SOME QUANTITATIVE RESULTS ON LIPSCHITZ INVERSE AND IMPLICIT FUNCTIONS THEOREMS

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Abstract

Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a Lipschitz mapping with generalized Jacobian at x_0, denoted by $\partial f(x_0)$, is of maximal rank. F. H. Clarke (1976) proved that f is locally invertible. In this paper, we give some quantitative assessments for Clarke's theorem on the Lipschitz inverse, and prove that the class of such mappings are open. Moreover, we also present a quantitative form for Lipschitz implicit function theorem.

1 Introduction

Classical inverse and implicit function theorems have attracted many researchers because of their applications in mathematics. These theorems are stated for the class of C^k mappings, and there have been researches for non-smooth mappings and global expansions. We recall some typical results of them.

F. H. Clarke (1976 - [1]) presented local inverse function theorem for Lipschitz mappings. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a Lipschitz mapping in a neighborhood of $x_0 \in \mathbf{R}^n$. If the generalized Jacobian $\partial f(x_0)$ at x_0 is of maximal rank (see Def. 2.5, 2.6 below), then there exist neighborhoods U and V of x_0 and $f(x_0)$, respectively, and a Lipschitz function $g : V \to \mathbf{R}^n$ such that

   (a) $g(f(u)) = u$ for all $u \in U$,

(b) $f(g(v)) = v$ for all $v \in V$.

More general, M. S. Gowda (2004 -[6]) considered inverse and implicit function theorems for the class of H - differentiable mappings. For the global case, J. Hadamard (1906 -[9]) presented global diffeomorphism conditions for C^1 mapping class. Generalizing the results of J. Hadamard, P. J. Rabier (1997 -[11]) demonstrated the results for C^1 mapping class on smooth manifolds. O. Gutú and J. A. Jaramillo (2007 -[7]) demonstrated global invertible conditions for the class of quasi-isometric mappings between complete metric spaces. Recently, T. Fukui, K. Kurdyka, and L. Paunescu (2010 -[5]) demonstrated some global inverse function theorems for the class of tame continuous mappings.

Most of the results on the inverse and implicit function theorems only show the existence of neighborhoods U and V to ensure $f : U \to V$ is invertible. The quantitative assessments for the subjects in the results were not considered. It is necessary to use the quantitative assessments for these theorems in several different fields such as: number theory, optimization, theory of measurement, assessment of complex algorithms, ...

Up to now, in general case, the problem of quantitative assessment for the classical inverse and implicit function theorems is unresolved. For the case $n \leq 2$, P. Henrici (1988 - [10]) gave a quantitative form for analytic function (one variable). Recently, D. Cohen (2005 - [13]) gave a different proof for the case of analytic functions. Under the result, a quantitative form for the theorems in the case of analytic function with two variables was given.

In this paper, we present a quantitative form for the Clarke inverse function theorem, where U, V and the Lipschitz constant of inverse mapping are evaluated quantitatively by $\partial f(x_0)$. Moreover, we also give a quantitative form for Lipschitz implicit function theorem and prove that the class of Lipschitz mappings satisfying Clarke's theorem are open: If f is perturbed by a mapping h with the Lipschitz constant small enough, then the mapping $f + h$ is locally invertible.

The remaining of the paper is organized as follows. In Section 2 we introduce necessary concepts and results. Section 3 presents the main results and examples.

2 Preliminaries

2.1 Perturbations and the Inverse

In this paper, we shall use the following notations:

$$\|x\| = (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}, \text{ where } x \in \mathbf{R}^n.$$

$\|A\| = \max_{\|x\|=1} \|Ax\|$, where A is a $m \times n$ matrix.

$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \|a_{ij}^2\| \right)^{\frac{1}{2}}$, where A is a $m \times n$ matrix.

If $A \in \mathbf{R}^{n \times n}$ is an invertible matrix, then

$$\|A^{-1}\| = \frac{1}{\min_{\|x\|=1} \|Ax\|}.$$

Matrix norms have some of the following properties:

(i) $\|AB\| \le \|A\|\|B\|$, $\|AB\|_F \le \|A\|_F \|B\|_F$.

(ii) $\|A\| \le \|A\|_F \le \sqrt{n}\|A\|$, where A is a $m \times n$ matrix.

(iii) For all $A \in \mathbf{R}^{m \times n}$ and $x \in \mathbf{R}^n$, we have $\|Ax\| \le \|A\|\|x\|$.

Let $\mathcal{M} = \mathbf{R}^{m \times n}$ denote the vector space of $m \times n$ matrices, we topologize $\mathcal{M}$ with the norm $\|\cdot\|$, and $\mathcal{M}(0,1)$ denote the unit ball in $\mathcal{M}$.

