East-West J. of Mathematics: Vol. 14, No. 2 (2012) pp. 144-153

THE GENERALIZED TRIBONACCI p-NUMBERS AND APPLICATIONS

Kantaphon Kuhapatanakul

Department of Mathematics, Faculty of Science
Kasetsart University, Bangkok 10900, Thailand
email: fscikpkk@ku.ac.th

Abstract

The generalized tribonacci p-numbers ($p \in \mathbb{N}$), which generalizes the tribonacci numbers, are integers satisfying

$$T_p(n+2) = T_p(n+1) + T_p(n) + T_p(n-p).$$

We construct the companion matrix for the generalized tribonacci p-numbers and derive some interesting identities. Several explicit formulas for the tribonacci p-numbers are also derived using the tribonacci p-triangle.

1 Introduction

As is well known, the sequences of Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ and tribonacci numbers $\{T_n\}_{n=0}^{\infty}$ are defined, respectively, by

$$F_{n+1} = F_n + F_{n-1},\ F_0 = 0,\ F_1 = 1$$
$$T_{n+2} = T_{n+1} + T_n + T_{n-1}, \quad T_0 = 0,\ T_1 = T_2 = 1.$$

In 1963, Alladi and Hoggatt [1] constructed the tribonacci triangle, see Figure 1, and used it to derive some interesting formulas involving tribonacci numbers.

Keywords: tribonacci p-numbers, tribonacci numbers, tribonacci p-triangle
(2010) AMS Classification: 11B37, 11B83
Supported by the Kasetsart University Research and Development Institute.

	0	1	2	3	4	5	6	7	⋯
0	1								
1	1	1							
2	1	3	1						
3	1	5	5	1					
4	1	7	13	7	1				
5	1	9	25	25	9	1			
6	1	11	41	63	41	11	1		
7	1	13	61	129	129	61	13	1	
⋮			⋮						

Figure 1 : Tribonacci triangle

Denoting by $B(n, i)$ the element in the nth row and ith column of the tribonacci triangle, they showed that the sum of elements on the rising diagonals in the tribonacci triangle is a tribonacci number (see also [5, Chapter 46]), i.e.,

$$T_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} B(n-i, i). \tag{1}$$

Since $B(n, i)$ can be written as binomial sums ([2, example 16]), we can rewrite (1) as

$$T_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/3 \rfloor} \binom{i}{j} \binom{n-i-j}{i}, \tag{2}$$

which is a known identity ([7], [8], [9]).

A good deal of identities involving elements satisfying linear recurrence sequences are often conveniently derived through matrix representation. For example, the generating matrix of the tribonacci numbers is given by ([3], [9])

$$A^n = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{bmatrix}.$$

One of the most interesting generalizations of the Fibonacci numbers is given by Stakhov [10], called the Fibonacci p-numbers $F_p(n)$, defined for $p \in \mathbb{N}$ by

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \qquad (n > p) \tag{3}$$

with initial conditions $F_p(0) = 0$ and $F_p(1) = F_p(2) = \cdots = F_p(p) = 1$.

Stakhov [10] also constructed the $(p+1) \times (p+1)$ companion matrix of the

Fibonacci p-numbers as

$$Q_p = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0. \end{bmatrix}.$$

It is not difficult to verify that the nth power of the matrix Q_p is

$$Q_p^n = \begin{bmatrix} F_p(n+1) & F_p(n-p+1) & \cdots & F_p(n-1) & F_p(n) \\ F_p(n) & F_p(n-p) & \cdots & F_p(n-2) & F_p(n-1) \\ \vdots & \vdots & & \vdots & \vdots \\ F_p(n-p+2) & F_p(n-2p+2) & \cdots & F_p(n-p) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-2p+1) & \cdots & F_p(n-p-1) & F_p(n-p) \end{bmatrix}, \tag{4}$$

which leads us to refer to Q_p as a generalized Fibonacci p-matrix. For futher properties of the Fibonacci p-numbers, we refer to [4] and [10]-[11].

Our first objective here is to introduce the notion of tribonacci p-numbers, which generalizes that of Fibonacci p-numbers.

