East-West J. of Mathematics: Vol. 14 , No.2 (2012) pp. 154-169

EXPANSIONS OF ELEMENTS WRITTEN WITH RESPECT TO A QUADRATIC GENERATING POLYNOMIAL

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Abstract

Let $p(x, y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x, y]$, where $\mathbb{F}_q$ is a finite field of q elements; $b_1, b_0 \in \mathbb{F}_q[x]$ and let $R := \mathbb{F}_q[x, y]/(p(x, y))$. A Scheicher-Thuswaldner algorithm enables us to represent each element of R through a digit system. All possible representations of elements in R are determined when $\deg b_1 \le \deg b_0$. As for the case $\deg b_1 > \deg b_0$, the same analysis is carried out subject to an assumption on the existence of a unique maximal term.

1 Introduction

In [2], Scheicher and Thuswaldner, devised a digit system for elements in a polynomial ring of two indeterminates detailed as follows: let $\mathbb{F}_q$ be a finite field of q elements, and

$$p(x, y) = y^n + b_{n-1}y^{n-1} + \cdots + b_1 y - b_0 \in \mathbb{F}_q[x, y],$$

where $b_i \in \mathbb{F}_q[x]$ and $\deg b_0 > 0$. Let

$$\mathcal{N} := \{g \in \mathbb{F}_q[x] : \ \deg g < \deg b_0\}$$
$$\mathcal{R} := \mathbb{F}_q[x, y]/(p(x, y)) = \left\{c_0 + c_1 y + \cdots + c_{n-1}y^{n-1} : \ c_i \in \mathbb{F}_q[x]\right\}.$$

Keywords: Finite field, Scheicher-Thuswaldner digit system
(2010) AMS Classification: 11T06, 12E20, 12E90
Supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

Then each $r \in \mathcal{R} \setminus \{0\}$ is uniquely represented as

$$r = r_0 + r_1 y + \cdots + r_{n-1} y^{n-1}, \quad r_j \in \mathbb{F}_q[x]. \tag{1}$$

We say that $r \in \mathcal{R} \setminus \{0\}$ has a finite *y-adic representation* if it admits a finite representation of the form

$$r = d_0 + d_1 y + \cdots + d_h y^h, \tag{2}$$

with all the $d_i \in \mathcal{N}$ and $h \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The polynomials d_i are called the *digits* of r. If each $r \in \mathcal{R} \setminus \{0\}$ has a unique finite y-adic representation (2), then $p(x, y)$ is referred to as a *digit system polynomial (DS-polynomial)* with y being the *base* and $\mathcal{N}$ being the *digit set*.

In order to determine those $p(x, y)$ which are DS-polynomials, Scheicher and Thuswaldner make use of the following algorithm. Given a general element as in (1), i.e.,

$$r := r^{(0)} = r_0^{(0)} + r_1^{(0)} y + \cdots + r_{n-1}^{(0)} y^{n-1} \in \mathcal{R} \setminus \{0\},$$

by the division algorithm there exist unique $d_0 \in \mathcal{N}$ and $\tilde{r}_0 := \left[r_0^{(0)} / b_0 \right] \in \mathbb{F}_q[x]$ such that

$$r_0^{(0)} = \tilde{r}_0 b_0 + d_0 \text{ with } \deg \tilde{r}_0 < \deg r_0^{(0)}. \tag{3}$$

Using

$$\frac{b_0}{y} = b_1 + b_2 y + \cdots + b_n y^{n-1}, \tag{4}$$

we define

$$\begin{aligned} r^{(1)} \left(:= r_0^{(1)} + r_1^{(1)} y + \cdots + r_{n-1}^{(1)} y^{n-1} \right) &= \frac{r^{(0)} - d_0}{y} \\ &= \left(\tilde{r}_0 b_1 + r_1^{(0)} \right) + \left(\tilde{r}_0 b_2 + r_2^{(0)} \right) y + \cdots + \left(\tilde{r}_0 b_n + r_n^{(0)} \right) y^{n-1} \end{aligned} \tag{5}$$

