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POLYNOMIAL-ARITHMETIC FUNCTIONS

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Abstract

Let $\mathcal{A}(\Omega)$ denote the ring of arithmetic functions $f : \Omega \to \mathbb{C}$, where Ω is the set of monic polynomials over a finite field. Some characterizations of completely multiplicative arithmetic functions in $\mathcal{A}(\Omega)$, using distributive property, are established. A necessary and sufficient condition for the α^{th} power function in $\mathcal{A}(\Omega)$ to be completely multiplicative is given for all nonzero real numbers α.

1 Introduction and preliminaries

Let $\mathbb{F}_{p^n}[x]$ be the set of all polynomials over a finite field $\mathbb{F}_{p^n}$ where p is a prime and n is a positive integer. For $M, N \in \mathbb{F}_{p^n}[x]$, we define a relation on $\mathbb{F}_{p^n}[x]$ by

$$M \sim N \text{ if and only if } M = aN \text{ for some } a \in \mathbb{F}_{p^n} \backslash \{0\}.$$

It is easily checked that this is an equivalence relation. Let Ω denote the set of all equivalence classes of nonzero polynomials in $\mathbb{F}_{p^n}[x]$. For convenience, we regard Ω as the set of monic polynomials over a finite field $\mathbb{F}_{p^n}$, with implicit understanding that these polynomials represent equivalence classes. Hence, a polynomial in Ω merely refers to a monic polynomial. It is well-known ([11]) that, each nonconstant polynomial $M \in \Omega$ can be uniquely written in the form

$$M = P_1^{a_1} P_2^{a_2} \cdots P_k^{a_k},$$

where $P_1, P_2, \ldots, P_k$ are irreducible polynomials in Ω and $a_1, a_2, \ldots, a_k,\ k \in \mathbb{N}$.

By a polynomial-arithmetic function, ([12]), we mean a mapping f from the set Ω into the field of complex numbers $\mathbb{C}$. Let $(\mathcal{A}(\Omega), +, *)$ denote the

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set of polynomial-arithmetic functions equipped with addition and Dirichlet convolution defined over Ω, respectively, by

$$(f+g)(M) = f(M) + g(M)$$
$$(f*g)(M) = \sum_{D|M}^{(\Omega)} f(D)\, g\left(\frac{M}{D}\right)$$

for all $M \in \Omega$, where the summation is over all $D \in \Omega$ which are divisors of M. Throughout, the notation $\sum^{(\Omega)}$ signifies a summation taken over monic polynomials in Ω. As in the case of classical arithmetic functions, we know that $(\mathcal{A}(\Omega), +, *)$ is an integral domain with identity I_Ω ([12]), defined by

$$I_\Omega(M) = \begin{cases} 1 & \text{if} \quad M = 1_\Omega \\ 0 & \text{otherwise,} \end{cases}$$

where 1_Ω is the identity element in $\mathbb{F}_{p^n}$.

We have shown in [5], that the set

$$\mathcal{U}(\Omega) := \{f \in \mathcal{A}(\Omega) : f(1_\Omega) \neq 0\}$$

is the group of units in $\mathcal{A}(\Omega)$. That is, for every $f \in \mathcal{U}(\Omega)$, there is $f^{-1} \in \mathcal{A}(\Omega)$, the inverse of f with respect to the Dirichlet convolution, such that $f * f^{-1} = I_\Omega$.

A function $f \in \mathcal{A}(\Omega)$ is said to be *multiplicative* if $f \neq 0$ and

$$f(MN) = f(M)\, f(N) \tag{1}$$

whenever $g.c.d.(M, N) = 1_\Omega$ and f is said to be *completely multiplicative* if (1) holds for all pairs of polynomials $M, N \in \Omega$ ([12]). We have seen in [12] that the set of multiplicative functions is a subgroup of the group of units $\mathcal{U}(\Omega)$. Hence if f is multiplicative, then so is f^{-1}. A polynomial-arithmetic function $a \in \mathcal{A}(\Omega)$ is said to be *completely additive* if

$$a(MN) = a(M) + a(N)$$

for all $M, N \in \Omega$ ([5]). Note that

- if $f(1_\Omega) \neq 0$, then $f^{-1}(1_\Omega) = 1$ and $f^{-1}(P) = -f(P)$ for all irreducible polynomial P in Ω;
- if f is multiplicative, then $f(1_\Omega) = 1$;
- if $a \in \mathcal{A}(\Omega)$ is completely additive, then $a(1_\Omega) = 0$.

Next, we recall the definitions of the polynomial-logarithmic operator in [5], the polynomial-exponential operator, the polynomial-power function and the generalized polynomial-Möbius function in [4].