Lemma 2.1. *If $F \in \mathbf{R}^{n \times n}$ and $\|F\| < 1$, then $I - F$ is nonsingular and*

$$(I - F)^{-1} = \sum_{k=0}^{\infty} F^k$$

with

$$\|(I - F)^{-1}\| \le \frac{1}{1 - \|F\|}.$$

Proof. See [8], Lemma 2.3.3. $\square$ Based on Lemma 2.1, we have the following theorem.

Theorem 2.2. *Let $A, E \in \mathbf{R}^{n \times n}$. If A is nonsingular and $r = \|A^{-1}E\| < 1$, then $A + E$ is nonsingular and $\|(A + E)^{-1} - A^{-1}\| \le \|E\|\|A^{-1}\|^2/(1 - r)$.*

Proof. See [8], Theorem 2.3.4. $\square$

2.2   Generalized Jacobians

Definition 2.3. A mapping $f : \mathbf{R}^m \to \mathbf{R}^n$ is called **Lipschitz** in a neighborhood of a point x_0 in $\mathbf{R}^n$ if there exist a constant K such that for all x and y near x_0, we have

$$\|f(x) - f(y)\| \le K\|x - y\|. \tag{1}$$

If $K \ge 1$ and

$$\frac{1}{K}\|x - y\| \le \|f(x) - f(y)\| \le K\|x - y\|,$$

then f is called **bi-Lipschitz** or K-bi-Lipschitz.

Theorem 2.4. (Rademacher). *If $f : \mathbf{R}^m \to \mathbf{R}^n$ is Lipschitz, then f is almost everywhere differentiable.*

Proof. See [4], Theorem 3.1.6.   □

The usual $m \times n$ Jacobian matrix of partial derivatives of f at x, when it exists, is denoted by $Jf(x)$. By Rademacher's theorem, we have the following definition.

Definition 2.5. (F. H. Clarke - [1],[2] The *Generalized Jacobian* of f at x_0, denoted by $\partial f(x_0)$, is the convex hull of all matrices M of the form

$$M = \lim_{i \to \infty} Jf(x_i),$$

where x_i converges to x_0 and f is differentiable at x_i for each i.
When $f : \mathbf{R}^n \to \mathbf{R}$, $\partial f(x_0)$ is called the generalized gradient of f at x_0.

Definition 2.6. $\partial f(x_0)$ is said to be of **maximal rank** if every M in $\partial f(x_0)$ is of maximal rank.

Remark 2.7. From (1), $\partial f(x)$ is bounded in the neighborhood of x_0.

Proposition 2.8 ([1]). *$\partial f(x_0)$ is a nonempty compact convex subset of $\mathcal{M}$.*

Lemma 2.9 ([1]). *Let ε be a positive number. Then for all x sufficiently near x_0,*

$$\partial f(x) \subset \partial f(x_0) + \varepsilon \mathcal{M}(0, 1).$$

2.3   Topology of Lipschitz mappings

Let $f : \mathbf{R}^m \to \mathbf{R}^n$. Then the Lipschitz constant of f is defined by

$$\mathrm{L}(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{\|x - y\|}, \ x \neq y \right\}.$$

Note that f is Lipschitz if and only if $\mathrm{L}(f) < \infty$. Set

$$\mathrm{Lip}(\mathbf{R}^m, \mathbf{R}^n) = \{f : \mathrm{L}(f) < +\infty\}.$$

For $f, g \in \mathrm{Lip}(\mathbf{R}^m, \mathbf{R}^n)$ and $\alpha \in \mathbf{R}$, we have the following properties:

 (i) $f + g, \alpha f \in \mathrm{Lip}(\mathbf{R}^m, \mathbf{R}^n)$,

 (ii) $\mathrm{L}(f) \geq 0$,

 (iii) $\mathrm{L}(f + g) \leq \mathrm{L}(f) + \mathrm{L}(g)$,

 (iv) $\mathrm{L}(\alpha f) = \alpha \mathrm{L}(f)$,

(v) $L(f) = 0 \Leftrightarrow f = \text{constant}$.

By (v), for $x_0 \in \mathbf{R}^m$, set

$$\text{Lip}_{x_0}(\mathbf{R}^m, \mathbf{R}^n) = \{f : f \text{ is Lipschitz and } f(x_0) = 0\}.$$

Then

$$L(f) = 0 \Leftrightarrow f \equiv 0, \quad \text{for all } f \in \text{Lip}_{x_0}(\mathbf{R}^m, \mathbf{R}^n).$$

Thus $\text{Lip}_{x_0}(\mathbf{R}^m, \mathbf{R}^n)$ is a normed vector space with the norm $L(\cdot)$.

