

THE GENERALIZED TRIBONACCI p -NUMBERS AND APPLICATIONS

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Abstract

The generalized tribonacci p -numbers ($p \in \mathbb{N}$), which generalizes the tribonacci numbers, are integers satisfying

$$T_p(n+2) = T_p(n+1) + T_p(n) + T_p(n-p).$$

We construct the companion matrix for the generalized tribonacci p -numbers and derive some interesting identities. Several explicit formulas for the tribonacci p -numbers are also derived using the tribonacci p -triangle.

1 Introduction

As is well known, the sequences of Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ and tribonacci numbers $\{T_n\}_{n=0}^{\infty}$ are defined, respectively, by

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1}, \quad F_0 = 0, \quad F_1 = 1 \\ T_{n+2} &= T_{n+1} + T_n + T_{n-1}, \quad T_0 = 0, \quad T_1 = T_2 = 1. \end{aligned}$$

In 1963, Alladi and Hoggatt [1] constructed the tribonacci triangle, see Figure 1, and used it to derive some interesting formulas involving tribonacci numbers.

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	0	1	2	3	4	5	6	7	...
0	1								
1	1	1							
2	1	3	1						
3	1	5	5	1					
4	1	7	13	7	1				
5	1	9	25	25	9	1			
6	1	11	41	63	41	11	1		
7	1	13	61	129	129	61	13	1	
⋮		⋮							

Figure 1 : Tribonacci triangle

Denoting by $B(n, i)$ the element in the n th row and i th column of the tribonacci triangle, they showed that the sum of elements on the rising diagonals in the tribonacci triangle is a tribonacci number (see also [5, Chapter 46]), i.e.,

$$T_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} B(n - i, i). \tag{1}$$

Since $B(n, i)$ can be written as binomial sums ([2, example 16]), we can rewrite (1) as

$$T_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/3 \rfloor} \binom{i}{j} \binom{n - i - j}{i}, \tag{2}$$

which is a known identity ([7], [8], [9]).

A good deal of identities involving elements satisfying linear recurrence sequences are often conveniently derived through matrix representation. For example, the generating matrix of the tribonacci numbers is given by ([3], [9])

$$A^n = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{bmatrix}.$$

One of the most interesting generalizations of the Fibonacci numbers is given by Stakhov [10], called the Fibonacci p -numbers $F_p(n)$, defined for $p \in \mathbb{N}$ by

$$F_p(n) = F_p(n - 1) + F_p(n - p - 1) \quad (n > p) \tag{3}$$

with initial conditions $F_p(0) = 0$ and $F_p(1) = F_p(2) = \dots = F_p(p) = 1$.

Stakhov [10] also constructed the $(p + 1) \times (p + 1)$ companion matrix of the

Fibonacci p -numbers as

$$Q_p = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0. \end{bmatrix}.$$

It is not difficult to verify that the n th power of the matrix Q_p is

$$Q_p^n = \begin{bmatrix} F_p(n+1) & F_p(n-p+1) & \cdots & F_p(n-1) & F_p(n) \\ F_p(n) & F_p(n-p) & \cdots & F_p(n-2) & F_p(n-1) \\ \vdots & \vdots & & \vdots & \vdots \\ F_p(n-p+2) & F_p(n-2p+2) & \cdots & F_p(n-p) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-2p+1) & \cdots & F_p(n-p-1) & F_p(n-p) \end{bmatrix}, \tag{4}$$

which leads us to refer to Q_p as a generalized Fibonacci p -matrix. For further properties of the Fibonacci p -numbers, we refer to [4] and [10]-[11].

Our first objective here is to introduce the notion of tribonacci p -numbers, which generalizes that of Fibonacci p -numbers.

Definition 1. Let $p \in \mathbb{N}$. The tribonacci p -numbers $T_p(n)$ are defined by

$$T_p(n+2) = T_p(n+1) + T_p(n) + T_p(n-p) \quad (n \in \mathbb{Z}),$$

where $T_p(1) = 1$ and $T_p(i) = 0$ for $-p \leq i \leq 0$.

If $p = 1$, then the tribonacci p -numbers $T_p(n)$ become the classical tribonacci numbers T_n .

The following table displays the first thirteen tribonacci p -numbers for $p = 1, 2, 3, 4$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504
$T_2(n)$	0	1	1	2	3	6	10	18	31	55	96	169	296
$T_3(n)$	0	1	1	2	3	5	9	15	26	44	75	128	218
$T_4(n)$	0	1	1	2	3	5	8	14	23	39	65	109	182

Table 1

It is easy to see that $T_p(i) = F_i$ for $0 \leq i \leq p+2$, where F_n is the n th Fibonacci number.