For notational convenience, let

$\mathcal{A}_1(\Omega) = \{f \in \mathcal{A}(\Omega) : f(1_\Omega) \in \mathbb{R}\}$ and $\mathcal{P}(\Omega) = \{f \in \mathcal{A}(\Omega) : f(1_\Omega) > 0\} \subseteq \mathcal{U}(\Omega)$.

It is not difficult to show that $(\mathcal{A}_1(\Omega), +)$ and $(\mathcal{P}(\Omega), *)$ are groups.

Definition 1. *([5]) Let $a \in \mathcal{A}(\Omega)$ be a completely additive polynomial-arithmetic function for which $a(M) \neq 0$ for all $M \in \Omega \backslash \{1_\Omega\}$. The polynomial-logarithmic operator (associated with a) is the map $Log_\Omega : \mathcal{P}(\Omega) \to \mathcal{A}_1(\Omega)$, defined by*

$$\begin{aligned} Log_\Omega f(1_\Omega) &= log f(1_\Omega), \\ Log_\Omega f(M) &= \frac{1}{a(M)} \sum_{D|M}^{(\Omega)} f(D) f^{-1}\left(\frac{M}{D}\right) a(D) \qquad (2) \\ &= \frac{1}{a(M)} \left(df * f^{-1}\right)(M) \qquad (3) \end{aligned}$$

for all $M \in \Omega \backslash \{1_\Omega\}$ where the right-hand side of the first equation denotes the real logarithmic value and $df(M) = f(M)a(M)$ for all $M \in \Omega$.

We have shown in [5] that Log_Ω is a group isomorphism from $(\mathcal{P}(\Omega), *)$ onto $(\mathcal{A}_1(\Omega), +)$, and hence

$$Log_\Omega(f * g) = Log_\Omega\, f + Log_\Omega\, g \qquad (f, g \in \mathcal{P}(\Omega)). \qquad (4)$$

Definition 2. *([4]) The polynomial-exponential operator is the map*

$$Exp_\Omega : \mathcal{A}_1(\Omega) \to \mathcal{P}(\Omega),$$

defined by $Exp_\Omega = (Log_\Omega)^{-1}$.

Note that

$$Exp_\Omega(f + g) = Exp_\Omega(f) * Exp_\Omega(g) \qquad (f, g \in \mathcal{A}_1(\Omega)). \qquad (5)$$

Definition 3. *([4]) For $f \in \mathcal{P}(\Omega)$ and $\alpha \in \mathbb{R}$, the α^{th} polynomial-power function is defined as*

$$f^\alpha = Exp_\Omega(\alpha Log_\Omega\, f) \in \mathcal{P}(\Omega). \qquad (6)$$

Clearly, $f^0 = I_\Omega$ and $f^1 = f$. For $r \in \mathbb{N}$, using (5) and (6), we obtain

$$\begin{aligned} f^r &= Exp_\Omega(r Log_\Omega\, f) \\ &= Exp_\Omega(Log_\Omega\, f + \cdots + Log_\Omega f) \\ &= Exp_\Omega(Log_\Omega\, f) * \cdots * Exp_\Omega(Log_\Omega\, f) \\ &= f * \cdots * f \quad (r \text{ factors}). \end{aligned}$$

We can show similarly that

$$f^{-r} = f^{-1} * f^{-1} * \cdots * f^{-1} \qquad (r \text{ factors}),$$

where f^{-1} is the inverse of f with respect to the Dirichlet convolution. It is easily checked that

$$f^{\alpha+\beta} = f^{\alpha} * f^{\beta} \tag{7}$$

$$(f^{\alpha})^{\beta} = f^{\alpha\beta} \tag{8}$$

and

$$(f^{\alpha})^{-1} = f^{-\alpha} \tag{9}$$

for all $f \in \mathcal{P}(\Omega)$ and $\alpha, \beta \in \mathbb{R}$.

We have shown in [4] that for $f \in \mathcal{A}_1(\Omega)$, $Exp_\Omega f$ is uniquely determined by the formulas

$$\begin{aligned} &Exp_\Omega f\,(1_\Omega) = exp(f\,(1_\Omega)),\\ &Exp_\Omega f\,(M) = \frac{1}{a\,(M)} \sum_{D|M}^{(\Omega)} Exp_\Omega\; f\,(D)\, f\left(\frac{M}{D}\right) a\left(\frac{M}{D}\right) \end{aligned} \tag{10}$$

for all $M \in \Omega\backslash\{1_\Omega\}$, where a is a completely additive polynomial-arithmetic function for which $a(M) \neq 0$ for all $M \in \Omega\backslash\{1_\Omega\}$. Observe that if $f(1_\Omega) = 1$, then

$$f^{\alpha}(1_\Omega) = Exp_\Omega(\alpha Log_\Omega\, f)(1_\Omega) = exp\left((\alpha Log_\Omega f)(1_\Omega)\right) = exp\left(\alpha \log f(1_\Omega)\right) = 1. \tag{11}$$

for all $\alpha \in \mathbb{R}$.