so that

$$r_i^{(1)} = \tilde{r}_0 b_{i+1} + r_{i+1}^{(0)} \quad (0 \le i \le n-1).$$

Similarly, there exist unique $d_1 \in \mathcal{N}$ and $\tilde{r}_1 = \left[r_0^{(1)} / b_0 \right] \in \mathbb{F}_q[x]$ such that

$$\tilde{r}_0 b_1 + r_1^{(0)} = r_0^{(1)} = \tilde{r}_1 b_0 + d_1.$$

Continuing in the same manner, for $k \ge 1$, we define

$$r^{(k)} := r_0^{(k)} + r_1^{(k)} y + \cdots + r_{n-1}^{(k)} y^{n-1} = \frac{r^{(k-1)} - d_{k-1}}{y} \tag{6}$$

so that, for $i = 0, 1, \ldots, n-1$, we have

$$\begin{aligned} r_i^{(k)} &= \tilde{r}_{k-1} b_{i+1} + r_{i+1}^{(k-1)} \\ &= \tilde{r}_{k-1} b_{i+1} + \tilde{r}_{k-2} b_{i+2} + \cdots + \tilde{r}_0 b_{i+k} + r_{i+k}^{(0)}. \end{aligned} \tag{7}$$

In particular, there exist unique $d_k \in \mathcal{N}$ and $\tilde{r}_k \in \mathbb{F}_q[x]$ such that

$$\tilde{r}_{k-1} b_1 + \tilde{r}_{k-2} b_2 + \cdots + \tilde{r}_0 b_k + r_k^{(0)} = r_0^{(k)} = \tilde{r}_k b_0 + d_k, \tag{8}$$

where

$$\tilde{r}_k = \left[r_0^{(k)} / b_0 \right] \tag{9}$$

and $\tilde{r}_j = 0$ if $j < 0,\ b_j = 0$ if $j > n,\ b_n = 1,\ r_j^{(0)} = 0$ if $j > n-1$. The y-adic representation of r is thus of the form

$$r = r^{(0)} = d_0 + d_1 y + \cdots + d_{k-1} y^{k-1} + y^k r^{(k)}. \tag{10}$$

If there exists $k \in \mathbb{N}$ such that $r^{(k)} = 0$, then (10) yields a finite y-adic representation for r of *length* k. If there are indices $j < k$ such that $r^{(j)} = r^{(k)}$, then (10) yields an ultimately *periodic* representation for r with *period* $k - j$.

The two main results of Scheicher-Thuswaldner are

I. $p(x, y)$ is a DS-polynomial if and only if $\max_{i=1,\ldots,n-1} \deg b_i < \deg b_0$;

II. for each $r \in \mathcal{R} \setminus \{0\}$, the sequence $\mathcal{U}_r := \left(r = r^{(0)}, r^{(1)}, r^{(2)}, \ldots\right)$ is ultimately periodic if and only if $\max_{i=1,\ldots,n-1} \deg b_i \le \deg b_0$.

We are interedted here in investigating what kind of expansions are possible in the simplest case when $n = 2$.

Throughout the rest of the paper, let

$$p(x, y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x, y]$$

with

$$B_0 := \deg b_0 > 0,\ B_1 := \deg b_1 = \beta_1 + B_0 \quad (\beta_1 \in \mathbb{Z}).$$

Here,

$$\mathcal{R} := \mathbb{F}_q[x, y] / (p(x, y)) = \{c_0 + c_1 y\ ;\ c_i \in \mathbb{F}_q[x]\}.$$

Take any starting element $r := r^{(0)} \in \mathcal{R} \setminus \{0\}$, and write it using Scheicher-Thuswaldner algorithm steps (7) and (8) recursively, we get

$$r^{(0)} = r_0^{(0)} + r_1^{(0)} y = (\tilde{r}_0 b_0 + d_0) + r_1^{(0)} y \tag{11}$$

$$r^{(1)} = r_0^{(1)} + r_1^{(1)} y = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = (\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y \tag{12}$$

$$r^{(2)} = r_0^{(2)} + r_1^{(2)} y = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y \tag{13}$$

$$\vdots$$

$$r^{(k)} = r_0^{(k)} + r_1^{(k)} y = (\tilde{r}_{k-1} b_1 + \tilde{r}_{k-2}) + \tilde{r}_{k-1} y = (\tilde{r}_k b_0 + d_k) + \tilde{r}_{k-1} y \quad (k \ge 2). \tag{14}$$

For each of presentation, let $\mathcal{U}_r = \left(r = r^{(0)}, r^{(1)}, r^{(2)}, \ldots\right)$. The two separate cases, $B_1 \leq B_0$ and $B_1 > B_0$, are analyzed in Sections 2 and 3, respectively.