3   Results - Examples

Applying the results of F. H. Clarke ([1]), perturbation matrix, the properties of Lipschitz mappings and differentiable mappings, in Theorem 3.1 we present a quantitative form of the Lipschitz inverse function theorem of F. H. Clarke, in Theorem 3.5 we give a quantitative form of the Lipschitz implicit function theorem, openness of the class of Lipschitz mappings satisfying Clarke's inverse function theorem is proved in Theorem 3.8 and Corollary 3.10.

We let $\mathbf{B}$ denote the unit ball in $\mathbf{R}^n$, and $\mathbf{S}$ denote the unit sphere in $\mathbf{R}^n$.

Theorem 3.1 (c.f. [1], Theorem 1). *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a Lipschitz mapping with Lipschitz constant K. If $\partial f(x_0)$ is of maximal rank, set*

$$\delta = \frac{1}{2} \inf_{M_0 \in \partial f(x_0)} \frac{1}{\|M_0^{-1}\|},$$

r be chosen so that f satisfies Lipschitz condition (1) and

$$\partial f(x) \subset \partial f(x_0) + \delta \mathcal{M}(0,1), \quad when \quad x \in \mathbf{B}(x_0, r),$$

then there exist neighborhoods U and V of x_0 and $f(x_0)$, respectively, and a Lipschitz mapping $g : V \to \mathbf{R}^n$ such that
   (a) $g(f(u)) = u$ for every $u \in U$,
   (b) $f(g(v)) = v$ for every $v \in V$,
where,

$$U = \mathbf{B}(x_0, \frac{r\delta}{2} \cdot \frac{1}{K}), \quad V = \mathbf{B}(f(x_0), \frac{r\delta}{2}), \quad and \quad L(g) = \frac{1}{\delta}.$$

Proof. First we have:

Lemma 3.2 (c.f. [1], Lemma 3). *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a Lipschitz mapping, $\partial f(x_0)$ has maximal rank, set*

$$\delta = \frac{1}{2} \inf_{M_0 \in \partial f(x_0)} \frac{1}{\|M_0^{-1}\|},$$

r be chosen so that in $\mathbf{B}(x_0, r)$ f satisfies the Lipschitz condition (1) and $\partial f(x) \subset \partial f(x_0) + \delta \mathcal{M}(0, 1)$. Then, for every unit vector v in $\mathbf{R}^n$, there exists a unit vector w in $\mathbf{R}^n$ such that, whenever x lies in $x_0 + r\mathbf{B}$ and M belongs to $\partial f(x)$,

$$w.(Mv) \geq \delta. \tag{2}$$

Proof of the lemma: By Proposition 2.8 and $\partial f(x_0)$ is of maximal rank, the subset $\partial f(x_0)\mathbf{S}$ of $\mathbf{R}^n$ is compact and not containing 0. For $M_0 \in \partial f(x_0)$, we have

$$\min_{\|x\|=1} \|M_0 x\| = \frac{1}{\|M_0^{-1}\|}.$$

Set

$$\delta = \frac{1}{2} \inf_{M_0 \in \partial f(x_0)} \frac{1}{\|M_0^{-1}\|},$$

we get $\partial f(x_0)\mathbf{S}$ distances 2δ from 0.
If $M \in G = \partial f(x_0) + \varepsilon \mathcal{M}(0, 1)$, then

$$\min_{\|x\|=1} \|Mx\| \geq \min_{\|x\|=1} \|M_0 x\| - \varepsilon.$$

Choosing

$$\varepsilon = \delta = \frac{1}{2} \inf_{M_0 \in \partial f(x_0)} \frac{1}{\|M_0^{-1}\|},$$

we get $G\mathbf{S}$ distances at least δ from 0. By Lemma 2.9, there exists a positive number r, such that

$$x \in x_0 + r\mathbf{B} \Rightarrow \partial f(x) \subset G. \tag{3}$$

Let r be chosen so that f satisfies (1) on $x_0 + r\mathbf{B}$.
Thus, let any unit vector v be given, apply the above results, the convex set Gv distances at least δ from 0. By the usual separation theorem for convex sets, there exists a unit vector w such that

$$w.(\gamma v) \geq \delta,$$

for every $\gamma \in G$. Hence, applying (3) we obtain (2).