Definition 4. *([4]) For $\alpha \in \mathbb{R}$, the generalized polynomial-Möbius function $\mu_\alpha^\Omega : \Omega \to \mathbb{C}$ is defined by*

$$\mu_\alpha^\Omega(M) = \prod_{i=1}^{k} \binom{\alpha}{a_i} (-1)^{a_i}, \quad \mu_\alpha^\Omega(1_\Omega) = 1, \tag{12}$$

where $M = P_1^{a_1} P_2^{a_2} \cdots P_k^{a_k}$, $P_1, P_2, \ldots, P_k$ are irreducible polynomials in Ω, $a_1, a_2, \ldots, a_k$, $k \in \mathbb{N}$ and

$$\binom{\alpha}{0} = 1, \ \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \qquad (n \in \mathbb{N}). \tag{13}$$

Observe that $\mu_0^\Omega = I_\Omega$, $\mu_{-1}^\Omega = u$, where $u(M) = 1$ for all $M \in \Omega$, and $\mu_1^\Omega = \mu^\Omega$, the polynomial-Möbius function ([12]) defined by

$$\mu^\Omega(M) = \begin{cases} 1 & \text{if } M = 1_\Omega,\\ 0 & \text{if } P^2|M,\ P \text{ irreducible polynomial in } \Omega,\\ (-1)^k & \text{if } M = P_1P_2\cdots P_k,\ \text{a product of distinct irreducible}\\ & \text{polynomials in } \Omega. \end{cases}$$

Clearly, μ_α^Ω is multiplicative for all real numbers α. It is easy to show that

$$\mu_\alpha^\Omega * \mu_\beta^\Omega = \mu_{\alpha+\beta}^\Omega \tag{14}$$

for all real numbers $\alpha,\ \beta$. Note that

$$\mu^\Omega * u = I_\Omega \quad \text{and} \quad \mu_{-\alpha}^\Omega = \left(\mu_\alpha^\Omega\right)^{-1}$$

for all real numbers α.

In the classical case, the distributive property of completely multiplicative functions discovered by J. Lambek ([6]), asserts that an arithmetic function f is completely multiplicative if and only if it distributes over every Dirichlet product. It is shown in [4] that such Lambek's result holds in the polynomial case. Afterward, E. Langford points out in [7] an interesting characterization of completely multiplicative functions f using partially discriminative products, which states that given a multiplicative function f, then f is completely multiplicative if and only if it distributes over some partially discriminative products. In 2004, V. Laohakosol and N. Pabhapote ([8]) gave a necessary and sufficient condition for the α^{th} power function f^α to be completely multiplicative. It states that given a multiplicative function f and $\alpha \in \mathbb{R}\backslash\{0\}$, then f^α is completely multiplicative if and only if $f(p^k) = \binom{-1/\alpha}{k}(-\alpha)^k f(p)^k$ for all primes p and all $k \in \mathbb{N}$.

The first objective of this paper is to establish some characterizations of completely multiplicative polynomial-arithmetic functions through their distributive property using polynomial-partially discriminative products and generalized polynomial-Möbius function. The second objective is to give a necessary and sufficient condition for the α^{th} polynomial-power function to be completely multiplicative for all nonzero real numbers α.

2 Basic results

To facilitate the proof of our main results, we recall the following results in [5] and [4].

Proposition 1. *([5]) If $f \in \mathcal{A}(\Omega)$ is multiplicative, then f is completely multiplicative if and only if $f\left(P^k\right) = f(P)^k$ for all irreducible polynomials $P \in \Omega$ and for all $k \in \mathbb{N}$.*

Proposition 2. *([5]) Let $f \in \mathcal{A}(\Omega)$ be multiplicative. Then f is completely multiplicative if and only if $f^{-1} = f\mu^\Omega$.*

Theorem 1. *([5]) Let $f \in \mathcal{P}(\Omega)$. Then f is multiplicative if and only if $Log f(M) = 0$ whenever M is not a power of an irreducible polynomial.*

Lemma 1. *([4]) A multiplicative function $f \in \mathcal{A}(\Omega)$ is completely multiplicative if and only if*

$$f(g * h) = fg * fh$$

for all $g,\ h \in \mathcal{A}(\Omega)$.

Theorem 2. *([4]) Let $f \in \mathcal{P}(\Omega)$ be multiplicative and $\alpha \in \mathbb{R}$. We have*

(i) if f is completely multiplicative then $f^\alpha = \mu^\Omega_{-\alpha} f$;

(ii) for $\alpha \notin \{0, 1\}$, if $f^\alpha = \mu^\Omega_{-\alpha} f$, then f is completely multiplicative.