2 The case $B_1 \leq B_0$

By the two main results of Scheicher-Thuswaldner, each element in $\mathcal{R} \setminus \{0\}$ has an ultimately periodic expansion, which also includes the case of finite expansions. We now determine all possible expansions of elements in $\mathcal{R} \setminus \{0\}$. *Note also that in this case,* $\beta_1 \leq 0$.

There is a **trivial case** when $\deg r_0^{(0)} < B_0$ and $\deg r_1^{(0)} < B_0$, which from the construction step (6) clearly gives $r_0^{(0)} = d_0,\ r^{(1)} = r_1^{(0)} = d_1,\ r^{(2)} = 0$, and so

$$\mathcal{U}_r = \left(r^{(0)}, r^{(1)}, r^{(2)} = 0, 0, \ldots\right)$$

is a finite sequence of length ≤ 2. Furthermore, if $r_1^{(0)} \neq 0$, then the sequence $\mathcal{U}_r$ has length 2, while if $r_1^{(0)} = 0$, the sequence is of length 1.

For the rest of this section, we assume $\deg r_0^{(0)} \geq B_0$ *or* $\deg r_1^{(0)} \geq B_0$.

From the relation (12), we treat separately two cases: $\deg r_0^{(1)} < B_0$ and $\deg r_0^{(1)} \geq B_0$.

Case 1. $\deg r_0^{(1)} = \deg\left(\tilde{r}_0 b_1 + r_1^{(0)}\right) < B_0$.

Then $\tilde{r}_1 = 0$ and $r^{(2)} = \tilde{r}_0$ with $\deg r_0^{(2)} = \deg \tilde{r}_0 = \deg r_0^{(0)} - B_0$.

If $\tilde{r}_0 = 0$, then $r^{(2)} = 0$, and so $r^{(0)}$ has a finite expansion of length 2.

If $\tilde{r}_0 \neq 0$, there are two subcases.

Subcase 1.1: $\deg \tilde{r}_0 < B_0$. We have $r^{(2)} = \tilde{r}_0 = d_2$, and (6) yields $r^{(3)} = 0$, i.e., $r^{(0)}$ has a finite expansion of length 3.

Subcase 1.2: $\deg \tilde{r}_0 \geq B_0$. From (13), we have

$$r^{(2)} = \tilde{r}_0 = \tilde{r}_2 b_0 + d_2 \neq 0$$

with

$$\tilde{r}_2 \neq 0,\ \ \deg \tilde{r}_2 = \deg \tilde{r}_0 - B_0 = \deg r_0^{(0)} - 2B_0. \tag{15}$$

From (14), we have

$$r^{(3)} = \tilde{r}_2 b_1 + \tilde{r}_2 y = (\tilde{r}_3 b_0 + d_3) + \tilde{r}_2 y \neq 0$$

with

$$\deg r_0^{(3)} = \deg \tilde{r}_2 b_1 = \deg r_0^{(0)} + \beta_1 - B_0,\ \ \deg \tilde{r}_3 = \deg r_0^{(0)} + \beta_1 - 2B_0. \tag{16}$$

From (14), we have

$$r^{(4)} = (\tilde{r}_3 b_1 + \tilde{r}_2) + \tilde{r}_3 y = (\tilde{r}_4 b_0 + d_4) + \tilde{r}_3 y. \tag{17}$$

Consider now the two possibilities $\beta_1 = 0$ and $\beta_1 \neq 0$.

