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ANOTHER PROOF OF THE POSITIVITY PROBLEM FOR FOURTH ORDER RECURRENCE SEQUENCES

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Abstract

An elementary proof is given for the decidability whether each element of a sequence satisfying a fourth order linear recurrence with integer coefficients is nonnegative.

1 Introduction

Given a k^{th} order linear recurrence of the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k} \quad (n \geq k), \tag{1}$$

where $a_1, a_2, \ldots, a_k (\neq 0) \in \mathbb{Z}$ and with initial values $u_0, u_1, \ldots, u_{k-1}$, we are interested in the *Positivity Problem*: Is it possible to decide whether the sequence $(u_n)_{n\geq 0}$ is nonnegative ?, i.e., is it decidable whether $u_n \geq 0$ for all $n \geq 0$? The Positivity Problem for $k = 2$ was solved in [3], for $k = 3$ in [4] and recently, for $k = 4$ in [5], where the following result is proved.

Theorem 1. *The Positivity Problem is decidable for each sequence of integers satisfying a linear fourth order recurrence with integer coefficients.*

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The proof given in [5] is short and non-elementary due to the use of a measure-theoretic result of Bell-Gerhold, [1]. We give here a longer yet purely elementary proof of this result. This is done by first classifying all possible explicit shapes of the sequence elements, which is done in the next section. In the last two sections, Theorem 1 is verified by case analysis. For cases whose elementary proofs have already been given in [5], we merely refer to such proofs.

2 Classification of roots

Following the elaboration in Section 5.2.g on pages 170-172 of [2], consider a fourth order linear recurrence of the form (1) (with $k = 4$). Its characteristic polynomial is

$$p(x) := x^4 - a_1x^3 - a_2x^2 - a_3x - a_4 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4) \in \mathbb{Z}[x] \quad (2)$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are all the four roots of $p(x)$. Write

$$p\,(x + a_1/4) = x^4 + \alpha x^2 + \beta x + \gamma = (x - x_1)(x - x_2)(x - x_3)(x - x_4) \in \mathbb{Q}[x],$$

where x_1, x_2, x_3, x_4 are all the four roots of $p(x + a_1/4) \in \mathbb{Q}[x]$ and

$$\alpha = -\frac{3a_1^2 + 8a_2}{8},\ \beta = -\frac{a_1^3 + 4a_1a_2 + 8a_3}{8},$$
$$\gamma = -\frac{3a_1^4 + 16a_1^2a_2 + 64a_1a_3 + 256a_4}{256}.$$

Then, $x_i = \lambda_i - a_1/4 \ \ (i = 1, 2, 3, 4)$ and $x_1 + x_2 + x_3 + x_4 = 0$. Let

$$u = (x_1 + x_2)(x_3 + x_4) = -(x_1 + x_2)^2, \quad (3)$$
$$v = (x_1 + x_3)(x_2 + x_4) = -(x_1 + x_3)^2, \quad (4)$$
$$w = (x_1 + x_4)(x_2 + x_3) = -(x_1 + x_4)^2. \quad (5)$$

We construct a polynomial of degree 3 whose roots are u, v, and w. It coefficients are found from the elementary symetric functions $\sum_{i<j} x_ix_j = \alpha,\ \sum_{i<j<k} x_ix_jx_k = -\beta,\ x_1x_2x_3x_4 = \gamma$ by computing $u + v + w = 2\alpha,\ uv + uw + vw = \alpha^2 - 4\gamma,\ uvw = -\beta^2$. This polynomial is

$$q(x) := x^3 - 2\alpha x^2 + (\alpha^2 - 4\gamma)x + \beta^2 = (x - u)(x - v)(x - w) \in \mathbb{Q}[x]. \quad (6)$$

Since

$$u - v = (x_1 - x_4)(x_3 - x_2) = (\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2),$$
$$u - w = (x_1 - x_3)(x_4 - x_2) = (\lambda_1 - \lambda_3)(\lambda_4 - \lambda_2),$$
$$v - w = (x_1 - x_2)(x_4 - x_3) = (\lambda_1 - \lambda_2)(\lambda_4 - \lambda_3),$$

the three polynomials $q(x)$, $p(x+a_1/4)$ and $p(x)$ have the same discriminant

$$\begin{aligned}\mathcal{D} &= \operatorname{disc}(q) = \operatorname{disc}\left(p(x+a_1/4)\right) = \operatorname{disc}(p)\\ &= 16\alpha^4\gamma - 4\alpha^3\beta^2 - 128\alpha^2\gamma^2 + 144\alpha\beta^2\gamma - 27\beta^4 + 256\gamma^3 \quad ([6, p.\ 192])\\ &= (u-v)^2(u-w)^2(v-w)^2\end{aligned}$$

To find the roots x_1, x_2, x_3, x_4 of the equation $p(x+a_1/4) = 0$, we first solve equation (6) to determine the roots u, v, w of $q(x) = 0$. From (3), (4), (5), setting

$$x_1 + x_2 := u' = \text{square root of} - u \tag{7}$$

$$x_1 + x_3 := v' = \text{square root of} - v \tag{8}$$

$$x_1 + x_4 := w' = \text{square root of} - w. \tag{9}$$

where these square roots must be chosen so that

$$u'v'w' = (x_1+x_2)(x_1+x_3)(x_1+x_4) = x_1^2 \sum_i x_i + \sum_{i<j<k} x_i x_j x_k = -\beta, \tag{10}$$

we get

$$\begin{aligned} u' + v' + w' &= 3x_1 + x_2 + x_3 + x_4 = 2x_1, \quad -u' + v' - w' = 2x_3,\\ u' - v' - w' &= x_2 - x_1 - x_3 - x_4 = 2x_2, \quad -u' - v' + w' = 2x_4.\end{aligned}$$

The solutions of $p(x+a_1/4) = 0$ are given by

$$x_1 = \frac{1}{2}(u' + v' + w') \tag{11}$$

$$x_2 = \frac{1}{2}(u' - v' - w') \tag{12}$$

$$x_3 = \frac{1}{2}(-u' + v' - w') \tag{13}$$

$$x_4 = \frac{1}{2}(-u' - v' + w'). \tag{14}$$

To determine the nature of these roots, we subdivide our consideration into three cases depending on the signs of $\mathcal{D}$.