3 Completely multiplicative polynomial-arithmetic functions

In this section, we first define the definition of partially discriminative products for polynomial-arithmetic functions similar to the classical cases, defined by E. Langford ([7]).

Definition 5. *For polynomial-arithmetic functions g and h, a Dirichlet product $g * h$ is called polynomial-partially discriminative if for every irreducible polynomial $P \in \Omega$ and $k \in \mathbb{N}$,*

$$(g * h)\left(P^k\right) = g(1_\Omega)h\left(P^k\right) + g\left(P^k\right)h\left(1_\Omega\right),$$

then $k = 1$.

Note that $\mu^\Omega * u$ is a polynomial-partially discriminative product.

Using Proposition 1, Lemma 1 and the same proof as in [7], we have the following theorem.

Theorem 3. *Suppose that $f \in \mathcal{A}(\Omega)$ is multiplicative. Then f is completely multiplicative if and only if f distributes over some polynomial-partially discriminative product $g * h$.*

The following two corollaries are immediate consequences of Proposition 2 and Theorem 3, respectively.

Corollary 1. *For any multiplicative function $f \in \mathcal{A}(\Omega)$, f is completely multiplicative if and only if f distributes over $\mu^\Omega * u$ $(= I_\Omega)$.*

Proof. Let $f \in \mathcal{A}(\Omega)$ be a multiplicative function. Then $f(1_\Omega) = 1$ and using Proposition 2, we obtain

$$\begin{aligned} f \text{ distributes over } \mu^\Omega * u &\Leftrightarrow f(\mu^\Omega * u) = f\mu^\Omega * fu \\ &\Leftrightarrow fI_\Omega = f\mu^\Omega * fu \\ &\Leftrightarrow I_\Omega = f\mu^\Omega * f \\ &\Leftrightarrow f^{-1} = f\mu^\Omega \\ &\Leftrightarrow f \text{ is completey multiplicative.} \end{aligned}$$

□

Corollary 2. *Let f be a multiplicative polynomial-arithmetic function, g be a polynomial-arithmetic function with $g(1_\Omega) = 1$ and $\alpha \in \mathbb{R} \setminus \{0\}$.*

A. If f is completely multiplicative, then

$$f\left(g * \mu_\alpha^\Omega\right) = fg * f\mu_\alpha^\Omega = fg * f^{-\alpha}.$$

B. Assume that

$$\left(g * \mu_\alpha^\Omega\right)\left(P^k\right) - \mu_\alpha^\Omega\left(P^k\right) - g\left(P^k\right) \neq 0 \tag{15}$$

for all integers $k \geq 2$ and for all irreducible polynomials P in Ω. If

$$f\left(g * \mu_\alpha^\Omega\right) = fg * f\mu_\alpha^\Omega,$$

then f is completely multiplicative.

Proof. A. Assume that f is completely multiplicative. By Lemma 1 and Theorem 2(i) , we have

$$f\left(g * \mu_\alpha^\Omega\right) = fg * f\mu_\alpha^\Omega = fg * f^{-\alpha}.$$

B. Since $g(1_\Omega) = 1$, then the condition (15) implies that the Dirichlet product $g * \mu_\alpha^\Omega$ is a polynomial-partially discriminative. Hence part B. follows from Theorem 3. □

In the proof of the theorem in [3], P. Haukkanen proved that if f is an arithmetic function with $f(1) > 0$ and n is a positive integer, then $f^\alpha(n)$ is a polynomial in α. Now for the next main result, we prove this fact for the polynomial case as follows:

Lemma 2. *For $\alpha \in \mathbb{R}$, if f is a multiplicative function in $\mathcal{A}(\Omega)$, then $f^\alpha(M)$ is a polynomial in α for fixed $M \in \Omega$.*

Proof. Assume that $f \in \mathcal{A}(\Omega)$ is a multiplicative function. Then $f(1_\Omega) = 1$. Let $M \in \Omega$ be fixed. We will prove that $f^\alpha(M)$ is a polynomial in α by induction on $\deg(M)$. If $\deg(M) = 0$, then $M = 1_\Omega$ and so $f^\alpha(1_\Omega) = 1$ is a constant polynomial. Assume that $f^{\alpha}(D)$ is a polynomial in α for all $D \in \Omega$ such that $\deg(D) < \deg(M)$. Now

$$f^{\alpha} = Exp_{\Omega}(\alpha Log_{\Omega}\ f), \tag{16}$$

where Exp_{Ω} is the polynomial-exponential operator and Log_{Ω} is the polynomial-logarithmic operator (associated with a completely additive polynomial-arithmetic function a). Using (16), (10) and $a(1_{\Omega}) = 0$, we obtain