Definition 1. *Let $p \in \mathbb{N}$. The tribonacci p-numbers $T_p(n)$ are defined by*

$$T_p(n+2) = T_p(n+1) + T_p(n) + T_p(n-p) \qquad (n \in \mathbb{Z}),$$

where $T_p(1) = 1$ and $T_p(i) = 0$ for $-p \leq i \leq 0$.

If $p = 1$, then the tribonacci p-numbers $T_p(n)$ become the classical tribonacci numbers T_n.

The following table displays the first thirteen tribonacci p-numbers for $p = 1, 2, 3, 4$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504
$T_2(n)$	0	1	1	2	3	6	10	18	31	55	96	169	296
$T_3(n)$	0	1	1	2	3	5	9	15	26	44	75	128	218
$T_4(n)$	0	1	1	2	3	5	8	14	23	39	65	109	182

Table 1

It is easy to see that $T_p(i) = F_i$ for $0 \leq i \leq p+2$, where F_n is the nth Fibonacci number.

We proceed next to construct a tribonacci p-matrix for tribonacci p-numbers and derive some identities of these numbers. We then construct the tribonacci p-triangle which generalizes the tribonacci triangle and employ it to derive an explicit formula for the tribonacci p-numbers.

2 Tribonacci p-matrix

Definition 2. *For $p \in \mathbb{N}$, a tribonacci p-matrix, denoted by A_p, is the $(p+2) \times (p+2)$ matrix whose elements a_{ij} are given by*

$$a_{11} = a_{12} = a_{1,p+2} = 1, \;\; a_{i,i-1} = 1 \;\; (2 \le i \le p+2) \text{ and } 0 \text{ otherwise, i.e.,}$$

$$A_p = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

We define a $(p+2) \times (p+2)$ matrix B_n, similar to the right-hand side of (4), by

$$B_n = \begin{bmatrix} T_p(n+1) & T_p(n) + T_p(n-p) & T_p(n-p+1) & \cdots & T_p(n) \\ T_p(n) & T_p(n-1) + T_p(n-p-1) & T_p(n-p) & \cdots & T_p(n-1) \\ T_p(n-1) & T_p(n-2) + T_p(n-p-2) & T_p(n-p-1) & \cdots & T_p(n-2) \\ \vdots & \vdots & \vdots & & \vdots \\ T_p(n-p) & T_p(n-p-1) + T_p(n-2p-1) & T_p(n-2p) & \cdots & T_p(n-p-1) \end{bmatrix}.$$

Motivated by the identity (4), we state and prove one of our main results.

Theorem 1. *Let $p \in \mathbb{N}$. We have*

$$A_p^n = B_n \qquad (n \in \mathbb{N}).$$

Proof. We prove by induction on n. It is easy to see that $B_1 = A_p$. Assume $A_p^n = B_n$ holds for some $n \ge 1$. By our assumption, we write

$$A_p^{n+1} = A_p^n A = B_n A_p.$$

From the matrix multiplication and the definition of the generalized tribonacci p-numbers, we have $B_{n+1} = B_n A_p$. Hence $A_p^{n+1} = B_{n+1}$. □

Since $A_p^{n+m} = A_p^n A_p^m$, equating the $(1,1)-$entry on both sides of this matrix equation, we obtain the following corollary.

Corollary 1. *For $m, n \in \mathbb{N}$, we have*

$$T_p(n+m+1) = T_p(n+1)T_p(m+1) + T_p(n)T_p(m) + \sum_{i=0}^{p} T_p(n-p+i)T_p(m-i). \quad (5)$$

Taking $m = n$ and $m = n-1$ in Corollary 1, we respectively get

$$T_p(2n+1) = T_p^2(n+1) + T_p^2(n) + \sum_{i=0}^{p} T_p(n-p+i)T_p(n-i) \tag{6}$$

and

$$T_p(2n) = T_p(n+1)T_p(n) + T_p(n)T_p(n-1) + \sum_{i=0}^{p} T_p(n-p+i)T_p(n-i-1). \tag{7}$$