We proceed next to construct a tribonacci p -matrix for tribonacci p -numbers and derive some identities of these numbers. We then construct the tribonacci p -triangle which generalizes the tribonacci triangle and employ it to derive an explicit formula for the tribonacci p -numbers.

2 Tribonacci p -matrix

Definition 2. For $p \in \mathbb{N}$, a tribonacci p -matrix, denoted by A_p , is the $(p + 2) \times (p + 2)$ matrix whose elements a_{ij} are given by

$$a_{11} = a_{12} = a_{1,p+2} = 1, \quad a_{i,i-1} = 1 \quad (2 \leq i \leq p + 2) \text{ and } 0 \text{ otherwise, i.e.,}$$

$$A_p = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

We define a $(p + 2) \times (p + 2)$ matrix B_n , similar to the right-hand side of (4), by

$$B_n = \begin{bmatrix} T_p(n+1) & T_p(n) + T_p(n-p) & T_p(n-p+1) & \cdots & T_p(n) \\ T_p(n) & T_p(n-1) + T_p(n-p-1) & T_p(n-p) & \cdots & T_p(n-1) \\ T_p(n-1) & T_p(n-2) + T_p(n-p-2) & T_p(n-p-1) & \cdots & T_p(n-2) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ T_p(n-p) & T_p(n-p-1) + T_p(n-2p-1) & T_p(n-2p) & \cdots & T_p(n-p-1) \end{bmatrix}.$$

Motivated by the identity (4), we state and prove one of our main results.

Theorem 1. Let $p \in \mathbb{N}$. We have

$$A_p^n = B_n \quad (n \in \mathbb{N}).$$

Proof. We prove by induction on n . It is easy to see that $B_1 = A_p$. Assume $A_p^n = B_n$ holds for some $n \geq 1$. By our assumption, we write

$$A_p^{n+1} = A_p^n A = B_n A_p.$$

From the matrix multiplication and the definition of the generalized tribonacci p -numbers, we have $B_{n+1} = B_n A_p$. Hence $A_p^{n+1} = B_{n+1}$. \square

Since $A_p^{n+m} = A_p^n A_p^m$, equating the $(1, 1)$ -entry on both sides of this matrix equation, we obtain the following corollary.

Corollary 1. For $m, n \in \mathbb{N}$, we have

$$T_p(n+m+1) = T_p(n+1)T_p(m+1) + T_p(n)T_p(m) + \sum_{i=0}^p T_p(n-p+i)T_p(m-i). \quad (5)$$

Taking $m = n$ and $m = n - 1$ in Corollary 1, we respectively get

$$T_p(2n + 1) = T_p^2(n + 1) + T_p^2(n) + \sum_{i=0}^p T_p(n - p + i)T_p(n - i) \quad (6)$$

and

$$T_p(2n) = T_p(n + 1)T_p(n) + T_p(n)T_p(n - 1) + \sum_{i=0}^p T_p(n - p + i)T_p(n - i - 1). \quad (7)$$

Taking $p = 1$ in (5), (6) and (7), we respectively get ([6, page 461])

$$\begin{aligned} T_{n+m+1} &= T_{n+1}T_{m+1} + T_nT_m + T_{n-1}T_m + T_nT_{m-1}, \\ T_{2n+1} &= T_{n+1}^2 + T_n^2 + 2T_{n-1}T_n \end{aligned}$$

and

$$T_{2n} = T_{n+1}T_n + T_nT_{n-1} + T_{n-1}^2 + T_nT_{n-2}.$$

Generalizing Theorem 1 further, we define the $(p + 3) \times (p + 3)$ matrices C_p and D_n ($n \in \mathbb{N}$) by

$$C_p = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & A_p & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \text{ and } D_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ S_p(n) & & & & \\ S_p(n-1) & & & & \\ S_p(n-2) & & B_n & & \\ \vdots & & & & \\ S_p(n-p+1) & & & & \end{bmatrix},$$

where

$$S_p(n) = \sum_{i=0}^n T_p(i) \text{ for } n \geq 0 \text{ and } S_p(k) = 0 \text{ for } k < 0.$$

We now give an extension of theorem 1.