$\oplus$ **If** $\beta_1 = 0$, using (16) and (15), we get $\deg \tilde{r}_3 b_1 = \deg r_0^{(0)} - B_0 > \deg r_0^{(0)} - 2B_0 = \deg \tilde{r}_2$, and so

$$\deg r_0^{(4)} = \deg \tilde{r}_3 b_1 = \deg r_0^{(0)} - B_0, \ \ \deg \tilde{r}_4 = \deg \tilde{r}_3 b_1 - \deg b_0 = \deg r_0^{(0)} - 2B_0.$$

Continuing in the same manner, we obtain

$$r^{(k)} = r_0^{(k)} + r_1^{(k)} y = (\tilde{r}_{k-1} b_1 + \tilde{r}_{k-2}) + \tilde{r}_{k-1} y$$

with

$$\deg r_0^{(k)} = \deg r_0^{(0)} - B_0, \ \ \deg \tilde{r}_k = \deg r_0^{(0)} - 2B_0 \ \ (k \geq 2).$$

Since $r_0^{(k)}$ and $\tilde{r}_k$ are polynomials over a finite field of fixed degree, there must exist two integers j_1, j_2 with $2 \leq j_1 < j_2$ such that $r^{(j_1)} = r^{(j_2)}$. This implies $r^{(0)}$ has a periodic, non-finite, expansion with period $j_2 - j_1$.

$\oplus$ **If** $\beta_1 < 0$ then $r^{(4)}$ is of the same shape as in (17), but using (16) and (15), we get

$$\deg \tilde{r}_3 b_1 = \deg r_0^{(0)} + 2\beta_1 - B_0, \quad \deg \tilde{r}_2 = \deg r_0^{(0)} - 2B_0.$$

There are three possibilities depending on $\deg \tilde{r}_3 b_1$ and $\deg \tilde{r}_2$.

- $\deg \tilde{r}_3 b_1 > \deg \tilde{r}_2$ (i.e., $\deg r_0^{(0)} + 2\beta_1 - B_0 > \deg r_0^{(0)} - 2B_0 \iff B_0 + 2\beta_1 > 0$). The relations (17) and (16) give then

$$\deg r_0^{(4)} = \deg \tilde{r}_3 b_1 = \deg r_0^{(0)} + 2\beta_1 - B_0,$$

$$\deg \tilde{r}_4 = \deg r_0^{(0)} + 2\beta_1 - 2B_0 = \deg \tilde{r}_3 + \beta_1. \tag{18}$$

From the relation (14),

$$r^{(5)} = (\tilde{r}_4 b_1 + \tilde{r}_3) + \tilde{r}_4 y = (\tilde{r}_5 b_0 + d_5) + \tilde{r}_4 y, \tag{19}$$

using (18), we get

$$\deg r_0^{(5)} = \deg \tilde{r}_4 b_1 = \deg r_0^{(0)} + 3\beta_1 - B_0, \ \ \deg \tilde{r}_5 = \deg r_0^{(0)} + 3\beta_1 - 2B_0.$$

Continuing in the same manner, for $r^{(k)} = r_0^{(k)} + r_1^{(k)} y$, we have

$$\deg r_0^{(k)} = \deg r_0^{(0)} + (k-2)\beta_1 - B_0 \qquad (k \geq 4).$$

Thus, $r^{(k)} = 0$ for some $k \geq 5$, observing that $r^{(4)} \neq 0$, i.e., $r^{(0)}$ has a finite expansion of length $k \geq 5$.

- $\deg \tilde{r}_3 b_1 < \deg \tilde{r}_2$ (i.e., $\deg r_0^{(0)} + 2\beta_1 - B_0 < \deg r_0^{(0)} - 2B_0 \iff B_0 + 2\beta_1 < 0$). Then (17) and (16) give

$$\deg r_0^{(4)} = \deg \tilde{r}_2 = \deg r_0^{(0)} - 2B_0,$$

$$\deg \tilde{r}_4 = \deg \tilde{r}_2 - B_0 = \deg r_0^{(0)} - 3B_0 = \deg \tilde{r}_3 - \beta_1 - B_0.$$

Using the same shape of $r^{(5)}$ as in (19), since $\deg \tilde{r}_4 b_1 = \deg \tilde{r}_3$, we get

$$\deg r_0^{(5)} \le \deg r_0^{(0)} + \beta_1 - 2B_0, \ \ \deg \tilde{r}_5 \le \deg r_0^{(0)} + \beta_1 - 3B_0.$$