Taking $p = 1$ in (5), (6) and (7), we respectively get ([6, page 461])

$$T_{n+m+1} = T_{n+1}T_{m+1} + T_nT_m + T_{n-1}T_m + T_nT_{m-1},$$

$$T_{2n+1} = T_{n+1}^2 + T_n^2 + 2T_{n-1}T_n$$

and

$$T_{2n} = T_{n+1}T_n + T_nT_{n-1} + T_{n-1}^2 + T_nT_{n-2}.$$

Generalizing Theorem 1 further, we define the $(p+3)\times(p+3)$ matrices C_p and D_n $(n \in \mathbb{N})$ by

$$C_p = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & A_p & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \text{ and } D_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ S_p(n) & & & & \\ S_p(n-1) & & & & \\ S_p(n-2) & & B_n & & \\ \vdots & & & & \\ S_p(n-p+1) & & & & \end{bmatrix},$$

where

$$S_p(n) = \sum_{i=0}^{n} T_p(i) \text{ for } n \geq 0 \text{ and } S_p(k) = 0 \text{ for } k < 0.$$

We now give an extension of theorem 1.

Theorem 2. *Let $p \in \mathbb{N}$. We have*

$$C_p^n = D_n \qquad (n \in \mathbb{N}).$$

Proof. We prove by induction on n. Clearly, $D_1 = C_p$. Now, assume $C_p^n = D_n$ for some $n \geq 1$. From the definition of $S_p(n)$, we can write

$$S_p(n+1) = S_p(n) + T_p(n+1).$$

Using $A_p^n = B_n$, matrix multiplication and inductive hypothesis, we get $D_{n+1} = D_nC_p = C_p^{n+1}$. So the proof is complete. □

Theorem 2 gives

$$D_{n+m} = C_p^{n+m} = C_p^n C_p^m = D_n D_m,$$

which, by equating the $(2,1)-$entry of both sides, yields an extension of [3, Corollary 1].

Corollary 2. *For all $n, m \in \mathbb{N}$, we have*

$$S_p(n+m) = S_p(n)+T_p(n+1)S_p(m)+T_p(n)S_p(m-1)+\sum_{i=0}^{p} T_p(n-p+i)S_p(m-i-1).$$

Taking $m = n$ and $m = n+1$ in Corollary 2, we respectively get

$$S_p(2n) = S_p(n)+T_p(n+1)S_p(n)+T_p(n)S_p(n-1)+\sum_{i=0}^{p} T_p(n-p+i)S_p(n-i-1)$$

and

$$S_p(2n+1) = S_p(n)+T_p(n+1)S_p(n+1)+T_p(n)S_p(n)+\sum_{i=0}^{p} T_p(n-p+i)S_p(n-i).$$

Taking $n = 1$ in Corollary 2, we get an extension of [3, Lemma 1] as

$$S_p(m+1) = 1 + S_p(m) + S_p(m-1) + S_p(m-p-1),$$

which is equivalent to

$$T_p(m+1) = 1 + 2S_p(m-1) - \sum_{i=1}^{p} T_p(m-i).$$

This yields the following corollary.

Corollary 3. *For $n \in \mathbb{N}$, we have*

$$S_p(n-1) = \frac{1}{2}\left(T_p(n+1) - 1 + \sum_{i=1}^{p} T_p(n-i)\right).$$

For $p = 1$, Corollary 3 yields

$$\sum_{i=0}^{n-1} T_i = \frac{1}{2}\left(T_{n+1} + T_{n-1} - 1\right),$$

which is a formula found in [3, Theorem 2].

3 Tribonacci p-triangle

In this last section, we introduce the tribonacci p-triangle. We start with the following definition.

Definition 3. *Let $p \in \mathbb{N}$ and n, i two non-negative integers. Define*

$$B_p(n,i) = \begin{cases} 1 & \text{if } i = 0, n, \\ 0 & \text{if } i > n, \\ B_p(n-1,i) + B_p(n-1,i-1) + B_p(n-p-1,i-1) & \text{if } 1 \leq i < n. \end{cases}$$

Now we are ready to introduce the main object of this section.