Case I: $\mathcal{D} > 0$

As seen in [4], the three roots u, v, w of

$$q(x) = x^3 - 2\alpha x^2 + (\alpha^2 - 4\gamma)x + \beta^2 = (x-u)(x-v)(x-w) \in \mathbb{Q}[x]$$

are distinct real numbers. We subdivide into further subcases depending on whether $\beta = 0$.

I.1 $\beta = 0$.
Then $q(x)$ has a zero root, say $u = 0$, implying by (7) that $u' = 0$, and so

$$\begin{aligned} q(x) &= x(x^2 - 2\alpha x + (\alpha^2 - 4\gamma)) \\ &= x(x^2 - (v+w)x + vw) = x(x-v)(x-w). \end{aligned}$$

I.1.1 $\alpha^2 - 4\gamma \geq 0$. Then v, w have the same sign, distinct and nonzero.

I.1.1.1 $\alpha \leq 0$. Then v, w are two distinct negative real numbers which by (8), (9) implies that v', w' are nonzero real numbers with $v' \neq \pm\, w'$. By (11),(12),(13) and (14), we have $x_1 = \frac{1}{2}(v'+w'),\ x_2 = \frac{1}{2}(-v'-w'),\ x_3 = \frac{1}{2}(v'-w'),\ x_4 = \frac{1}{2}(-v'+w)$. Hence, $p\,(x + a_1/4)$ has four distinct nonzero real roots.

I.1.1.2 $\alpha > 0$. Then v, w are two distinct positive real numbers, which by (8), (9) implies that $v' = \Box i,\ w' = \heartsuit i$ are purely imaginary with $\Box \neq \pm\, \heartsuit$ being two nonzero real numbers. By (11),(12),(13) and (14), we have $x_1 = \frac{1}{2}(\Box + \heartsuit)i,\ x_2 = -\frac{1}{2}(\Box + \heartsuit)i = \bar{x}_1,\ x_3 = \frac{1}{2}(\Box - \heartsuit)i,\ x_4 = -\frac{1}{2}(\Box - \heartsuit)i = -\bar{x}_3$. Thus, the roots of $p(x + a_1/4)$ consists of two nonzero distinct purely imaginary conjugate pairs.

I.1.2 $\alpha^2 - 4\gamma < 0$. Then v and w are two nonzero real numbers of opposite sign, say $v \in \mathbb{R}^-$ and $w \in \mathbb{R}^+$, which by (8), (9) implies that v' is a nonzero real number and $w' = \heartsuit i$ is nonzero purely imaginary. By (11),(12),(13) and (14), we have $x_1 = \frac{1}{2}(v' + \heartsuit i),\ x_2 = \frac{1}{2}(-v' - \heartsuit i) = -x_1,\ x_3 = \frac{1}{2}(v' - \heartsuit i) = \bar{x}_1,\ x_4 = \frac{1}{2}(-v' + \heartsuit i) = -\bar{x}_1$. Thus, the roots of $p(x + a_1/4)$ consists of two distinct, nonzero, non-real complex conjugate pairs.

I.2 $\beta \neq 0$.
Then $q(x) = x^3 - 2\alpha x^2 + (\alpha^2 - 4\gamma)x + \beta^2 = (x-u)(x-v)(x-w) \in \mathbb{Q}[x]$ has all distinct three roots $u, v, w \in \mathbb{R} \setminus \{0\}$.

I.2.1 $\alpha^2 - 4\gamma \geq 0$.

I.2.1.1 $\alpha \leq 0$. Since the coefficients of $q(x)$ are all nonnegative, by Descartes' rule of signs, the three roots $u, v, w \in \mathbb{R}^-$, which by (7), (8), (9) implies that u', v', w' are three nonzero real numbers with $x_1 + x_2 = u' \neq \pm\, v' = \pm(x_1 + x_3),\ x_1 + x_2 = u' \neq \pm\, w' = \pm(x_1 + x_4),\ x_1 + x_3 = v' \neq \pm\, w' = \pm(x_1 + x_4)$. By (11),(12),(13) and (14), we have $x_1 = \frac{1}{2}(u' + v' + w'),\ x_2 = \frac{1}{2}(u' - v' - w'),\ x_3 = \frac{1}{2}(-u' + v' - w'),\ x_4 = \frac{1}{2}(-u' - v' + w')$. Hence, $p(x + a_1/4)$ has four distinct real roots x_1, x_2, x_3, x_4.

I.2.1.2 $\alpha > 0$. Since $\beta^2 > 0$, we know that $uvw > 0$ and so two roots of $q(x)$, say u, v, are negative real numbers and one root, say w, is a positive real number, which by (7), (8), (9) imply that $u', v' \in \mathbb{R},\ u' \neq \pm\, v'$ and $w' = \heartsuit i,\ \heartsuit \in \mathbb{R} \setminus \{0\}$. Thus, $x_1 =$

$\frac{1}{2}(u' + v' + \heartsuit i),\ x_2 = \frac{1}{2}(u' - v' - \heartsuit i),\ x_3 = \frac{1}{2}(-u' + v' - \heartsuit i),\ x_4 = \frac{1}{2}(-u' - v' + \heartsuit i)$. Since $p(x + a_1/4) \in \mathbb{Q}[x]$, its four complex roots x_1, x_2, x_3, x_4 must occur in two conjugate pairs. From their shapes, there are two possibilities, viz., ($\bar{x}_1 = x_2$ and $\bar{x}_4 = x_3$) or ($\bar{x_1} = x_3$ and $\bar{x}_4 = x_2$).

If $\bar{x}_1 = x_2$, then $v' = 0,\ u' \neq 0$ and so $x_1 = \frac{1}{2}(u' + i\heartsuit) \in \mathbb{C} \setminus \mathbb{R},\ x_2 = \bar{x}_1,\ x_3 = -x_1,\ x_4 = -\bar{x}_1$. If $\bar{x}_1 = x_3$, then $u' = 0,\ v' \neq 0$ and we deduce that the solutions are of the same form as in the last possibility, i.e., two distinct complex conjugate pairs.

I.2.2 $\alpha^2 - 4\gamma < 0$. Since $\beta^2 > 0$, we know that $uvw > 0$ and so two roots of $q(x)$, say u, v are negative real numbers, and one root, say w is a positive real number, which by (7), (8), (9) imply that $u', v' \in \mathbb{R},\ u' \neq \pm\ v'$ and $w' = \heartsuit i,\ \heartsuit \in \mathbb{R} \setminus \{0\}$. The roots of $p(x + a_1/4)$ are of the same shape as those in the sub-case [I.2.1.2], i.e., two distinct complex conjugate pairs.