Proof of the theorem: Using the proof of [1], Theorem 1 to replace [1], Lemma 3 by the preceding lemma.
Estimation of the neighborhood U, V and the Lipschitz constant $L(g)$:
According to the proof of [1], Theorem 1, we have

$$L(g) = \frac{1}{\delta},$$

$$V = f(x_0) + (r\delta/2)\mathbf{B} = \mathbf{B}\left(f(x_0), \frac{r\delta}{2}\right),$$

and choose U being an arbitrary neighborhood of x_0 and satisfying $f(U) \subset V$. Then for all $x \in U$ we have

$$\|f(x) - f(x_0)\| \leq K\|x - x_0\| \leq \frac{r\delta}{2}.$$

Hence,

$$\|x - x_0\| \leq \frac{r\delta}{2}\frac{1}{K}.$$

So

$$U = \mathbf{B}\left(x_0, \frac{r\delta}{2}\frac{1}{K}\right).$$

$\square$

Remark 3.3. When f is C^1, $\partial f(x_0)$ reduces to $Jf(x_0)$, and g is in the class C^1. Thus we get the quantitative form of the classical inverse function theorem.

Remark 3.4. If $F : A \to \mathbf{R}^n$ be a Lipschitz mapping in a neighborhood of (x_0, y_0), $A = U \times V$ be a open subset of $\mathbf{R}^m \times \mathbf{R}^n$, then the generalized Jacobian of F at (x_0, y_0) satisfies

$$\partial F(x_0, y_0) \subset \left\{ \left(\begin{matrix} M_1 & M_2 \end{matrix}\right) : M_1 \in \partial_1 F(x_0, y_0), M_2 \in \partial_2 F(x_0, y_0) \right\},$$

where $\partial_1 F(x_0, y_0)$ and $\partial_2 F(x_0, y_0)$ are generalized Jacobians of $F(\cdot, y_0) : U \to \mathbf{R}^n$ and $F(x_0, \cdot) : U \to \mathbf{R}^n$ at (x_0, y_0), respectively.

Theorem 3.5. *Let $F : A \to \mathbf{R}^n$ be a Lipschitz mapping in a neighborhood of (x_0, y_0) with Lipschitz constant K, $A = U \times V$ be a open subset of $\mathbf{R}^m \times \mathbf{R}^n$. Suppose that $\partial_2 F(x_0, y_0)$ is of maximal rank, $F(x_0, y_0) = 0$, set*

$$\delta = \frac{1}{2}\inf_{M_2 \in \partial_2 F(x_0, y_0)} \frac{1}{(m + (1 + mK^2)n\|M_2^{-1}\|^2)^{\frac{1}{2}}},$$

r be chosen so that F satisfies Lipschitz condition (1) and

$$\partial F(x, y) \subset \partial F(x_0, y_0) + \delta\mathcal{M}(0, 1), \quad when \; (x, y) \in \mathbf{B}((x_0, y_o), r).$$

Then there exist a Lipschitz mapping $g : U_0 \to V$ defined in a neighborhood $U_0 \subset \mathbf{R}^m$ of x_0 such that $g(x_0) = y_0$ and

$$F(x, g(x)) = 0,$$

for all $x \in U_0$.
Moreover,

$$U_0 = \mathbf{B}\left(x_0, \frac{r\delta}{2}\frac{1}{K+1}\right), \quad and \quad L(g) \leq \sup_{M_2 \in \partial_2 F(x_0, y_0)} K\|M_2^{-1}\|.$$

Proof.

1. Set $f(x,y) = (x, F(x,y))$, for all $(x,y) \in U \times V$. Since F is Lipschitz, according to Radamacher's theorem, there exists the generalized Jacobian of F at (x_0, y_0). In a neighborhood of (x_0, y_0), F is almost everywhere differentiable, therefore existing generalized Jacobian of f at (x_0, y_0) and

$$\partial f(x_0, y_0) \subset \left\{ \begin{pmatrix} I_m & 0 \\ M_1 & M_2 \end{pmatrix} : \quad M_1 \in \partial_1 F(x_0, y_0), \quad M_2 \in \partial_2 F(x_0, y_0) \right\}.$$

Then $\partial f(x_0, y_0)$ is of maximal rank, because $\partial_2 F(x_0, y_0)$ is of maximal rank.