Theorem 2. *Let $p \in \mathbb{N}$. We have*

$$C_p^n = D_n \quad (n \in \mathbb{N}).$$

Proof. We prove by induction on n . Clearly, $D_1 = C_p$. Now, assume $C_p^n = D_n$ for some $n \geq 1$. From the definition of $S_p(n)$, we can write

$$S_p(n + 1) = S_p(n) + T_p(n + 1).$$

Using $A_p^n = B_n$, matrix multiplication and inductive hypothesis, we get $D_{n+1} = D_n C_p = C_p^{n+1}$. So the proof is complete. \square

Theorem 2 gives

$$D_{n+m} = C_p^{n+m} = C_p^n C_p^m = D_n D_m,$$

which, by equating the $(2, 1)$ -entry of both sides, yields an extension of [3, Corollary 1].

Corollary 2. *For all $n, m \in \mathbb{N}$, we have*

$$S_p(n+m) = S_p(n) + T_p(n+1)S_p(m) + T_p(n)S_p(m-1) + \sum_{i=0}^p T_p(n-p+i)S_p(m-i-1).$$

Taking $m = n$ and $m = n + 1$ in Corollary 2, we respectively get

$$S_p(2n) = S_p(n) + T_p(n+1)S_p(n) + T_p(n)S_p(n-1) + \sum_{i=0}^p T_p(n-p+i)S_p(n-i-1)$$

and

$$S_p(2n+1) = S_p(n) + T_p(n+1)S_p(n+1) + T_p(n)S_p(n) + \sum_{i=0}^p T_p(n-p+i)S_p(n-i).$$

Taking $n = 1$ in Corollary 2, we get an extension of [3, Lemma 1] as

$$S_p(m+1) = 1 + S_p(m) + S_p(m-1) + S_p(m-p-1),$$

which is equivalent to

$$T_p(m+1) = 1 + 2S_p(m-1) - \sum_{i=1}^p T_p(m-i).$$

This yields the following corollary.

Corollary 3. *For $n \in \mathbb{N}$, we have*

$$S_p(n-1) = \frac{1}{2} \left(T_p(n+1) - 1 + \sum_{i=1}^p T_p(n-i) \right).$$

For $p = 1$, Corollary 3 yields

$$\sum_{i=0}^{n-1} T_i = \frac{1}{2} (T_{n+1} + T_{n-1} - 1),$$

which is a formula found in [3, Theorem 2].

3 Tribonacci p -triangle

In this last section, we introduce the tribonacci p -triangle. We start with the following definition.

Definition 3. Let $p \in \mathbb{N}$ and n, i two non-negative integers. Define

$$B_p(n, i) = \begin{cases} 1 & \text{if } i = 0, n, \\ 0 & \text{if } i > n, \\ B_p(n-1, i) + B_p(n-1, i-1) + B_p(n-p-1, i-1) & \text{if } 1 \leq i < n. \end{cases}$$

Now we are ready to introduce the main object of this section.

Definition 4. The *tribonacci p -triangle* is an array of numbers defined by

	0	1	2	3	4	...	n	...
0	$B_p(0, 0)$							
1	$B_p(1, 0)$	$B_p(1, 1)$						
2	$B_p(2, 0)$	$B_p(2, 1)$	$B_p(2, 2)$					
3	$B_p(3, 0)$	$B_p(3, 1)$	$B_p(3, 2)$	$B_p(3, 3)$				
4	$B_p(4, 0)$	$B_p(4, 1)$	$B_p(4, 2)$	$B_p(4, 3)$	$B_p(4, 4)$			
⋮			⋮					
n	$B_p(n, 0)$	$B_p(n, 1)$	$B_p(n, 2)$...		$B_p(n, n)$	
⋮			⋮					

When $p = 1$, we see that the tribonacci 1-triangle is indeed the tribonacci triangle (Figure 1). The tribonacci p -triangles for $p = 2, 3$ are given by

	0	1	2	3	4	5	6		0	1	2	3	4	5	6
0	1								0	1					
1	1	1							1	1					
2	1	2	1						2	2	1				
3	1	4	3	1					3	3	3	1			
4	1	6	8	4	1				4	5	6	4	1		
5	1	8	16	13	5	1			5	7	12	10	5	1	
6	1	10	28	32	19	6	1		6	9	21	23	15	6	1
⋮			⋮						⋮		⋮				

tribonacci 2-triangle

tribonacci 3-triangle

Observe that sums of elements on each rising diagonal in the tribonacci 2-triangle, and tribonacci 3-triangle seem to give the tribonacci 2-numbers $T_2(n)$, and tribonacci 3-numbers $T_3(n)$, respectively. The following table provides some more information.

n	$T_2(n)$	$T_3(n)$
1	1	1
2	1	1
3	1+1=2	1+1=2
4	1+2=3	1+2=3
5	1+4+1=6	1+3+1=5
6	1+6+3=10	1+5+3=9
7	1+8+8+1=18	1+7+6+1=15
8	1+10+16+4=31	1+9+12+4=26
9	1+12+28+13+1=55	1+11+21+10+1=44

Table 2

We proceed now to show that the sum of elements on each rising diagonal in the tribonacci p -triangle is equal to the tribonacci p -numbers $T_p(n)$. We need the following lemma.