Using (14), from $r^{(6)} = (\tilde{r}_5 b_1 + \tilde{r}_4) + \tilde{r}_5 y = (\tilde{r}_6 b_0 + d_6) + \tilde{r}_5 y$, since

$$\deg \tilde{r}_5 b_1 \le \deg r_0^{(0)} + 2\beta_1 - 2B_0 < \deg r_0^{(0)} - 3B_0 = \deg \tilde{r}_4,$$

we see that

$$\deg r_0^{(6)} = \deg \tilde{r}_4 = \deg r_0^{(0)} - 3B_0, \ \ \deg \tilde{r}_6 = \deg r_0^{(0)} - 4B_0.$$

Continuing in the same manner, from $r^{(7)} = (\tilde{r}_6 b_1 + \tilde{r}_5) + \tilde{r}_6 y = (\tilde{r}_7 b_0 + d_7) + \tilde{r}_6 y$, we get

$$\deg r_0^{(7)} \le \deg r_0^{(0)} + \beta_1 - 3B_0, \ \ \deg \tilde{r}_7 \le \deg r_0^{(0)} + \beta_1 - 4B_0,$$

and from $r^{(8)} = (\tilde{r}_7 b_1 + \tilde{r}_6) + \tilde{r}_7 y = (\tilde{r}_8 b_0 + d_8) + \tilde{r}_7 y$, we get

$$\deg r_0^{(8)} = \deg r_0^{(0)} - 4B_0, \ \ \deg \tilde{r}_8 = \deg r_0^{(0)} - 5B_0.$$

In general, for $k \ge 2$, we have

$$\begin{aligned} \deg r_0^{(2k)} &= \deg r_0^{(0)} - kB_0 \to -\infty \qquad (k \to \infty) \\ \deg r_0^{(2k+1)} &\le \deg r_0^{(0)} + \beta_1 - kB_0 \to -\infty \qquad (k \to \infty). \end{aligned}$$

Thus, there must exist $k \ge 5$ such that $r^{(k)} = 0$, noting that $r^{(4)} \ne 0$, i.e., $r^{(0)}$ has a finite expansion of length $k \ge 5$.

- $\deg \tilde{r}_3 b_1 = \deg \tilde{r}_2$ (i.e., $\deg r_0^{(0)} + 2\beta_1 - B_0 = \deg r_0^{(0)} - 2B_0 \iff 2\beta_1 = -B_0$).
Then (17) and (16) give

$$\deg r_0^{(4)} \le \deg r_0^{(0)} + 2\beta_1 - B_0, \ \ \deg \tilde{r}_4 \le \deg r_0^{(0)} + 2\beta_1 - 2B_0. \tag{20}$$

Using the same shape of $r^{(5)}$ as in (19), since

$$\deg \tilde{r}_4 b_1 \le \deg r_0^{(0)} + 3\beta_1 - B_0 = \deg \tilde{r}_3$$

we see that

$$\deg r_0^{(5)} \leq \deg r_0^{(0)} + 3\beta_1 - B_0, \;\; \deg \tilde{r}_5 \leq \deg r_0^{(0)} + 3\beta_1 - 2B_0.$$

Using (14), from $r^{(6)} = (\tilde{r}_5 b_1 + \tilde{r}_4) + \tilde{r}_5 y = (\tilde{r}_6 b_0 + d_6) + \tilde{r}_5 y$, since $\deg \tilde{r}_5 b_1 \leq \deg r_0^{(0)} + 4\beta_1 - B_0$, together with (20) we get

$$\deg r_0^{(6)} \leq \deg r_0^{(0)} + 4\beta_1 - B_0, \;\; \deg \tilde{r}_6 \leq \deg r_0^{(0)} + 4\beta_1 - 2B_0.$$

Continuing in the same manner, we have for $k \geq 4$

$$\deg r_0^{(k)} \leq \deg r_0^{(0)} + (k-2)\beta_1 - B_0 \to -\infty \qquad (k \to \infty),$$

and so there exists $k \geq 5$ such that $r^{(k)} = 0$, noting that $r^{(4)} \neq 0$, i.e., $r^{(0)}$ has a finite expansion of length $k \geq 5$.