Definition 4. *The **tribonacci p-triangle** is an array of numbers defined by*

	0	1	2	3	4	$\cdots$	n	$\cdots$
0	$B_p(0,0)$							
1	$B_p(1,0)$	$B_p(1,1)$						
2	$B_p(2,0)$	$B_p(2,1)$	$B_p(2,2)$					
3	$B_p(3,0)$	$B_p(3,1)$	$B_p(3,2)$	$B_p(3,3)$				
4	$B_p(4,0)$	$B_p(4,1)$	$B_p(4,2)$	$B_p(4,3)$	$B_p(4,4)$			
$\vdots$			$\vdots$					
n	$B_p(n,0)$	$B_p(n,1)$	$B_p(n,2)$		$\cdots$		$B_p(n,n)$	
$\vdots$			$\vdots$					

When $p = 1$, we see that the tribonacci 1-triangle is indeed the tribonacci triangle (Figure 1). The tribonacci p-triangles for $p = 2, 3$ are given by

	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	4	3	1			
4	1	6	8	4	1		
5	1	8	16	13	5	1	
6	1	10	28	32	19	6	1
$\vdots$			$\vdots$				

tribonacci 2-triangle

	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	5	6	4	1		
5	1	7	12	10	5	1	
6	1	9	21	23	15	6	1
$\vdots$			$\vdots$				

tribonacci 3-triangle

Observe that sums of elements on each rising diagonal in the tribonacci 2-triangle, and tribonacci 3-triangle seem to give the tribonacci 2-numbers $T_2(n)$, and tribonacci 3-numbers $T_3(n)$, respectively. The following table provides some more information.

n	$T_2(n)$	$T_3(n)$
1	1	1
2	1	1
3	1+1=2	1+1=2
4	1+2=3	1+2=3
5	1+4+1=6	1+3+1=5
6	1+6+3=10	1+5+3=9
7	1+8+8+1=18	1+7+6+1=15
8	1+10+16+4=31	1+9+12+4=26
9	1+12+28+13+1=55	1+11+21+10+1=44

Table 2

We proceed now to show that the sum of elements on each rising diagonal in the tribonacci p-triangle is equal to the tribonacci p-numbers $T_p(n)$. We need the following lemma.

Lemma 1. *Let $p \in \mathbb{N}$ and n a non-negative integer. Then*

$$B_p(n,i) = \sum_{j=0}^{i} \binom{i}{j}\binom{n-pj}{i}. \tag{8}$$

Proof. We prove by induction on n. The identity (8) clearly holds when $n = 0$. Assume (8) is true for $n \geq 0$ and $0 \leq i \leq n$. By Definition 3 and the inductive hypothesis, we get

$$\begin{aligned}
B_p(n+1,i) &= B_p(n,i) + B_p(n,i-1) + B_p(n-p,i-1) \\
&= \sum_{j=0}^{i} \binom{i}{j}\binom{n-pj}{i} + \sum_{j=0}^{i-1} \binom{i-1}{j}\binom{n-pj}{i-1} + \sum_{j=0}^{i-1} \binom{i-1}{j}\binom{n-pj-p}{i-1} \\
&= \binom{n}{i} + \sum_{j=1}^{i-1} \binom{i}{j}\binom{n-pj}{i} + \binom{n-pi}{i} + \binom{n}{i-1} + \sum_{j=1}^{i-1} \binom{i-1}{j}\binom{n-pj}{i-1} \\
&\quad + \sum_{j=1}^{i-1} \binom{i-1}{j-1}\binom{n-pj}{i-1} + \binom{n-pi}{i-1} \\
&= \binom{n+1}{i} + \sum_{j=1}^{i-1} \binom{i}{j}\binom{n-pj+1}{i} + \binom{n-pi+1}{i} \\
&= \sum_{j=0}^{i} \binom{i}{j}\binom{n-pj+1}{i}.
\end{aligned}$$

Then (8) holds for $n+1$, thereby proving the lemma. □

Since all the terms in the summation of $B_p(n,i)$ are zero when $j > \min\{\frac{n-i}{p}, i\}$, the identity (8) can be rewritten as

$$B_p(n,i) = \sum_{j=0}^{\lfloor \frac{n+i}{p+2} \rfloor} \binom{i}{j}\binom{n-pj}{i}. \tag{9}$$

We are now ready to verify our identity.