Case II: $\mathcal{D} = 0$

As seen in [4], the three roots u, v, w are real numbers and since

$$\begin{aligned}\mathcal{D} &= 16\alpha^4\gamma - 4\alpha^3\beta^2 - 128\alpha^2\gamma^2 + 144\alpha\beta^2\gamma - 27\beta^4 + 256\gamma^3 \quad ([6, p.\ 192])\\ &= (u-v)^2(u-w)^2(v-w)^2,\end{aligned}$$

we must have

$$u = v \quad \text{or} \quad u = w \quad \text{or} \quad v = w. \tag{15}$$

Recall that $q(x) = x^3 - 2\alpha x^2 + (\alpha^2 - 4\gamma)x + \beta^2 = (x-u)(x-v)(x-w) \in \mathbb{Q}[x]$. Let $\alpha^* := -(\alpha^2 + 12\gamma)/3$.

II.1 If $\alpha^* = 0$, then $\gamma = -\alpha^2/12$ and putting this into the expression for $\mathcal{D}$, we get $\beta^2 = -8\alpha^3/27$. Substituting these values into the expression for $q(x)$, we find that $q(x) = (x - 2\alpha/3)^3$ showing that $u = v = w = 2\alpha/3$.

II.1.1 If $\beta = 0$, then $u = v = w = 0$. By (7),(8) and (9), we get $u' = v' = w' = 0$. Thus, (11),(12),(13) and (14) yield $x_1 = x_2 = x_3 = x_4 = 0$.

II.1.2 If $\beta \neq 0$, then $\beta^2 = -uvw = -u^3 > 0$, so that $u = v = w \in \mathbb{R}^-$. By (7),(8) and (9), we get $u' = v' = w' \in \mathbb{R} \setminus \{0\}$. Thus, (11),(12),(13) and (14) yield $x_1 = \frac{3u'}{2},\ x_2 = \frac{-u'}{2} = x_3 = x_4$.

II.2 If $\alpha^* \neq 0$, we have u, v and w are not all the same.

II.2.1 If $\beta = 0$, then $uvw = 0$; without loss of generality, assume $u = 0$. By (15), we deduce $u = v = 0$. From the shape of $q(x)$, we must have $\alpha^2 - 4\gamma = 0$ and $w = 2\alpha$. By (7),(8) and (9), we get $u' = v' =$

0, w' = square root of -2α, so that (11),(12),(13) and (14) yield $x_1 = w'/2 = x_4,\ x_2 = -w'/2 = x_3$. Clearly, $\alpha \neq 0$, for otherwise $u = v = w = 0$. Let $r = \frac{1}{2}\sqrt{|-2\alpha|} \in \mathbb{R} \setminus \{0\}$. Taking into account that α can be either positive or negative, we find that there are two possible set of solutions $\{x_1, x_2, x_3, x_4\} = \{r,\ r,\ -r,\ -r\}$ or $\{ir,\ ir,\ -ir,\ -ir\}$.

II.2.2 If $\beta \neq 0$, since $\beta \in \mathbb{Q}$, we deduce that $-uvw > 0$ so that two elements, say $u, v \in \mathbb{R} \setminus \{0\}$, are of the same sign (and so by (15), $u = v$), while $w \in \mathbb{R} \setminus \{0\}$ is negative.

If $u = v \in \mathbb{R}^-,\ w \in \mathbb{R}^-$, then (7),(8), (9) show that $u' = v', w' \in \mathbb{R} \setminus \{0\}$. Thus, (11),(12),(13),(14) yield four real roots $x_1 = \frac{1}{2}(2u' + w'),\quad x_2 = -\frac{w'}{2} = x_3,\quad x_4 = \frac{1}{2}(-2u' + w')$. Apart from $x_2 = x_3$, all other roots are distinct for if $x_1 = x_2$, then $u' = -w'$ implying that $(v =)\ u = w$, contradicting the fact that all three roots are not the same; if $x_1 = x_4$, then $u' = 0$, implying that $u = v = 0$ which contradicts $\beta \neq 0$; if $x_2 = x_4$, then $u' = w'$ implying that $(v =)\ u = w$, again a contradiction.

If $u = v \in \mathbb{R}^+,\ w \in \mathbb{R}^-$, then (7),(8), (9) show that $w' = 2W_1 \in \mathbb{R} \setminus \{0\}$, and we have $u' = \pm v' = ir_2$ ($r_2 \in \mathbb{R} \setminus \{0\}$). If $u' = v'$, then (11),(12),(13),(14) yield $\{x_1,\ x_2,\ x_3,\ x_4\} = \{W_1 + ir_2,\ -W_1,\ -W_1,\ W_1 - ir_2\}$, while if $u' = -v'$, we get $\{x_1,\ x_2,\ x_3,\ x_4\} = \{W_1,\ -W_1 + ir_2,\ -W_1 - ir_2,\ W_1\}$. Clearly, $x_1 \neq x_2,\ x_1 \neq x_4,\ x_2 \neq x_4$ in both situations.

Case III: $\mathcal{D} < 0$

As seen in [4], $q(x)$ has one real root and two complex conjugate roots, say $u \in \mathbb{R},\ v = \overline{w} \in \mathbb{C} \setminus \mathbb{R}$.

III.1 $\beta = 0$ (so that $uvw = 0$).

If $u \neq 0$, then $v = w = 0$, contradicting $v = \overline{w} \in \mathbb{C} \setminus \mathbb{R}$ and so we must have $u = 0$. Thus, (7) yields $u' = 0$. Since $v = \overline{w}$, (8) and (9) show that either $v' = \overline{w'} \in \mathbb{C} \setminus \mathbb{R}$ or $v' = -\overline{w'} \in \mathbb{C} \setminus \mathbb{R}$. As (11),(12),(13) and (14) are symmetric in v' and w', we can assume that $v' = \overline{w'} = r_3 + ir_4$ ($r_3, r_4 (\neq 0) \in \mathbb{R}$), as the other alternative yields roots of the same shape. Thus, (11),(12),(13) and (14) yield $x_1 = (v' + w')/2 = r_3 = -x_2,\ x_3 = (v' - w')/2 = ir_4 = -x_4$. Here, $r_3 \neq 0$, for otherwise $x_1 = x_2 = 0$ implying $D = 0$ which is a contradiction.