$$\begin{aligned} f^{\alpha}(M) &= Exp_{\Omega}(\alpha Log_{\Omega}\ f)(M) \\ &= \frac{1}{a(M)} \sum_{D|M}^{(\Omega)} Exp_{\Omega}\left(\alpha Log_{\Omega}\ f\right)(D)\left(\alpha Log_{\Omega}\ f\right)\left(\frac{M}{D}\right) a\left(\frac{M}{D}\right) \\ &= \frac{1}{a(M)} \sum_{D|M}^{(\Omega)} f^{\alpha}(D)\left(\alpha Log_{\Omega}\ f\right)\left(\frac{M}{D}\right) a\left(\frac{M}{D}\right) \\ &= \frac{1}{a(M)} \sum_{D|M, D\neq M}^{(\Omega)} f^{\alpha}(D)\left(\alpha Log_{\Omega}\ f\right)\left(\frac{M}{D}\right) a\left(\frac{M}{D}\right), \end{aligned}$$

and the desired result follows from the induction hypothesis. □

The following Theorem is our second main result.

Theorem 4. *Let $f \in \mathcal{A}(\Omega)$ be a multiplicative function, g a polynomial-arithmetic function with $g(1_{\Omega}) = 1$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Assume that*

$$\left(g * \mu_{\alpha}^{\Omega}\right)\left(P^k\right) - g\left(P^k\right) + \alpha \neq 0 \tag{17}$$

for all integers $k \geq 2$ and for all irreducible polynomials P in Ω. If

$$f\left(g * \mu_{\alpha}^{\Omega}\right) = fg * f^{-\alpha}, \tag{18}$$

then f is completely multiplicative.

Proof. Since $f \in \mathcal{A}(\Omega)$ is a multiplicative function, we have $f(1_{\Omega}) = 1$. To show that f is completely multiplicative, it suffices to show that

$$f\left(P^k\right) = f(P)^k \tag{19}$$

for all irreducible polynomials $P \in \Omega$ and for all $k \in \mathbb{N}$. By Proposition 1, we prove (19) by induction on k. It is clear for $k = 1$, so assume that $k \geq 2$ and $f\left(P^i\right) = f\left(P\right)^i$ for all $i \in \{1, \ldots, k-1\}$. Using (7) and $f^0 = I_{\Omega}$, rewrite (18) in an equivalent form as

$$f(g * \mu_{\alpha}^{\Omega}) * f^{\alpha} = fg.$$

Since $f(1_{\Omega}) = g(1_{\Omega}) = \mu_{\alpha}^{\Omega}(1_{\Omega}) = f^{\alpha}(1_{\Omega}) = 1$, we get

$$\begin{aligned} fg(P^k) =& f(g * \mu_{\alpha}^{\Omega})(P^k) + f(g * \mu_{\alpha}^{\Omega})(P^{k-1}) f^{\alpha}(P) \\ &+ \cdots + f(g * \mu_{\alpha}^{\Omega})(P) f^{\alpha}(P^{k-1}) + f^{\alpha}(P^k). \end{aligned} \tag{20}$$

We pause to note two important facts.
Fact 1. If $f(P^i) = f(P)^i$ for $1 \le i \le j$, then

$$f^r(P^j) = \sum_{i_1+\cdots+i_r=j} f(P^{i_1})\cdots f(P^{i_r}) = f(P)^j \binom{r+j-1}{j}$$

for all $r \in \mathbb{N}$.
Fact 2. For $j \in \mathbb{N}$, we have

$$\mu_{-\alpha}^{\Omega}(P^j) = \binom{-\alpha}{j}(-1)^j = \binom{\alpha+j-1}{j}$$

for all $\alpha \in \mathbb{R}$.
Using induction hypothesis, Fact 1 and Fact 2, we get that

$$f^r(P^i) = f(P)^i \binom{r+i-1}{i} = f(P)^i \mu_{-r}^{\Omega}\left(P^i\right) \qquad (i = 1, \ldots, k-1) \tag{21}$$

for all $r \in \mathbb{N}$. It follows by induction hypothesis and Fact 2 that

$$\begin{aligned} f^r(P^k) &= \sum_{i_1+\cdots+i_r=k} f(P^{i_1})\cdots f(P^{i_r}), \\ &= \sum_{i_1+\cdots+i_r=k, i_j\neq k} f(P^{i_1})\cdots f(P^{i_r}) + rf(P^k), \\ &= \sum_{i_1+\cdots+i_r=k, i_j\neq k} f(P)^{i_1}\cdots f(P)^{i_r} + rf(P^k), \\ &= f(P)^k \sum_{i_1+\cdots+i_r=k, i_j\neq k} 1 + rf(P^k), \\ &= f(P)^k \left[\binom{r+k-1}{k} - r\right] + rf(P^k) \\ &= f(P)^k \left(\mu_{-r}^{\Omega}(P^k) - r\right) + rf(P^k) \end{aligned} \tag{22}$$