Theorem 3. *Let $p \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$. Then*

$$T_p(n+1) = \sum_{i=0}^{\lfloor n/2 \rfloor} B_p(n-i,i). \tag{10}$$

Proof. For $0 \le i \le n \le p+1$, since

$$B_p(n-i,i) = \sum_{j=0}^{i} \binom{i}{j}\binom{n-i-pj}{i} = \binom{n-i}{i},$$

we get

$$\sum_{i=0}^{\lfloor n/2 \rfloor} B_p(n-i,i) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} = F_{n+1} = T_p(n+1),$$

i.e., (10) holds for $0 \le n \le p+1$. Now assume the identity (10) holds for some $n \ge 0$. By Definition 1 and the inductive hypothesis, we get

$$\begin{aligned}
&T_p(n+2) = T_p(n+1) + T_p(n) + T_p(n-p) \\
&\quad = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} B_p(n-i,i) + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} B_p(n-i-1,i) + \sum_{i=0}^{\lfloor \frac{n-p-1}{2} \rfloor} B_p(n-i-p-1,i) \\
&\quad = B_p(n,0) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} B_p(n-i,i) + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} B_p(n-i,i-1) \\
&\qquad + \sum_{i=1}^{\lfloor \frac{n-p+1}{2} \rfloor} B_p(n-i-p-1,i-1) \\
&\quad = \begin{cases} B_p(n+1,0) + \displaystyle\sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} B_p(n-i+1,i) & \text{if } n \text{ is even} \\ B_p(n+1,0) + \displaystyle\sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} B_p(n-i+1,i) + B_p\left(\frac{n-1}{2}, \frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases} \\
&\quad = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} B_p(n-i+1,i),
\end{aligned}$$

i.e., (10) holds for $n+1$. Hence the proof is complete. $\square$

Using (9), we see that the identity (10) can also be written in terms of binomial coefficients as stated in the following corollary.

Corollary 4. *Let* $p \in \mathbb{N}$ *and* $n \in \mathbb{N} \cup \{0\}$. *Then*

$$T_p(n+1) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n+i}{p+2} \rfloor} \binom{i}{j} \binom{n-i-pj}{i}. \tag{11}$$

Taking $p=1$ in (11), we obtain (2) which gives an expansion of the classical tribonacci numbers.

References

[1] K. Alladi and V. E. Hoggatt, Jr., *On tribonacci numbers and related functions*, Fibonacci Quarterly, **15**(1) (1977), 42-45.

[2] P. Barry, *On integer-sequence-based constructions of generalized Pascal triangles*, Journal of Integer Sequences, **9** (2006), Article 06.2.4.

[3] E. Kilic, *Tribonacci sequences with certain indices and their sums*, Ars Combinatoria, **86** (2008), 13-22.

[4] E. Kilic, *The Binet fomula, sums and representations of generalized Fibonacci p-numbers*, Euro. J. Combin., **29** (2008), 701-711.

[5] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, New York, 2001.

[6] S. Rabinowitz, *Algorithmic manipulation of third-order linear recurrences*, Fibonacci Quarterly, **34**(5) (1996), 447-464.

[7] A. G. Shannon, *Iterative formulas associated with generalized third-order recurrence relations*, SIAM J. Appl. Math., **23**(3) (1972), 364-367.

[8] A. G. Shannon, *Tribonacci numbers and Pascal's pyramid*, Fibonacci Quarterly, **15**(3) (1977), 268 and 275.

[9] A. G. Shannon and A. F. Horadam, *Some propertirs of third-order recurrence relations*, Fibonacci Quarterly, **10**(2) (1972), 135-145.

[10] A. P. Stakhov, *Fibonacci matrices, a generalization of the 'Cassini formula', and a new coding theory*, Chaos, Solitons and Fractals, **30** (2006), 56-66.

[11] A. P. Stakhov and B. Rozin, *Theory of Binet formulas for Fibonacci and Lucas p-numbers*, Chaos, Solitons and Fractals, **27** (2006), 1162-1177.