2. For $M = \begin{pmatrix} I_m & 0 \\ M_1 & M_2 \end{pmatrix} \in \partial f(x_0, y_0)$,

$$M^{-1} = \begin{pmatrix} I_m & 0 \\ -M_2^{-1} M_1 & M_2^{-1} \end{pmatrix}.$$

Therefore, we have

$$\begin{aligned}
\|M^{-1}\| \le \|M^{-1}\|_F &= \left(m + \|M_2^{-1} M_1\|_F^2 + \|M_2^{-1}\|_F^2 \right)^{\frac{1}{2}} \\
&\le \left(m + \|M_2^{-1}\|_F^2 (\|M_1\|_F^2 + 1) \right)^{\frac{1}{2}} \\
&\le \left(m + \|M_2^{-1}\|_F^2 (m\|M_1\|^2 + 1) \right)^{\frac{1}{2}} \\
&\le \left(m + (1 + mK^2) \|M_2^{-1}\|_F^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$\frac{1}{\|M^{-1}\|} \ge \frac{1}{\left(m + (1 + mK^2)\|M_2^{-1}\|_F^2 \right)^{\frac{1}{2}}} \ge \frac{1}{\left(m + (1 + mK^2)n\|M_2^{-1}\|^2 \right)^{\frac{1}{2}}}.$$

Set

$$\Delta = \frac{1}{2} \inf_{M \in \partial f(x_0, y_0)} \frac{1}{\|M^{-1}\|},$$

we get

$$\Delta \ge \delta. \tag{4}$$

3. According to the theorem, we have

$$\partial F(x,y) \subset \partial F(x_0, y_0) + \delta \mathcal{M}(0, 1), \text{ when } (x,y) \in \mathbf{B}((x_0, y_o), r),$$

by (4), we get

$$\partial F(x,y) \subset \partial F(x_0, y_0) + \Delta \mathcal{M}(0, 1), \text{ when } (x,y) \in \mathbf{B}((x_0, y_o), r).$$

Therefore, we can chose r so that f satisfies Lipschitz condition (1) and

$$\partial f(x,y) \subset \partial f(x_0, y_0) + \Delta \mathcal{M}(0, 1), \text{ when } (x,y) \in \mathbf{B}((x_0, y_o), r).$$

4. Since F is a Lipschitz mapping with coefficient K, f is Lipschitz with coefficient $K + 1$. Applying Theorem 3.1, f is locally invertible and

$$f^{-1}(x, z) = (x, h(x, z)),$$

with h is a Lipschitz mapping. Define

$$g(x) = h(x, 0).$$

Then g is Lipschitz and

$$(x, F(x, g(x))) = f(x, g(x)) = f(x, h(x, 0)) = f(f^{-1}(x, 0)) = (x, 0).$$

This indicates the existence of g satisfying the requirements of the theorem.

5. Estimation of the neighborhood U_0 of x_0 and $L(g)$:
Applying Theorem 3.1, we obtain

$$(x, h(x, z)) \in U = \mathbf{B}\left((x_0, y_0), \frac{r\Delta}{2}\frac{1}{k+1}\right).$$

Thus

$$(x, g(x)) \in U' = \mathbf{B}\left((x_0, 0), \frac{r\Delta}{2}\frac{1}{k+1}\right).$$

Hence, by (4), the theorem is satisfied for all $(x, g(x)) \in U'' = \mathbf{B}\left((x_0, 0), \frac{r\delta}{2}\frac{1}{k+1}\right).$
Projecting U'' onto the space $\mathbf{R}^m$, we get

$$U_0 = \mathbf{B}\left(x_0, \frac{r\delta}{2}\frac{1}{k+1}\right).$$

Moreover, applying the formula of implicit function derivative,

$$Dg = -\frac{\partial F^{-1}}{\partial y} \cdot \frac{\partial F}{\partial x}, \quad \text{whenever} \quad \frac{\partial F^{-1}}{\partial y} \cdot \frac{\partial F}{\partial x} \quad \text{exist.}$$

Therefore, we get

$$L(g) \le \sup_{M_2 \in \partial_2 F(x_0, y_0)} K\|M_2^{-1}\|.$$

$\square$

Remark 3.6. When F is C^1, $\partial_2 F(x_0, y_0)$ reduces to $J_2 F(x_0, y_0)$, and g is in the class C^1. Thus we get the quantitative form of the classical implicit function theorem.