III.2 $\beta \neq 0$ (so that $u \in \mathbb{R} \setminus \{0\}$).

(a) If $u < 0$, then $u' = r \in \mathbb{R} \setminus \{0\}$. Similar to the last case, we can assume that $v' = \overline{w'} = r_3 + ir_4$ ($r_3, r_4(\neq 0) \in \mathbb{R}$), as the possibility $v' = -\overline{w'}$ yields roots of the same shape. Thus, (11),(12),(13) and (14) yield $x_1 = (r + 2r_3)/2,\ x_2 = (r - 2r_3)/2,\ x_3 = (-r + i2r_4)/2,\ x_4 = (-r - i2r_4)/2$. Here again $r_3 \neq 0$ (so that $x_1 \neq x_2$), for otherwise $x_1 = x_2$ implying $\mathcal{D} = 0$, a contradiction

(b) If $u > 0$, then $u' = ir$ $(r \in \mathbb{R} \setminus \{0\})$. Again by choosing the sign of u' appropriately, we can assume that $v' = \overline{w'} = r_3 + ir_4$ $(r_3, r_4 (\neq 0) \in \mathbb{R})$. Thus, (11),(12),(13) and (14) yield $x_1 = (2r_3 + ir)/2,\ x_2 = (-2r_3 + ir)/2,\ x_3 = i(-r + 2r_4)/2,\ x_4 = i(-r - 2r_4)/2$. Since $p(x + a_1/4) \in \mathbb{Q}[x]$, its four complex roots x_1, x_2, x_3, x_4 must occur in two conjugate pairs. From their shapes, there are two possibilities, viz., $\overline{x_1} = x_3$ or $\overline{x_1} = x_4$. In either case, we deduce that $r_4 = 0$, a contradiction. There is no solution in this case.

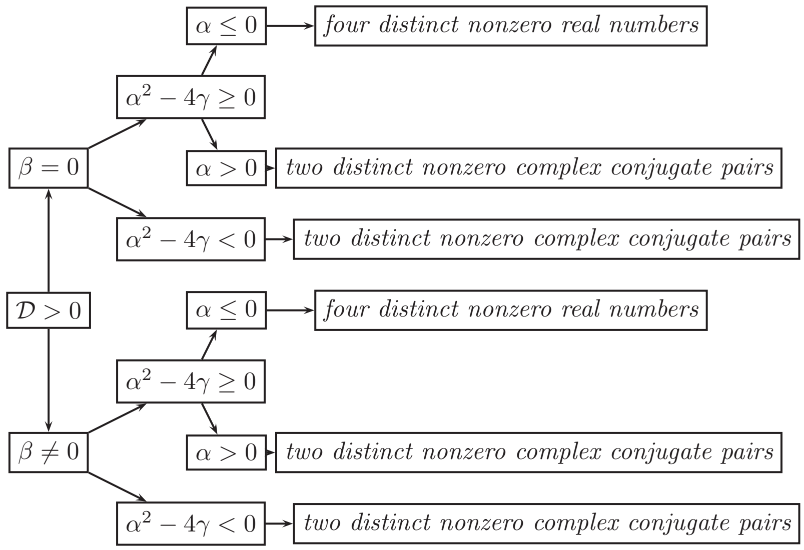
From the above information about the nature of the four zeros of $p(x + a_1/4)$, we deduce the following result about the four zeros of $p(x)$.

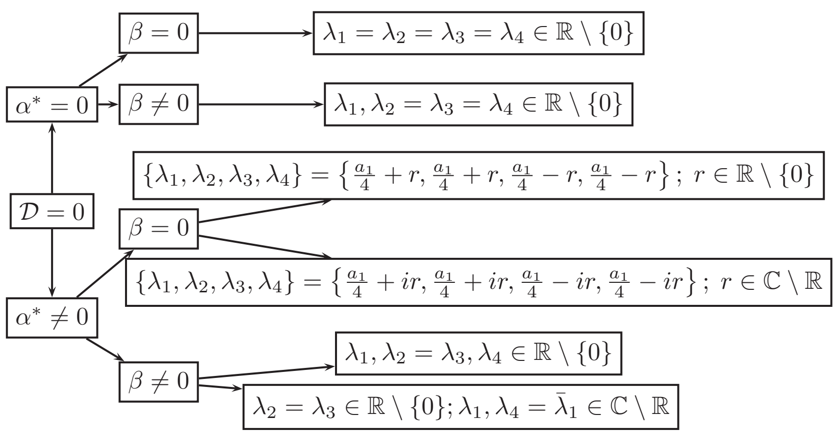
Theorem 2. *Let*

$$\begin{aligned}
p(x) :=& x^4 - a_1x^3 - a_2x^2 - a_3x - a_4 \\
=& (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4) \in \mathbb{Z}[x],\ a_4 \neq 0 \\
p\left(x + a_1/4\right) :=& x^4 + \alpha x^2 + \beta x + \gamma = (x - x_1)(x - x_2)(x - x_3)(x - x_4) \in \mathbb{Q}[x] \\
\alpha =& -\frac{3a_1^2 + 8a_2}{8},\ \alpha^* := -\frac{\alpha^2 + 12\gamma}{3},\ \beta = -\frac{a_1^3 + 4a_1a_2 + 8a_3}{8}, \\
\gamma =& -\frac{3a_1^4 + 16a_1^2a_2 + 64a_1a_3 + 256a_4}{256} \\
\mathcal{D} =& \operatorname{disc}(p(x)) = \operatorname{disc}\left(p\left(x + \frac{a_1}{4}\right)\right) \\
=& 16\alpha^4\gamma - 4\alpha^3\beta^2 - 128\alpha^2\gamma^2 + 144\alpha\beta^2\gamma - 27\beta^4 + 256\gamma^3.
\end{aligned}$$

Then the nature of the four zeros of $p(x)$ can be classified as

$$\boxed{\mathcal{D} < 0} \longrightarrow \boxed{\text{two distinct real roots and two complex conjugate roots}}$$





Using the information about the four zeros of $p(x)$, we now summarize the shapes of the sequence elements.