for all $r \in \mathbb{N}$.
Claim. 1

$$f^{\alpha}(P^i) = f(P)^i \mu_{-\alpha}^{\Omega}\left(P^i\right) \qquad (i = 1, \ldots, k-1) \tag{23}$$

for all real numbers α.
Proof of Claim 1. For $\alpha \in \mathbb{R}$, we have that $f^{\alpha}(P^i)$ is a polynomial in α for all $i \in \{1, \ldots, k-1\}$, by Lemma 2. By (12) and (13), $\mu_{-\alpha}^{\Omega}\left(P^i\right)$ is also a polynomial in α for all $i \in \{1, \ldots, k-1\}$. Thus, both sides of (23) are polynomials in α. It follows from (21) that (23) is true for infinitely many values of α, so $f^{\alpha}(P^i) - f(P)^i \mu_{-\alpha}^{\Omega}\left(P^i\right)$ is the zero polynomial. Therefore, (23)

holds for all real numbers α.
Claim. 2

$$f^{\alpha}(P^k) = f(P)^k \left(\mu_{-\alpha}^{\Omega}(P^k) - \alpha\right) + \alpha f(P^k)$$

for all real numbers α.
Proof of Claim 2. This is proved in a manner similar to Claim 1 using (22).

Returning to (20) and using the induction hypothesis together with Claim 1 and Claim 2, we obtain

$$\begin{aligned}
fg(P^k) =& f(P^k)(g*\mu_\alpha^\Omega)(P^k) + f(P)^{k-1}(g*\mu_\alpha^\Omega)(P^{k-1})f(P)\mu_{-\alpha}^\Omega(P) \\
&+ \cdots + f(P)(g*\mu_\alpha^\Omega)(P)f(P)^{k-1}\mu_{-\alpha}^\Omega(P^{k-1}) + f(P)^k\left(\mu_{-\alpha}^\Omega(P^k)-\alpha\right) \\
&+ \alpha f(P^k) \\
=& f(P^k)(g*\mu_\alpha^\Omega)(P^k) \\
&+ f(P)^k\left\{(g*\mu_\alpha^\Omega)(P^{k-1})\mu_{-\alpha}^\Omega(P) + \cdots + (g*\mu_\alpha^\Omega)(P)\mu_{-\alpha}^\Omega(P^{k-1}) + \mu_{-\alpha}^\Omega(P^k) - \alpha\right\} \\
&+ \alpha f(P^k) \\
=& f(P^k)(g*\mu_\alpha^\Omega)(P^k) + f(P)^k\left\{\left(g*\mu_\alpha^\Omega*\mu_{-\alpha}^\Omega\right)(P^k) - \left(g*\mu_\alpha^\Omega\right)(P^k) - \alpha\right\} + \alpha f(P^k) \\
=& f(P^k)(g*\mu_\alpha^\Omega)(P^k) + f(P)^k\left\{g(P^k) - \left(g*\mu_\alpha^\Omega\right)(P^k) - \alpha\right\} + \alpha f(P^k), \text{ by (14).}
\end{aligned}$$

Thus,

$$f(P^k)\left\{(g*\mu_\alpha^\Omega)(P^k) + \alpha - g(P^k)\right\} = f(P)^k\left\{(g*\mu_\alpha^\Omega)(P^k) + \alpha - g(P^k)\right\},$$

and the assertion follows from the assumption (17). □

4 Completely multiplicative polynomial-arithmetic functions and the α^{th} polynomial-power functions

In this section, we give a necessary and sufficient condition for the α^{th} polynomial-power function to be completely multiplicative, which is our last main result (Theorem 7). To facilitate the proof, we first prove Lemma 3, Theorem 5 and Theorem 6.

Lemma 3. *For an irreducible polynomial $P \in \Omega$, if $f(1_\Omega) = 1$ and $f(P^i) = f(P)^i$ for all $i \in \{2, \ldots, n\}$, then*

$$f^{-1}(P^i) = 0$$

for all $i \in \{2, \ldots, n\}$.

Proof. This is easily proved by induction on n.

□

In 1974, T. B. Carroll gave a characterization of completely multiplicative arithmetic functions in the classical case ([2]), which states that if f is an arithmetic function such that $f(1) > 0$, then f is completely multiplicative if and only if

$$Logf(n) = \begin{cases} (\log p)f(p)^a & \text{if } n = p^a,\ p \text{ prime},\ a \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where *Log* is the Rearick's logarithmic operator ([9],[10]). For the polynomial case, we prove such result using Lemma 3.