Example 3.7. For $m = 1, n = 2$, consider $F(x, y, z) = (2x + |y| + 3y, 2x + |z| + 3z)$ in $\mathbf{B}\left((0, 0, 0), 1\right)$. Then

$$\|F(x, y, z) - F(x', y', z')\| \leq \sqrt{24}\|(x, y, z) - (x', y', z')\|.$$

Thus F is Lipschitz with Lipschitz constant $K = \sqrt{24}$.
We have

$$J_2 F\left(x_i, y_i, z_i\right) = \begin{pmatrix} \frac{|y_i|}{y_i} + 3 & 0 \\ 0 & \frac{|z_i|}{z_i} + 3 \end{pmatrix}, \quad (x_i, y_i, z_i) \text{ near } (0, 0, 0).$$

Hence,

$$\partial_2 F(0, 0, 0) = \left\{ \begin{pmatrix} s + 3 & 0 \\ 0 & t + 3 \end{pmatrix} : -1 \leq s \leq 1, -1 \leq t \leq 1 \right\},$$

and $\partial_2 F(0, 0, 0)$ is of maximal rank.
Let $M_2 \in \partial_2 F(0, 0, 0)$. Then there exist M_2^{-1} defined by

$$M_2^{-1} = \begin{pmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{t+3} \end{pmatrix}.$$

So

$$\delta = \frac{1}{2} \inf_{M_2 \in \partial_2 F(x_0, y_0)} \frac{1}{(m + (1 + mK^2)n\|M_2^{-1}\|^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{54}}.$$

By the preceding theorem, there exist a Lipschitz mapping g such that $g(0) = (0, 0)$ and

$$F(x, g(x)) = (0, 0), \quad \text{for all } x \in U_0.$$

Moreover, for (x, y, z) near $(0, 0, 0)$, we have

$$JF(x, y, z) = \begin{pmatrix} 2 & \frac{|y|}{y} + 3 & 0 \\ 2 & 0 & \frac{|z|}{z} + 3 \end{pmatrix}.$$

We can chose $r = 1$, and then

$$\partial F(x, y, z) \subset \partial F(0, 0, 0) + \frac{1}{\sqrt{54}}\mathcal{M}(0, 1), \quad \text{every } (x, y, z) \in \mathbf{B}((0, 0, 0), r).$$

Hence, we obtain

$$U_0 = B\left(0, \frac{1}{6\sqrt{6}} \cdot \frac{1}{1 + 2\sqrt{6}}\right), \quad \text{and } L(g) \leq \sqrt{6}.$$

Theorem 3.8. *Let $f_0 : \mathbf{R}^n \to \mathbf{R}^n$ be a Lipschitz mapping in the neighborhood of x_0 so that $\partial f_0(x_0)$ is of maximal rank and satisfies*

$$K\|x - y\| \leq \|f_0(x) - f_0(y)\| \leq K'\|x - y\|.$$

Let $f = f_0 + h$, with $h : \mathbf{R}^n \to \mathbf{R}^n$ is a Lipschitz mapping with Lipschitz constant L so that

$$L < K.$$

Set

$$\delta = \frac{1}{2} \inf_{M_0 \in \partial f(x_0)} \frac{1}{\|M_0^{-1}\|}.$$

Suppose that r was chosen so that f satisfies Lipschitz condition (1) and $\partial f(x) \subset \partial f(x_0) + \delta \mathcal{M}(0,1)$ in $\mathbf{B}(x_0, r)$. Then there exist neighborhoods U and V of x_0 and $f(x_0)$, respectively, and a Lipschitz mapping $g : V \to \mathbf{R}^n$ such that

(a) $g(f(u)) = u$ for every $u \in U$,

(b) $f(g(v)) = v$ for every $v \in V$.
Moreover, U, V and $L(g)$ are determined by

$$U = \mathbf{B}\left(x_0, \frac{r\delta}{2} \frac{1}{K' + L}\right), \quad V = \mathbf{B}\left(f(x_0), \frac{r\delta}{2}\right), \quad \text{and} \quad L(g) = \frac{1}{\delta}.$$

Proof.
Remark: f is differentiable at x_i if and only if f can be approximated in a neighborhood of x_i by the affine mapping

$$T(x) = f(x_i) + Jf(x_i)(x - x_i).$$

Hence,

$$K\|x - x_i\| \leq \|f(x) - f(x_i)\| \approx \|Jf(x_i)(x - x_i)\| \leq K'\|x - x_i\|.$$

From which we obtain

$$K \leq \inf_{y \neq 0} \frac{\|Jf(x_i)y\|}{\|y\|}, \forall Jf(x_i) \Rightarrow K \leq \inf_{Jf(x_i)} \left(\min_{\|x\|=1} \|Jf(x_i)x\|\right), \quad (5)$$

$$\sup_{y \neq 0} \frac{\|Jf(x_i)y\|}{\|y\|} \leq K', \forall Jf(x_i) \quad \Rightarrow \quad \sup_{Jf(x_i)} \left(\max_{\|x\|=1} \|Jf(x_i)x\|\right) \leq K'$$

$$\Rightarrow \quad \sup_{Jf(x_i)} \|Jf(x_i)\| \leq K'. \tag{6}$$

From the remarks above we will prove that f is a Lipschitz mapping and $\partial f(x_0)$ is of maximal rank.