Theorem 3. *Let $\{u_n\}_{n\geq 0}$ be a sequence of elements satisfying a fourth order linear recurrence of the form* (1) *(with $k = 4$), let $p(x)$ as in Theorem 2 be its characteristic polynomial having $\lambda_1, \ldots, \lambda_4$ as all its (non-vanishing) zeros and let $\mathcal{D}$ be the discriminant of $p(x)$. Then the general term of the sequence takes one of the following forms*

$\mathbf{C(r_1r_2r_3r_4)}$: $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + D\lambda_4^n,$ *where* $A, B, C, D \in \mathbb{R}$

$\mathcal{D} > 0$

$\mathbf{C(z_1\bar{z}_1z_2\bar{z}_2)}$: $u_n = A\lambda_1^n + \bar{A}\bar{\lambda}_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n,$ *where* $A, B \in \mathbb{C}$

$\mathcal{D} = 0$

$\mathbf{C(r_1^4)}$

$u_n = (A + Bn + Cn^2 + Dn^3)\lambda_1^n,$ *where* $A, B, C, D \in \mathbb{R}$

$\mathbf{C(r_1r_2^3)}$

$u_n = A\lambda_1^n + (B + Cn + Dn^2)\lambda_2^n,$ *where* $A, B, C, D \in \mathbb{R}$

$\mathbf{C(r_1^2r_2^2)}$

$u_n = (A + Bn)\lambda_1^n + (C + Dn)\lambda_2^n,$ *where* $A, B, C, D \in \mathbb{R}$

$\mathbf{C(z^2\bar{z}^2)}$

$u_n = (A + Bn)\lambda^n + (\bar{A} + \bar{B}n)\bar{\lambda}^n,$ *where* $A, B \in \mathbb{C}$

$\mathbf{C(r_1r_2r_3^2)}$

$u_n = A\lambda_1^n + B\lambda_2^n + (C + Dn)\lambda_3^n,$ *where* $A, B, C, D \in \mathbb{R}$

$\mathbf{C(r_1^2z\bar{z})}$

$u_n = (A + Bn)\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n,$ *where* $A, B, \in \mathbb{R},\ C \in \mathbb{C}$

$\mathcal{D} < 0$

$\mathbf{C(r_1r_2z\bar{z})}$

$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n,$ *where* $A, B, \in \mathbb{R},\ C \in \mathbb{C}$

3 Proof of Theorem 1 when Char(z) has only real roots.

In this case, the general term of the sequence is

$$u_n = P_1(n)\lambda_1^n + P_2(n)\lambda_2^n + \cdots + P_m(n)\lambda_m^n \quad (n \geq 0,\ m \leq 4),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct nonzero real numbers and $P_i(n) = A_{i,1}+A_{i,2}n+\cdots+A_{i,\ell_i}n^{\ell_i-1} \in \mathbb{R}[x]$ $(\ell_i \in \mathbb{N};\ i = 1, 2, \ldots, m;\ A_{i,\ell_i} \neq 0)$, with $\ell_1 + \ell_2 + \cdots + \ell_m = 4$. We have two possibilities to consider.

3.1 There are two roots λ_i, λ_j $(i, j \in \{1, 2, \ldots, m;\ i \neq j)$ such that $|\lambda_i| = |\lambda_j|$.

See the simple proof given in [5, Section 3.1].

3.2 All roots have different absolute values.

Without loss of generality, assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_m|$. Here,

$$u_n = \lambda_1^n \{P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n\} \quad (n \geq 0).$$

We treat two subcases depending on the sign of λ_1.

3.2.1 $\lambda_1 < 0$.

Since $\operatorname{sign}(P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n) = \operatorname{sign}(A_{1,\ell_1})$ when n is large enough and $\operatorname{sign}(\lambda_1^n)$ oscillates, the sequence (u_n) is nonnegative only when $A_{1,\ell_1} = 0$, which contradicts the definition of $A_{1,\ell_1} \neq 0$.

3.3 $\lambda_1 > 0$.

See the simple proof given in [5, Section 3.1].

4 Proof of Theorem 1 when $\mathrm{Char}(z)$ has non-real roots.

4.1 $\mathrm{C}(z_1\bar{z}_1z_2\bar{z}_2)$

In this case, the general term of the sequence is

$$u_n = A\lambda_1^n + \bar{A}\bar{\lambda}_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n \quad (n \geq 0), \tag{16}$$

where $A, B \in \mathbb{C}$ and $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_1 = |\lambda_1|e^{i\theta_1}$, $\lambda_2 = |\lambda_2|e^{i\theta_2}$, $A = |A|e^{i\varphi_1}$ and $B = |B|e^{i\varphi_2}$ where $\theta_1, \theta_2, \varphi_1, \varphi_2 \in [-\pi, \pi]$, $\theta_1, \theta_2 \notin \{-\pi, 0\}$ so that

$$u_n = 2\{|\lambda_1|^n|A|\cos(\varphi_1 + n\theta_1) + |B||\lambda_2|^n\cos(\varphi_2 + n\theta_2)\}.$$

Without loss of generality, we need only treat two possibilities $|\lambda_1| > |\lambda_2|$ and $|\lambda_1| = |\lambda_2|$.

4.1.1 $|\lambda_1| > |\lambda_2|$ (> 0).

Here, $u_n = 2|\lambda_1|^n\{|A|\cos(\varphi_1 + n\theta_1) + |B|(|\lambda_2/\lambda_1|)^n\cos(\varphi_2 + n\theta_2)\}$. Thus, (u_n) is nonnegative if and only if

$$|A|\cos(\varphi_1 + n\theta_1) + |B|(|\lambda_2|/|\lambda_1|)^n\cos(\varphi_2 + n\theta_2) \geq 0. \tag{17}$$

By Lemma 2.1 of [5], $|A|\cos(\varphi_1+n\theta_1)$ takes some fixed positive and negative values for infinitely many n provided $A\neq 0$, and since $|B|\left(|\lambda_2/\lambda_1|\right)^n\cos(\varphi_2+n\theta_2)\to 0$ $(n\to\infty)$, the requirement (17) holds if and only if $A=0$, and so $u_n=B\lambda_2^n+\bar{B}\bar{\lambda}_2^n$ $(n\geq 0)$, which is of the form (HHH3) ([5, Lemma 2.3]).

4.1.2 $|\lambda_1|=|\lambda_2|$.

