodular.

Proof Let L be a join-distributive lattice and $x \wedge y \prec x, y$. Since $[x \wedge y, x \vee y] \subseteq [x \wedge y, j(x \wedge y)]$. If $x, y \prec x \vee y$ does not hold, then there exists $z \in [x \wedge y, x \vee y]$ satisfying $x < z \prec x \vee y$, and so y has at least two complements x and z in $[x \wedge y, x \vee y]$, this is a contradiction to lemma 1 because $[x \wedge y, x \vee y]$ is an interval in bounded distributive lattice $[x \wedge y, j(x \vee y)]$. (Or say, and so $\{x \wedge y, x, y, z, x \vee y\}$ is a sublattice of $[x \wedge y, j(x \wedge y)]$ and is a pentagon, a contradiction to [2, p.80, Theorem 1] because $[x \wedge y, j(x \wedge y)]$ is a distributive.)

Hence $x, y \prec x \vee y$ is correct.

2 Poset Properties

In this section, the infinite antimatroid lattice shown in remark 1(3) is characterized in purely lattice-theoretical terms. In addition, we will see that poset infinite antimatroids generate all other infinite antimatroids as images.

Theorem 1 Let $\mathcal{F} \subseteq 2^E$ be a set system. Then the following conditions are equivalent.

(1) $(E, \mathcal{F})$ is an infinite antimatroid.

(2) $(\mathcal{F}, \subseteq)$ is a join-distributive lattice with $\emptyset, E \in \mathcal{F}$ and $Y \prec X \Rightarrow X = Y \cup x$ for some $x \in E \setminus Y$.

(3) $(\mathcal{F}, \subseteq)$ is a pre-semimodular lattice with $\emptyset, E \in \mathcal{F}$ and $Y \prec X \Rightarrow X = Y \cup x$ for some $x \in E \setminus Y$.

Proof (1)$\Rightarrow$(2). The sets that cover $X \in \mathcal{F}$ in an infinite antimatroid lattice $(\mathcal{F}, \subseteq)$ are of the form $X \cup x_i$ for some $x_i \in E \setminus X, (i \in \mathcal{I})$. Since $\mathcal{F}$ is closed under union, $X \cup (\bigcup\limits_{i \in \mathcal{I}} x_i) \in \mathcal{F}$. For any $Y \in [X, j(X) = X \cup (\bigcup\limits_{i \in \mathcal{I}} x_i)]$, one has $Y = X \cup (\bigcup\limits_{i \in \mathcal{I}_Y} x_i)$ for some $\mathcal{I}_Y \subseteq \mathcal{I}$. Hence, $(\mathcal{F}, \subseteq)$ is join-distributive.

The (III) pledges the hold of $Y \prec X \Rightarrow X = Y \cup x$ for some $x \in E \setminus Y$.

(2)⇒(3). Routine verification from lemma 2.

(3)⇒(1). $E \in \mathcal{F}$ implies that $\mathcal{F}$ is normal.

Let $X_\alpha \in \mathcal{F}$ $(\alpha \in \mathcal{A})$. Since $X_\alpha \subseteq \bigcup_{\alpha\in\mathcal{A}} X_\alpha$, one has $\bigvee_{\alpha\in\mathcal{A}} X_\alpha \subseteq \bigcup_{\alpha\in\mathcal{A}} X_\alpha$, besides, $X_\alpha \subseteq \bigvee_{\alpha\in\mathcal{A}} X_\alpha$ is evident, and so $\bigcup_{\alpha\in\mathcal{A}} X_\alpha \subseteq \bigvee_{\alpha\in\mathcal{A}} X_\alpha$. Therefore, $\bigcup_{\alpha\in\mathcal{A}} X_\alpha = \bigvee_{\alpha\in\mathcal{A}} X_\alpha \in \mathcal{F}$, that is, $\mathcal{F}$ is closed under union.

Let $X, Y \in \mathcal{F}$ and $Y \subset X$. When $Y \prec X$. By (3), (III) holds. When $Y \not\prec X$. There exists $Z \in \mathcal{F}, Y \prec Z \subset X$ because $(\mathcal{F}, \subseteq)$ is a lattice. By (3), there is $z \in Z \setminus Y \subset X \setminus Y$ satisfying $Y \cup z \in \mathcal{F}$. That is , (III) holds.

Summing up, $(E, \mathcal{F})$ is an infinite antimatroid.