We have

$$\begin{aligned}
\|f(x) - f(y)\| &= \|(f_0(x) - f_0(y)) + (h(x) - h(y))\| \\
&\leq \|(f_0(x) - f_0(y))\| + \|(h(x) - h(y))\| \\
&\leq (K' + L)\|x - y\|.
\end{aligned}$$

So f is Lipschitzian with Lipschitz constant $K' + L$.

According to Rademacher's theorem, f is almost everywhere differentiable near x_0, so that the general Jacobian $\partial f(x_0)$ exits. Moreover,

$$\partial f(x_0) \subset \partial f_0(x_0) + \partial h(x_0).$$

Indeed, let E be the set of points where f_0 or h is fail to be differentiable. Then every $M \in \partial f(x_0), M$ has the form

$$M = \lim_{i \to \infty} J(f_0 + h)(x_0 + h_i), h_i \to 0 \text{ when } i \to \infty,$$

here the sequence $\{x_0 + h_i\}$ lies in the complement of E, and admits a subsequence $\{x_0 + h_{n_i}\}$ such that $Jf_0(x_0 + h_{n_i})$ and $Jh(x_0 + h_{n_i})$ both exist and converge. Hence

$$M = \lim_{i \to \infty} Jf_0(x_0 + h_{n_i}) + \lim_{i \to \infty} Jh(x_0 + h_{n_i}) = M_0 + H,$$

where $M_0 \in \partial f_0(x_0), H \in \partial h(x_0)$.

Next, we prove $\partial f(x_0)$ is of maximal rank:
Using (5) and (6) we have

$$\begin{aligned}
L < K \quad &\Rightarrow \quad \sup_{H \in \partial h(x_0)} \|H\| \; < \; \inf_{M_0 \in \partial f_0(x_0)} \left(\min_{\|x\|=1} \|M_0 x\| \right) \\
&\Rightarrow \quad \sup_{H \in \partial h(x_0)} \|H\| \; < \; \inf_{M_0 \in \partial f_0(x_0)} \left(\frac{1}{\frac{1}{\min_{\|x\|=1} \|M_0 x\|}} \right) \\
&\Rightarrow \quad \sup_{H \in \partial h(x_0)} \|H\| \; < \; \frac{1}{\sup_{M_0 \in \partial f_0(x_0)} \left(\frac{1}{\min_{\|x\|=1} \|M_0 x\|} \right)} \\
&\Rightarrow \quad \sup_{H \in \partial h(x_0)} \|H\| \; < \; \frac{1}{\sup_{M_0 \in \partial f_0(x_0)} \|M_0^{-1}\|},
\end{aligned}$$

so we get

$$\sup_{H \in \partial h(x_0), M_0 \in \partial f_0(x_0)} \|M_0^{-1} H\| < 1.$$

According to Theorem 2.2, $M_0 + H$ is of maximal rank for all $H \in \partial h(x_0), M_0 \in \partial f_0(x_0)$.
According to the proof above, if $M \in \partial f(x_0)$ then $M = M_0 + H$, with $M_0 \in$

$\partial f_0(x_0), H \in \partial h(x_0)$. Hence, M is of maximal rank for every $M \in \partial f(x_0)$. So $\partial f(x_0)$ is of maximal rank.

Thus f is Lipschitz and $\partial f(x_0)$ is of maximal rank, applying Theorem 3.1, we get the results of the theorem.

Moreover, according to the proof of Theorem 3.1, we have

$$L(g) = \frac{1}{\delta}, \quad V = f(x_0) + (r\delta/2)\mathbf{B} = \mathbf{B}\left(f(x_0), \frac{r\delta}{2}\right),$$

and choose U being an arbitrary neighborhood of x_0 and satisfying $f(U) \subset V$. Then for all $x \in U$ we have

$$\|f(x) - f(x_0)\| \le (K' + L)\|x - x_0\| \le \frac{r\delta}{2}.$$

Hence,

$$\|x - x_0\| \le \frac{r\delta}{2}\frac{1}{K' + L}.$$

So

$$U = \mathbf{B}\left(x_0, \frac{r\delta}{2}\frac{1}{K' + L}\right).$$

$\square$

Remark 3.9. If $L > K$, then $f = f_0 + h$ may not satisfy local invertible condition.

Corollary 3.10. *The class of Lipschitz mappings satisfying Clarke's inverse function theorem is open in the space* $Lip_{x_0}(\mathbf{R}^m, \mathbf{R}^n)$.