Here, $u_n=2|\lambda_1|^n\left\{|A|\cos(\varphi_1+n\theta_1)+|B|\cos(\varphi_2+n\theta_2)\right\}$. Notice that the arguments used in the preceding subcase do not work here. This is the first situation where our analysis has to get back to the shape of the roots of the characteristic polynomial. Tracing back to the proofs of Theorem 2 and 3, we see that the case $\mathrm{C}(z_1\bar{z}_1z_2\bar{z}_2)$ occurs in Case I: $\mathcal{D}>0$, with two distinct complex conjugate pairs, which are labeled I.1.1.2, I.1.2, I.2.1.2 and I.2.2. Using the notation of Theorem 2, the corresponding information is as follows:

- I.1.1.2 $\beta=0,\ \alpha^2-4\gamma\geq 0$ and $\alpha>0$. We have $x_1=\frac{1}{2}(\Box+\heartsuit)i,\ x_2=-\frac{1}{2}(\Box+\heartsuit)i=\bar{x}_1,\ x_3=\frac{1}{2}(\Box-\heartsuit)i,\ x_4=-\frac{1}{2}(\Box-\heartsuit)i=-\bar{x}_3$, with $\Box\neq\pm\heartsuit$ being two nonzero real numbers. The roots of the characteristic polynomial, denoted for unambiguity by $\tilde{\lambda}$, are $\tilde{\lambda}_1=\frac{a_1}{4}+\frac{1}{2}(\Box+\heartsuit)i,\ \tilde{\lambda}_2=\frac{a_1}{4}-\frac{1}{2}(\Box+\heartsuit)i=\bar{\tilde{\lambda}}_1,\ \tilde{\lambda}_3=\frac{a_1}{4}+\frac{1}{2}(\Box-\heartsuit)i,\ \tilde{\lambda}_4=\frac{a_1}{4}-\frac{1}{2}(\Box-\heartsuit)i=-\bar{\tilde{\lambda}}_3$. Combining with the condition and notation of our on-going case, we get

$$|\tilde{\lambda}_1|^2=|\tilde{\lambda}_3|^2\Longleftrightarrow\left(\frac{a_1}{4}\right)^2+\frac{1}{4}(\Box+\heartsuit)^2=\left(\frac{a_1}{4}\right)^2+\frac{1}{4}(\Box-\heartsuit)^2$$
$$\Longleftrightarrow\Box\heartsuit=0,$$

  which is a contradiction and we are done in this case.

- I.1.2 $\beta=0,\ \alpha^2-4\gamma<0$. We have $x_1=\frac{1}{2}(v'+\heartsuit i),\ x_2=\frac{1}{2}(-v'-\heartsuit i)=-x_1,\ x_3=\frac{1}{2}(v'-\heartsuit i)=\bar{x}_1,\ x_4=\frac{1}{2}(-v'+\heartsuit i)=-\bar{x}_1$. Then $\tilde{\lambda}_1=\left(\frac{a_1}{4}+\frac{v'}{2}\right)+\frac{\heartsuit}{2}i,\ \tilde{\lambda}_2=\left(\frac{a_1}{4}-\frac{v'}{2}\right)-\frac{\heartsuit}{2}i,\ \tilde{\lambda}_3=\left(\frac{a_1}{4}+\frac{v'}{2}\right)-\frac{\heartsuit}{2}i=\bar{\tilde{\lambda}}_1,\ \tilde{\lambda}_4=\left(\frac{a_1}{4}-\frac{v'}{2}\right)+\frac{\heartsuit}{2}i=\bar{\tilde{\lambda}}_2$. Combining with the condition and notation of our on-going case, we get

$$|\tilde{\lambda}_1|^2=|\tilde{\lambda}_2|^2\Longleftrightarrow\left(\frac{a_1}{4}+\frac{v'}{2}\right)^2+\left(\frac{\heartsuit}{2}\right)^2=\left(\frac{a_1}{4}-\frac{v'}{2}\right)^2+\left(\frac{\heartsuit}{2}\right)^2$$
$$\Longleftrightarrow\frac{a_1v'}{2}=0\ \ (v'\neq 0)\Longleftrightarrow a_1=0.$$

  Thus, $\tilde{\lambda}_1=\frac{v'}{2}+\frac{\heartsuit}{2}i,\quad\tilde{\lambda}_2=-\frac{v'}{2}-\frac{\heartsuit}{2}i=-\tilde{\lambda}_1$.

- I.2.1.2 $\beta\neq 0,\ \alpha^2-4\gamma\geq 0$ and $\alpha>0$. We have $x_1=\frac{1}{2}(u'+i\heartsuit)\in\mathbb{C}\setminus\mathbb{R},\ x_2=\bar{x}_1,\ x_3=-x_1,\ x_4=-\bar{x}_1$ yielding $\frac{a_1}{4}+\frac{u'}{2}+i\frac{\heartsuit}{2}=\tilde{\lambda}_1=$

$\bar{\tilde{\lambda}}_2$, $\frac{a_1}{4} - \frac{u'}{2} - i\frac{\heartsuit}{2} = \tilde{\lambda}_3 = \bar{\tilde{\lambda}}_4$. Combining with the condition and notation of our on-going case, we get

$$|\tilde{\lambda}_1|^2 = |\tilde{\lambda}_3|^2 \Longleftrightarrow \left(\frac{a_1}{4} + \frac{u'}{2}\right)^2 + \left(\frac{\heartsuit}{2}\right)^2 = \left(\frac{a_1}{4} - \frac{u'}{2}\right)^2 + \left(\frac{\heartsuit}{2}\right)^2$$
$$\Longleftrightarrow \frac{a_1 u'}{2} = 0 \ \ (u' \neq 0) \Longleftrightarrow a_1 = 0.$$

Thus, $\tilde{\lambda}_1 = \frac{u'}{2} + \frac{\heartsuit}{2}i$, $\tilde{\lambda}_3 = -\frac{u'}{2} - \frac{\heartsuit}{2}i = -\tilde{\lambda}_1$.

- I.2.2 $\beta \neq 0$, $\alpha^2 - 4\gamma < 0$. The roots of $p(x + a_1/4)$ are of the same shape as those in the subcase I.2.1.2.

Collecting together all possibilities in I.1.1.2, I.1.2, I.2.1.2 and I.2.2, the two roots in (16) can only be of the form $\lambda_2 = -\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$, and the general term of the sequence in (16) is thus of the form $u_n = A\lambda_1^n + \bar{A}\bar{\lambda}_1^n + B(-\lambda_1)^n + \bar{B}(-\bar{\lambda}_1)^n$, i.e., for $k \geq 0$,

$$u_{2k} = (A+B)\lambda_1^{2k} + (\bar{A}+\bar{B})\bar{\lambda}_1^{2k}, \quad u_{2k+1} = (A-B)\lambda_1^{2k+1} + (\bar{A}-\bar{B})\bar{\lambda}_1^{2k+1}.$$

Both u_{2k} and u_{2k+1} are of the form (HHH3) and so are decidable by [5, Lemma 2.3].