Theorem 5. *Let $f \in \mathcal{P}(\Omega)$. Then f is completely multiplicative if and only if for all irreducible polynomials $P \in \Omega$ and all integers $k \geq 1$.*

$$Log_{\Omega} f(M) = \begin{cases} \frac{f(P)^k}{k} & \text{if } M = P^k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If f is completely multiplicative, then f is multiplicative and so by Theorem 1, $Logf(M) = 0$ whenever M is not a power of an irreducible polynomial. If $M = P^k$ for some irreducible polynomial P in Ω and $k \in \mathbb{N}$, then using Proposition 1, Proposition 2, $f^{-1}(1_\Omega) = 1$ and $f^{-1}(P) = -f(P)$, we obtain

$$\begin{aligned} Log_{\Omega} f(M) &= Log_{\Omega} f(P^k) \\ &= \frac{1}{a(P^k)} \sum_{i=0}^{k} f(P^i) f^{-1}(P^{k-i}) a(P^i) \\ &= \frac{1}{a(P^k)} \left[f(P^k) f^{-1}(1_\Omega) a(P^k) + f(P^{k-1}) f^{-1}(P) a(P^{k-1}) \right] \\ &= \frac{1}{ka(P)} \left[kf(P^k) a(P) - (k-1) f(P)^{k-1} f(P) a(P) \right] \\ &= \frac{1}{k} \left[kf(P)^k - (k-1) f(P)^k \right] \\ &= \frac{f(P)^k}{k}. \end{aligned}$$

Conversely, by Theorem 1, we have f is multiplicative and so $f(1_\Omega) = f^{-1}(1_\Omega) = 1$ and $f^{-1}(P) = -f(P)$ for all irreducible polynomials $P \in \Omega$. To show that f is completely multiplicative, it suffices to show

$$f(P^k) = f(P)^k$$

for all irreducible polynomials $P \in \Omega$ and all positive integers k. Let $P \in \Omega$ be an arbitrary irreducible polynomial. By assumption, we have

$$\begin{aligned}\frac{f(P)^2}{2} &= Log_\Omega f(P^2)\\ &= \frac{1}{a(P^2)}\left[f(1_\Omega)f^{-1}(P^2)a(1_\Omega) + f(P)f^{-1}(P)a(P) + f(P^2)f^{-1}(1_\Omega)a(P^2)\right]\\ &= \frac{1}{2a(P)}\left[2f(P^2)a(P) - f(P)^2a(P)\right]\\ &= f(P^2) - \frac{1}{2}f(P)^2.\end{aligned}$$

It follows that

$$f(P^2) = f(P)^2.$$

Let k be a positive integer greater than two. Assume that

$$f(P^i) = f(P)^i$$

for all $i \in \{1, \dots, k-1\}$. By Lemma 3, $f^{-1}(P^i) = 0$ for all $i \in \{2, \dots, k-1\}$. It follows from the assumption that

$$\begin{aligned}\frac{f(P)^k}{k} &= Log_\Omega f(P^k)\\ &= \frac{1}{a(P^k)}\sum_{i=0}^{k} f(P^i)f^{-1}(P^{k-i})a(P^i)\\ &= \frac{1}{ka(P)}\left[f(1_\Omega)f^{-1}(P^k)a(1_\Omega) + f(P^{k-1})f^{-1}(P)a(P^{k-1}) + f(P^k)f^{-1}(1_\Omega)a(P^k)\right]\\ &= \frac{1}{ka(P)}\left[-(k-1)f(P)^k a(P) + kf(P^k)a(P)\right]\\ &= \frac{1}{k}\left[(1-k)f(P)^k + kf(P^k)\right],\end{aligned}$$

which implies that $f(P^k) = f(P)^k$ for all irreducible polynomials $P \in \Omega$ and all positive integers k. □

By the same proof as in [1], we have the following theorem.

Theorem 6. *Let $f \in \mathcal{A}(\Omega)$ be multiplicative. Then f is completely multiplicative if and only if*

$$f^{-1}(P^a) = 0 \tag{24}$$

for all irreducible polynomials $P \in \Omega$ and all integers $a \geq 2$.

Our last main result reads:

Theorem 7. *Let $f \in \mathcal{P}(\Omega)$ be multiplicative and $\alpha \in \mathbb{R}\backslash\{0\}$. Then f^α is completely multiplicative if and only if*

$$f(P^k) = \binom{-1/\alpha}{k}(-\alpha)^k f(P)^k \tag{25}$$

for all irreducible polynomials $P \in \Omega$ and all positive integers k.