Example 3.11. For $n = 2$, consider $f_0(x, y) = (|x| + 2x, |y| + 2y)$ in $\mathbf{B}((0,0), 1)$. We have

$$Jf_0(x_i, y_i) = \begin{pmatrix} \frac{|x_i|}{x_i} + 2 & 0 \\ 0 & \frac{|y_i|}{y_i} + 2 \end{pmatrix}, \quad (x_i, y_i) \text{ near } (0, 0).$$

Thus

$$\partial f_0(0, 0) = \left\{\begin{pmatrix} s + 2 & 0 \\ 0 & t + 2 \end{pmatrix} : -1 \le s \le 1, -1 \le t \le 1\right\},$$

and $\partial f_0(0, 0)$ is of maximal rank.
We have

$$\begin{aligned}\|f_0(x, y) - f_0(x', y')\| &= \|(|x| + 2x, |y| + 2y) - (|x'| + 2x', |y'| + 2y')\| \\ &= \|((|x| - |x'|) + 2(x - x'), (|y| - |y'|) + 2(y - y'))\|.\end{aligned}$$

Hence,

$$\|(x,y) - (x',y')\| \leq \|f_0(x,y) - f_0(x',y')\| \leq 3\|(x,y) - (x',y')\|.$$

Thus, f_0 is Lipschitz with Lipschitz constants $K = 1, K' = 3$.

1. Case $L < K$. We find $h(x,y) = (\frac{1}{2}|x|, \frac{1}{2}|y|)$ is Lipschitz with Lipschitz constant $L = \frac{1}{2}$ satisfying

$$L < K.$$

Set $f = f_0 + h$, we have

$$f(x,y) = (\frac{3}{2}|x| + 2|x|, \frac{3}{2}|y| + 2y),$$

and f is Lipschitz in $\mathbf{B}((0,0),1)$ with Lipschitz constant $\frac{7}{2}$.
Then,

$$Jf(x_i, y_i) = \begin{pmatrix} \frac{3}{2}\frac{|x_i|}{x_i} + 2 & 0 \\ 0 & \frac{3}{2}\frac{|y_i|}{y_i} + 2 \end{pmatrix}, \quad (x_i, y_i) \text{ near } (0,0),$$

$$\partial f(0,0) = \left\{ \begin{pmatrix} s+2 & 0 \\ 0 & t+2 \end{pmatrix} : -\frac{3}{2} \leq s \leq \frac{3}{2}, -\frac{3}{2} \leq t \leq \frac{3}{2} \right\},$$

and $\partial f(0,0)$ is of maximal rank.
So $f = f_0 + h$ is locally invertible.
We have

$$\delta = \frac{1}{2} \inf_{M_0 \in \partial f(x_0)} \frac{1}{\|M_0^{-1}\|} = 1.$$

Moreover,

$$Jf(x,y) = \begin{pmatrix} \frac{3}{2}\frac{|x|}{x} + 2 & 0 \\ 0 & \frac{3}{2}\frac{|y|}{y} + 2 \end{pmatrix}, \quad (x,y) \in \mathbf{B}((0,0),r), \; r \leq 1.$$

We can chose $r = 1$, and then

$$\partial f(x,y) \subset \partial f(0,0) + \mathcal{M}(0,1), \quad \text{when} \quad (x,y) \in \mathbf{B}((0,0),r).$$

Hence, we obtain

$$U = \mathbf{B}\left((0,0), \frac{1}{7}\right), \quad V = \mathbf{B}\left((0,0), \frac{1}{2}\right), \quad \text{and} \quad L(g) = 1.$$

2. Case $L > K$. We find $h(x,y) = (\frac{3}{2}|x|, \frac{3}{2}|y|)$ is Lipschitz with Lipschitz constant $L = \frac{3}{2} > K$. Set $f = f_0 + h$, we have

$$f(x,y) = (\frac{5}{2}|x| + 2x, \frac{5}{2}|y| + 2y),$$

and f is Lipschitz with Lipschitz constant $\frac{9}{2}$.
Then,

$$Jf(x_i, y_i) = \begin{pmatrix} \frac{5}{2}\frac{|x_i|}{x_i} + 2 & 0 \\ 0 & \frac{5}{2}\frac{|y_i|}{y_i} + 2 \end{pmatrix}, \quad (x_i, y_i) \text{ near } (0,0),$$

$$\partial f(0,0) = \left\{ \begin{pmatrix} s+2 & 0 \\ 0 & t+2 \end{pmatrix} : -\frac{5}{2} \leq s \leq \frac{5}{2}, -\frac{5}{2} \leq t \leq \frac{5}{2} \right\},$$

and $\partial f(0,0)$ is not of maximal rank.

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