Theorem 2 Let P be a poset such that for any two *ideals* X, Y in P(i.e. $A \subseteq E$ is an ideal in P if $y \in A, x \leq y$ implies $x \in A$), if $Y \subset X$, then there is a minimal element $x \in X \setminus Y$, . Let $f : P \to E$ be a function from a poset P to a set E, and let $\mathcal{H} = \{f(A) \subseteq E : A \text{ is an } \textit{ideal} \text{ in } P\}$. Then $(E, \mathcal{H})$ is an infinite antimatroid(called a *poset infinite antimatroid*).

To prove theorem 2, we need the following preparations.

Lemma 3 Let $(E, \mathcal{F})$ be an infinite antimatroid. $\tau : 2^E \to 2^E$ is defined as $\tau(A) = \cap\{X : A \subseteq X, E \setminus X \in \mathcal{F}\}$ for $A \in 2^E$. Then τ satisfies
(i) $\tau(\emptyset) = \emptyset$;
(ii) $A \subseteq \tau(A)$;
(iii) $A \subseteq B \Rightarrow \tau(A) \subseteq \tau(B)$;
(iv) $\tau(\tau(A)) = \tau(A)$;
(AE) if $y, z \notin \tau(X)$ and $z \in \tau(X \cup y)$, then $y \notin \tau(X \cup z)$.

Proof Because all of the proof are easy, we give the sketch of the proof. Since $\emptyset, E \in \mathcal{F}$ and (I)-(II), it is not difficult to have the hold of (i)-(iv), we only have to prove the (AE).

Let B be the (unique)basis of $E \setminus X$ and A the basis of $E \setminus (X \cup y)$. Then $A \subseteq B \setminus \{y, z\}$ where $y, z \notin \tau(X)$ and $z \in \tau(X \cup y)$. Hence, one has $y, z \in B$ and $z \notin A$. In light of (III), we can augment A to some set $A \cup x \in \mathcal{F}$, where $x \in B$. Since A is a basis of $E \setminus (X \cup y)$, we must have $x = y$, i.e. $A \cup y$ is a feasible subset of $E \setminus (X \cup z)$, and hence $y \notin \tau(X \cup z)$.

Lemma 4 Let τ be defined as in lemma 3. Then

(1) $(E, \leq)$ is a poset, where $x \leq y \Leftrightarrow x \in \tau(y)$.

(2) The set of ideals of $(E, \leq)$ is $\{X \subseteq E : X = \bigcup_{x\in X} \tau(x)\}$.

(3) $E \setminus X \in \mathcal{F}$ if and only if $X = \bigcup_{x\in X} \tau(x)$.

Proof (1) Routine verification.

(2) Let $X = \bigcup\limits_{x\in X} \tau(x)$. One has $X = \bigcup\limits_{x\in X} \tau(x) = \{y : y \leq x$ for some $x \in X\}$. Conversely, let X be an ideal in $(E, \leq)$. Then $X = \{y : y \leq x$ for some $x \in X\} = \bigcup\limits_{x\in X} \tau(x)$. Therefore, the need holds.

(3) Claim 1. $E \setminus X \in \mathcal{F}$.

For any $x \in X$, by (1), $x \in \tau(x)$, and so $X \subseteq \bigcup\limits_{x\in X} \tau(x)$. In view of $E\setminus X \in \mathcal{F}$ and $x \in X$, one has $\tau(x) = \cap\{Y : x \in Y, E \setminus Y \in \mathcal{F}\} \subseteq \{X : x \in X, E \setminus X \in \mathcal{F}\} = X$, and so $\bigcup\limits_{x\in X} \tau(x) \subseteq X$. Hence $X = \bigcup\limits_{x\in X} \tau(x)$.

Claim 2. $X = \bigcup\limits_{x\in X} \tau(x)$.

Suppose $E \setminus X \notin \mathcal{F}$. Then one has $\cup\{A : A \subseteq E \setminus X, A \in \mathcal{F}\} = (\cup\{A : A \subset E \setminus X, A \in \mathcal{F}\}) \cup \{A : A = E \setminus X, A \in \mathcal{F}\} \subseteq E \setminus X$. Since $\mathcal{F}$ is closed under union, one gets $\cup\{A : A \subset E \setminus X, A \in \mathcal{F}\} \in \mathcal{F}$, besides, by the supposition, $\{A : A = E \setminus X, A \in \mathcal{F}\} = \emptyset$, and so $\cup\{A : A \subseteq E \setminus X, A \in \mathcal{F}\} \subset E \setminus X$.