4.2 $\mathbf{C}(z^2\bar{z}^2)$

In this case, the general term of the sequence is $u_n = (A + Bn)\lambda_1^n + (\bar{A} + \bar{B}n)\bar{\lambda}_1^n$ $(n \geq 0)$, where $A, B \in \mathbb{C}$ and $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_1 = |\lambda_1|e^{i\theta}$, $A = |A|e^{i\varphi_1}$, $B = |B|e^{i\varphi_2}$ where $\theta, \varphi_1, \varphi_2 \in [-\pi, \pi)$, $\theta \neq \{-\pi, 0\}$. Thus,

$$u_n = 2|\lambda_1|^n \left\{|A|\cos(\varphi_1 + n\theta) + n|B|\cos(\varphi_2 + n\theta)\right\}.$$

Then the sequence (u_n) is nonnegative if and only if for all n,

$$|A|\cos(\varphi_1 + n\theta) + n|B|\cos(\varphi_2 + n\theta) \geq 0. \tag{18}$$

Since $\mathrm{sign}(|A|\cos(\varphi_1 + n\theta) + n|B|\cos(\varphi_2 + n\theta)) = \mathrm{sign}(\cos(\varphi_2 + n\theta))$ when n is large enough, provided $B\cos(\varphi_2 + n\theta) \neq 0$. By [5, Lemma 2.2], $\cos(\varphi_2 + n\theta)$ takes some positive and some negative values for infinitely many $n \in \mathbb{N}$. Thus, (18) holds only when $B = 0$, and so $u_n = A\lambda_1^n + \bar{A}\bar{\lambda}_1^n$, which is of the form (HHH3) ([5, Lemma 2.3]).

4.3 $\mathbf{C}(r_1^2 z\bar{z})$

In this case, the general term of the sequence is $u_n = (A + Bn)\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$ $(n \geq 0)$, where $A, B, \lambda_1 (\neq 0) \in \mathbb{R}$, $C \in \mathbb{C}$ and $\lambda_3 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_3 = |\lambda_3|e^{i\theta}$, $C = |C|e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi)$, $\theta \notin \{-\pi, 0\}$ so that

$$u_n = (A + Bn)\lambda_1^n + 2|C||\lambda_3|^n \cos(\varphi + n\theta).$$

We subdivide into three subcases depending on the absolute values of $|\lambda_1|$ and $|\lambda_3|$.

4.3.1 $|\lambda_1| = |\lambda_3|$.

There are two further possibilities.

4.3.1(1) $\lambda_1 < 0$.
Here, $u_n = |\lambda_1|^n\{(-1)^n(A + Bn) + 2|C|\cos(\varphi + n\theta)\}$ $(n \geq 0)$. Since

$$(-1)^n(A + Bn) + 2|C|\cos(\varphi + n\theta) \to \pm\infty \quad (n \to \infty)$$

according as n is even or odd provided $B \neq 0$, the sequence (u_n) is nonnegative only when $B = 0$ and so $u_n = A\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4) ([5, Lemma2.3]).

4.3.1(2) $\lambda_1 > 0$. See subcase 1 of $C(r_1^2 z\bar{z})$ in [5].

4.3.2 $|\lambda_1| > |\lambda_3|$.

Rewrite the general term of the sequence as

$$u_n = \lambda_1^n\left\{A + Bn + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta)\right\} \quad (n \geq 0).$$

We consider two possibilities corresponding to the signs of λ_1.

4.3.2(1) $\lambda_1 < 0$. Since $\text{sign}(A+Bn+2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi+n\theta)) = \text{sign}(B)$ when n is large enough provided $B \neq 0$, and $\text{sign}(\lambda_1^n)$ oscillates, the sequence (u_n) is nonnegative only when $B = 0$, and so $u_n = A\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4) [Lemma 2.3 of [5]].

4.3.2(2) $\lambda_1 > 0$. See subcase 2 of $C(r_1^2 z\bar{z})$ in [5].

4.3.3 $|\lambda_1| < |\lambda_3|$.

Rewrite the general term of the sequence as

$$u_n = |\lambda_3|^n\left\{(A + Bn)(\lambda_1/|\lambda_3|)^n + 2|C|\cos(\varphi + n\theta)\right\} \quad (n \geq 0).$$

Observe that $(A + Bn)(\lambda_1/|\lambda_3|)^n \to 0$ $(n \to \infty)$. By Lemma 2.2 of [5], $\cos(\varphi + n\theta)$ takes some fixed positive and some fixed negative values for infinitely many n. Thus, the sequence (u_n) is nonnegative only when $C = 0$, yielding $u_n = (A + Bn)\lambda_1^n$, which is of the form (HHH2) [Lemma 2.3 of [5]].

4.4 C$(r_1 r_2 z\bar{z})$

In this case, the general term of the sequence is $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$ $(n \geq 0)$, where $A, B \in \mathbb{R}$, $C \in \mathbb{C}$, $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ and $\lambda_3 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_3 = |\lambda_3|e^{i\theta}$, $C = |C|e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi)$, $\theta \notin \{-\pi, 0\}$ so that $u_n = A\lambda_1^n + B\lambda_2^n + 2|C||\lambda_3|^n \cos(\varphi + n\theta)$. We split our consideration into three possibilities.

1. The three λ's have the same absolute values, i.e., $|\lambda_1| = |\lambda_2| = |\lambda_3|$.
2. There are two λ_i's having the same absolute value, i.e., $|\lambda_1| = |\lambda_2|$ or $|\lambda_1| = |\lambda_3|$ or $|\lambda_2| = |\lambda_3|$.
3. All three roots λ_1, λ_2 and λ_3 have different absolute values.

4.4.1 $|\lambda_1| = |\lambda_2| = |\lambda_3|$.

See subcase 1, of $\mathrm{C}(r_1 r_2 z\bar{z})$ in [5].

4.4.2 $|\lambda_1| = |\lambda_2|$ or $|\lambda_1| = |\lambda_3|$ or $|\lambda_2| = |\lambda_3|$.

We need only treat the first two cases as the third is similar to the second.