Lemma 1. *Let $p \in \mathbb{N}$ and n a non-negative integer. Then*

$$B_p(n, i) = \sum_{j=0}^i \binom{i}{j} \binom{n-pj}{i}. \quad (8)$$

Proof. We prove by induction on n . The identity (8) clearly holds when $n = 0$. Assume (8) is true for $n \geq 0$ and $0 \leq i \leq n$. By Definition 3 and the inductive hypothesis, we get

$$\begin{aligned} B_p(n+1, i) &= B_p(n, i) + B_p(n, i-1) + B_p(n-p, i-1) \\ &= \sum_{j=0}^i \binom{i}{j} \binom{n-pj}{i} + \sum_{j=0}^{i-1} \binom{i-1}{j} \binom{n-pj}{i-1} + \sum_{j=0}^{i-1} \binom{i-1}{j} \binom{n-pj-p}{i-1} \\ &= \binom{n}{i} + \sum_{j=1}^{i-1} \binom{i}{j} \binom{n-pj}{i} + \binom{n-pi}{i} + \binom{n}{i-1} + \sum_{j=1}^{i-1} \binom{i-1}{j} \binom{n-pj}{i-1} \\ &\quad + \sum_{j=1}^{i-1} \binom{i-1}{j-1} \binom{n-pj}{i-1} + \binom{n-pi}{i-1} \\ &= \binom{n+1}{i} + \sum_{j=1}^{i-1} \binom{i}{j} \binom{n-pj+1}{i} + \binom{n-pi+1}{i} \\ &= \sum_{j=0}^i \binom{i}{j} \binom{n-pj+1}{i}. \end{aligned}$$

Then (8) holds for $n+1$, thereby proving the lemma. \square

Since all the terms in the summation of $B_p(n, i)$ are zero when $j > \min\{\frac{n-i}{p}, i\}$, the identity (8) can be rewritten as

$$B_p(n, i) = \sum_{j=0}^{\lfloor \frac{n+i}{p+2} \rfloor} \binom{i}{j} \binom{n-pj}{i}. \quad (9)$$

We are now ready to verify our identity.

Theorem 3. *Let $p \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$. Then*

$$T_p(n+1) = \sum_{i=0}^{\lfloor n/2 \rfloor} B_p(n-i, i). \quad (10)$$

Proof. For $0 \leq i \leq n \leq p+1$, since

$$B_p(n-i, i) = \sum_{j=0}^i \binom{i}{j} \binom{n-i-pj}{i} = \binom{n-i}{i},$$

we get

$$\sum_{i=0}^{\lfloor n/2 \rfloor} B_p(n-i, i) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} = F_{n+1} = T_p(n+1),$$

i.e., (10) holds for $0 \leq n \leq p+1$. Now assume the identity (10) holds for some $n \geq 0$. By Definition 1 and the inductive hypothesis, we get

$$\begin{aligned} T_p(n+2) &= T_p(n+1) + T_p(n) + T_p(n-p) \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} B_p(n-i, i) + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} B_p(n-i-1, i) + \sum_{i=0}^{\lfloor \frac{n-p-1}{2} \rfloor} B_p(n-i-p-1, i) \\ &= B_p(n, 0) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} B_p(n-i, i) + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} B_p(n-i, i-1) \\ &\quad + \sum_{i=1}^{\lfloor \frac{n-p+1}{2} \rfloor} B_p(n-i-p-1, i-1) \\ &= \begin{cases} B_p(n+1, 0) + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} B_p(n-i+1, i) & \text{if } n \text{ is even} \\ B_p(n+1, 0) + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} B_p(n-i+1, i) + B_p\left(\frac{n-1}{2}, \frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases} \\ &= \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} B_p(n-i+1, i), \end{aligned}$$

i.e., (10) holds for $n + 1$. Hence the proof is complete. \square

Using (9), we see that the identity (10) can also be written in terms of binomial coefficients as stated in the following corollary.

Corollary 4. *Let $p \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$. Then*

$$T_p(n+1) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n+i}{p+2} \rfloor} \binom{i}{j} \binom{n-i-pj}{i}. \quad (11)$$

Taking $p = 1$ in (11), we obtain (2) which gives an expansion of the classical tribonacci numbers.

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