Let $D = \cup\{A : A \subset E\setminus X, A \in \mathcal{F}\}$. We see that D is the maximum element of $\mathcal{F}$ contained in $E \setminus X$. Besides, $X \subset E \setminus D = E \setminus \cup\{A : A \subset E \setminus X, A \in \mathcal{F}\} = \cap\{E \setminus A : A \subset E \setminus X, A \in \mathcal{F}\} = \cap\{Y : X \subset Y, E \setminus Y \in \mathcal{F}\}$. Because $X = \bigcup\limits_{x\in X} \tau(x) \subset \bigcup\limits_{d\in E\setminus D} \tau(d) = E \setminus D = \cap\{Y : X \subset Y, E\setminus Y \in \mathcal{F}\}$, one has that for $\forall x \in X, x \in \cap\{Y : X \subset Y, E \setminus Y \in \mathcal{F}\} \subseteq \cap\{Y : x \in Y, E \setminus Y \in \mathcal{F}\} = \tau(x)$, and so $X \subseteq \cap\{Y : X \subset Y, E \setminus Y \in \mathcal{F}\} \subseteq \bigcup\limits_{x\in X} \tau(x) = X$. Say, $X = E \setminus D$, a contradiction to $D \subset E \setminus X$. In other words, $E \setminus X \in \mathcal{F}$.

Summing up the above two claims, the required is proved.

Proof of theorem 2 Based on the above preparatory work and (i.e. lemma 3, lemma 4 and remark 2), we only have to prove that $(E, \mathcal{H})$ is an infinite antimatroid.

By the definition of ideal in P, it is easy to see $f(\emptyset) = \emptyset$ and $f(P) \in \mathcal{H}$. Let A_α ($\alpha \in \mathcal{A}$) be ideals in P. For any $x \leq y \in \bigcup\limits_{\alpha\in\mathcal{A}} A_\alpha$, it follows $y \in A_{\alpha_0}$ for some $\alpha_0 \in \mathcal{A}$, and so $x \in A_{\alpha_0}$, further $x \in \bigcup\limits_{\alpha\in\mathcal{A}} A_\alpha$, that is to say, $\bigcup\limits_{\alpha\in\mathcal{A}} A_\alpha$ is an ideal of P. In addition, since $f(A_\alpha) \subseteq f(\bigcup\limits_{\alpha\in\mathcal{A}} A_\alpha)$ implies $\bigcup\limits_{\alpha\in\mathcal{A}} f(A_\alpha) \subseteq f(\bigcup\limits_{\alpha\in\mathcal{A}} A_\alpha)$. Besides, for every $y \in f(\bigcup\limits_{\alpha\in\mathcal{A}} A_\alpha)$, there is $\alpha_1 \in \mathcal{A}$ and $x \in A_{\alpha_1}$ satisfying $y = f(x)$, and so $y \in f(A_{\alpha_1}) \subseteq \bigcup\limits_{\alpha\in\mathcal{A}} f(A_\alpha)$. Hence, $f(\bigcup\limits_{\alpha\in\mathcal{A}} A_\alpha) \subseteq \bigcup\limits_{\alpha\in\mathcal{A}} f(A_\alpha)$ is correct. In the other words, $\mathcal{H}$ is closed under union.

Let $H_X = f(X), H_Y = f(Y), H_Y \subset H_X$ and $H_X, H_Y \in \mathcal{H}$. Then we get that both X and Y are ideals in P and $Y \subset X$. Let x be a minimal element in $X \setminus Y$. By the definition of ideal in a poset, one has that $Y \cup x$ is an ideal in P. That is, $f(Y \cup x) = f(Y) \cup f(x) \in \mathcal{H}$, and hence $\mathcal{H}$ satisfies (III).

The following example expresses that the supposition "for any two ideals X, Y in P, if $Y \subset X$, then there is a minimal element $x \in X \setminus Y$" is necessary for theorem 2.

Example 1 Let P be the poset on the set $\mathbb{R}$ of real numbers with the ordinary order. Let $X = (-\infty, 1)$ and $Y = (-\infty, 0]$. It is obviously that both X and Y are ideals in P and $Y \subset X$. But there is not $x \in X \setminus Y$ such that $Y \cup x$ is an ideal in P. Hence for identity map $f : P \to \mathbb{R}$, $(\mathbb{R}, \mathcal{H} = \{f(A) \subseteq E : A$ is an ideal in $P\})$ is not an infinite antimatroid.

Remark 2 (1) Theorem 1 and theorem 2 tell us that poset theory will be a key source for the research on infinite antimatroids.

(2) [3,4] tells us that every finite antimatroids is induced in the way provided by theorem 2 by a map from some poset. Unfortunately, this is not true for infinite antimatroids though theorem 2 is the extension of the corresponding results for finite antimatorids. This view hints that our infinite antimatroid does not simply extend the definition of finite antimatroid, it has its intent.

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