4.4.2(1) $|\lambda_1| = |\lambda_2|$**.** See subcase 3.1, of $\mathrm{C}(r_1 r_2 z\bar{z})$ in [5].

4.4.2(2) $|\lambda_1| = |\lambda_3|$**.** Here, $u_n = A\lambda_1^n + 2|C|\cos(\varphi+n\theta)|\lambda_1|^n + B\lambda_2^n \ (n \geq 0)$. We subdivide into two further sub-cases depending on whether $|\lambda_1| > |\lambda_2|$.

4.4.2(2.1) $|\lambda_1| > |\lambda_2|$**.**

We consider two possibilities corresponding to the signs of λ_1.

- $\lambda_1 < 0$. Rewrite the general term of the sequence as

$$u_n = |\lambda_1|^n \left\{(-1)^n A + 2|C|\cos(\varphi + n\theta) + B(\lambda_2/|\lambda_1|)^n\right\} \quad (n \geq 0).$$

The sequence (u_n) is nonnegative if and only if for all $k \in \mathbb{N} \cup \{0\}$ we must have

$$2|C|\cos(\varphi + (2k+1)\theta) + B(\lambda_2/|\lambda_1|)^{2k+1} \geq A \tag{19}$$

and

$$A \geq -2|C|\cos(\varphi + 2k\theta) - B(\lambda_2/|\lambda_1|)^{2k}. \tag{20}$$

We consider first the case $\theta \neq -\pi/2$. Since $B(\lambda_2/|\lambda_1|)^n \to 0 \ (n \to \infty)$, by Lemma 2.2 of [5] and the remark after it, as k varies over $\mathbb{N} \cup \{0\}$, both $\cos(\varphi + 2k\theta)$ and $\cos(\varphi + (2k+1)\theta)$ take some fixed positive and some fixed negative values infinitely often. Thus, the relations (19) and (20) hold only when $A = C = 0$ yielding $u_n = B\lambda_2^n$, and so is nonnegative only when $B \geq 0$ and $\lambda_2 > 0$. Next, for the case $\theta = -\pi/2$, by Lemma 2.2 of [5] and the remark after it, (19) and (20) become when k_1 is even

$$-2|C|\cos(\varphi) - B(\lambda_2/|\lambda_1|)^{2k_1} \leq A \leq 2|C|\cos(\varphi + \theta) + B(\lambda_2/|\lambda_1|)^{2k_1+1} \tag{21}$$

and when k_2 is odd

$$2|C|\cos(\varphi) - B(\lambda_2/|\lambda_1|)^{2k_2} \leq A \leq -2|C|\cos(\varphi + \theta) + B(\lambda_2/|\lambda_1|)^{2k_2+1}. \tag{22}$$

Combining (21) with (22), we get

$$\begin{aligned} -B(\lambda_2/|\lambda_1|)^{2k_1} - B(\lambda_2/|\lambda_1|)^{2k_2} &\le 2A \\ &\le B(\lambda_2/|\lambda_1|)^{2k_1+1} + B(\lambda_2/|\lambda_1|)^{2k_2+1}. \end{aligned}$$

Since the bounds on both sides tend to 0 as $k_1, k_2 \to \infty$, we deduce that $A = 0$ and so $u_n = B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4) [Lemma 2.3 of [5]].

- $\lambda_1 > 0$. See subcase 3.3 in C($r_1 r_2 z\bar{z}$) of [5].

4.4.2(2.2) $|\lambda_1| < |\lambda_2|$**.**
We consider two possibility corresponding to the sign of λ_2.
• $\lambda_2 < 0$. Here, $u_n = \lambda_2^n \{A(\lambda_1/\lambda_2)^n + 2|C|(|\lambda_1|/\lambda_2)^n \cos(\varphi + n\theta) + B\}$, $(n \ge 0)$. Observe that $A(\lambda_1/\lambda_2)^n + 2|C|(|\lambda_1|/\lambda_2)^n \cos(\varphi + n\theta) + B \to B$ $(n \to \infty)$, provided $B \ne 0$. Since λ_2^n oscillates between $\pm|\lambda_2|$, then the sequence (u_n) is nonnegative only when $B = 0$, and so $u_n = A\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4) [Lemma 2.3 of [5]].
• $\lambda_2 > 0$. See subcase 3.2 in C($r_1 r_2 z\bar{z}$) of [5].

4.4.3 All three roots λ_1, λ_2 and λ_3 have different absolute values.

Without loss of generality, assume $|\lambda_1| > |\lambda_2|$. Here,

$$u_n = \lambda_1^n \left(A + B(\lambda_2/\lambda_1)^n\right) + 2|C||\lambda_3|^n \cos(\varphi + n\theta) \quad (n \ge 0).$$

We subdivide into two further subcases depending on whether $|\lambda_1| > |\lambda_3|$.
4.4.3(1) $|\lambda_1| > |\lambda_3|$. Rewrite the general term of the sequence as

$$u_n = \lambda_1^n \left\{A + B(\lambda_2/\lambda_1)^n + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta)\right\} \quad (n \ge 0).$$

We consider to possibilities corresponding to the signs of λ_1.
• $\lambda_1 < 0$. Since $\mathrm{sign}(A + B(\lambda_2/\lambda_1)^n + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta)) = \mathrm{sign}(A)$ when n is large enough provided $A \ne 0$, and $\mathrm{sign}(\lambda_1^n)$ oscillates the sequence (u_n) is nonnegative only when $A = 0$ and so $u_n = B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4) [Lemma 2.3 of [5]].
• $\lambda_1 > 0$. See subcase 2 in C($r_1 r_2 z\bar{z}$) of [5].
4.4.3(2) $|\lambda_1| < |\lambda_3|$. Rewrite the general term of the sequence as

$$u_n = |\lambda_3|^n \left\{(A + B(\lambda_2/\lambda_1)^n)(\lambda_1/|\lambda_3|)^n + 2|C|\cos(\varphi + n\theta)\right\} \quad (n \ge 0).$$

Observe that $(A + B(\lambda_2/\lambda_1)^n)(\lambda_1/|\lambda_3|)^n \to 0 \quad (n \to \infty)$. Next, by Lemma 2.2 of [5], $\cos(\varphi + n\theta)$ takes some fixed positive and some fixed negative values infinitely often. Thus, the sequence (u_n) is nonnegative only when $C = 0$ yielding $u_n = A\lambda_1^n + B\lambda_2^n$, which is of the form (HHH1) [Lemma 2.3 of [5]].

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