Proof. Assume that f^α is completely multiplicative. By Theorem 2 (i) and (8), we have

$$\mu^{\Omega}_{-1/\alpha} f^\alpha = (f^\alpha)^{1/\alpha} = f.$$

Let $P \in \Omega$ be any irreducible polynomial and $k \in \mathbb{N}$. Then

$$f(P^k) = \mu^{\Omega}_{-1/\alpha}(P^k) f^\alpha(P^k) = \binom{-1/\alpha}{k}(-1)^k f^\alpha(P)^k.$$

By Theorem 5, we have

$$\begin{aligned} f^\alpha(P) &= Log_\Omega f^\alpha(P) \\ &= Log_\Omega \left(Exp_\Omega(\alpha Log_\Omega f)\right)(P) \\ &= (\alpha Log_\Omega f)(P) \\ &= \frac{\alpha}{a(P)}\left[f(1_\Omega)f^{-1}(P)a(1_\Omega) + f(P)f^{-1}(1_\Omega)a(P)\right] \\ &= \alpha f(P), \end{aligned}$$

since $a(1_\Omega) = 0$ and $f^{-1}(1_\Omega) = 1$. Hence,

$$f(P^k) = \binom{-1/\alpha}{k}(-1)^k \left(\alpha f(P)\right)^k = \binom{-1/\alpha}{k}(-\alpha)^k f(P)^k.$$

To prove the converse, we assume (25) and prove that f^α is completely multiplicative. Since $f \in \mathcal{P}(\Omega)$ is multiplicative, $Log_\Omega f(M) = 0$, whenever M is not a power of an irreducible polynomial, by Theorem 1. Using (11) and (6), we have $f^\alpha \in \mathcal{P}(\Omega)$ and

$$Log_\Omega f^\alpha(M) = \alpha Log_\Omega f(M) = 0,$$

whenever M is not a power of an irreducible polynomial. By Theorem 1 again, f^α is multiplicative. To prove that f^α is completely multiplicative, it suffices to prove, by Theorem 6, that

$$f^{-\alpha}(P^k) = 0$$

for all irreducible polynomials $P \in \Omega$ and all integers $k \geq 2$.

Let $g = f^{-\alpha}$. Then

$$Log_{\Omega} g = Log_{\Omega} f^{-\alpha} = Log\left(Exp_{\Omega}\left(-\alpha Log_{\Omega} f\right)\right) = -\alpha Log_{\Omega} f.$$

We must show that $g(P^k) = 0$ for all irreducible polynomials $P \in \Omega$ and all integers $k \geq 2$. Since

$$(Log_{\Omega} f)(M) = \frac{1}{a(M)}(df * f^{-1})(M) \qquad (f \in \mathcal{P}(\Omega),\ M \in \Omega\backslash\{1_{\Omega}\}),$$

we have

$$f * dg = -\alpha(g * df).$$

Hence,

$$\sum_{i=0}^{k} f(P^i)dg(P^{k-i}) = -\alpha \sum_{i=0}^{k} g(P^i)df(P^{k-i}) \tag{26}$$

for all irreducible polynomials $P \in \Omega$ and all positive integers k. Let P be an irreducible polynomial in Ω and $k \in \mathbb{N}$. Using (11) and taking $k = 1$ in (26), we get

$$g(P) = -\alpha f(P).$$

Taking $k = 2$ in (26) and using $g(P) = -\alpha f(P)$, we obtain

$$\begin{aligned} 2g(P^2) &= -\alpha\left[2f(P^2) - (\alpha+1)f(P)^2\right] \\ &= -\alpha\left[2\binom{-1/\alpha}{2}(-\alpha)^2 f(P)^2 - (\alpha+1)f(P)^2\right] \\ &= -\alpha\left[(\alpha+1)f(P)^2 - (\alpha+1)f(P)^2\right] \\ &= 0, \end{aligned}$$

so $g(P^2) = 0$. Consider $k \geq 3$ and assume that $g(P^i) = 0$ for all $i \in \{2, , \ldots, k-1\}$. Returning to (26), we get that

$$kg(P^k) = -k\alpha f(P^k) + \left[(k-1)\alpha^2 + \alpha\right] f(P^{k-1})f(P).$$

It follows from (25) that

$$kg(P^k) = -k\alpha\binom{-1/\alpha}{k}(-\alpha)^k f(P)^k + \left[(k-1)\alpha^2 + \alpha\right]\binom{-1/\alpha}{k-1}(-\alpha)^{k-1} f(P)^{k-1} f(P) = 0,$$

so $g(P^k) = 0$ for all irreducible polynomials $P \in \Omega$ and all integers $k \geq 2$. □

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