Summarizing the Case 1, we see that $r^{(0)}$ always has a finite expansion, except only in the case $\deg \tilde{r}_0 \geq B_0, \; \beta_1 = 0$, where it has a non-finite, ultimately periodic expansion.

Case 2. $\deg r_0^{(1)} = \deg \left(\tilde{r}_0 b_1 + r_1^{(0)} \right) \geq B_0$.
Then (12) gives $\tilde{r}_1 \neq 0$, and we have two subcases depending upon whether $\tilde{r}_0 = 0$.

Subcase 2.1: $\tilde{r}_0 = 0$ ($\Longleftrightarrow \deg r_0^{(0)} < B_0$).
The relation (12) gives $r^{(1)} = r_1^{(0)} = \tilde{r}_1 b_0 + d_1$, and so $\deg \tilde{r}_1 = \deg r_1^{(0)} - B_0$. From (13), we have $r^{(2)} = \tilde{r}_1 b_1 + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y \neq 0$ and so

$$\deg r_0^{(2)} = \deg \tilde{r}_1 b_1 = \deg r_1^{(0)} + \beta_1, \;\; \deg \tilde{r}_2 = \deg r_1^{(0)} + \beta_1 - B_0.$$

$\oplus$ **If** $\beta_1 = 0$, from (14), we get $r^{(3)} = (\tilde{r}_2 b_1 + \tilde{r}_1) + \tilde{r}_2 y = (\tilde{r}_3 b_0 + d_3) + \tilde{r}_2 y$, with

$$\deg r_0^{(3)} = \deg \tilde{r}_2 b_1 = \deg r_1^{(0)}, \;\; \deg \tilde{r}_3 = \deg r_1^{(0)} - B_0.$$

From (14) again, we get $r^{(4)} = (\tilde{r}_3 b_1 + \tilde{r}_2) + \tilde{r}_3 y = (\tilde{r}_4 b_0 + d_4) + \tilde{r}_3 y$, with $\deg \tilde{r}_3 b_1 = \deg r_1^{(0)}$ and so

$$\deg r_0^{(4)} = \deg \tilde{r}_3 b_1 = \deg r_1^{(0)}, \;\; \deg \tilde{r}_4 = \deg r_1^{(0)} - B_0.$$

Continuing in the same manner, for $r^{(k)} = r_0^{(k)} + \tilde{r}_{k-1} y$, we have

$$\deg r_0^{(k)} = \deg r_1^{(0)}, \;\; \deg \tilde{r}_{k-1} = \deg r_1^{(0)} - B_0 \quad (k \geq 2).$$

Since $r_0^{(k)}, \; \tilde{r}_{k-1}$ both $\in \mathbb{F}_q[x]$, there are indexes $1 \leq j_1 < j_2$ such that $r^{(j_1)} = r^{(j_2)}$, i.e., $r^{(0)}$ has a non-finite ultimately periodic expansion with period $j_2 - j_1$.

⊕ **If** $\beta_1 < 0$, using the same shape of $r^{(3)}$ as in (14) we have $\deg \tilde{r}_2 b_1 = \deg r_1^{(0)} + 2\beta_1$ and $\deg \tilde{r}_1 = \deg r_1^{(0)} - B_0$. Checking three separate possibilities

$$\deg r_1^{(0)} + 2\beta_1 > \deg r_1^{(0)} - B_0, \ \ \deg r_1^{(0)} + 2\beta_1 < \deg r_1^{(0)} - B_0,$$

$$\deg r_1^{(0)} + 2\beta_1 = \deg r_1^{(0)} - B_0,$$

using the same analysis as Subcase 1.2 (with $\beta_1 < 0$), we arrive at the conclusion that $r^{(0)}$ has a finite expansion.

Subcase 2.2. $\tilde{r}_0 \neq 0$.

Then (11) yields $\deg r_0^{(0)} = \deg \tilde{r}_0 b_0$, and so $\deg \tilde{r}_0 b_1 = \deg r_0^{(0)} + \beta_1$. Taking from (12), i.e., $r^{(1)} = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y$ into account, we need to treat three possibilities $\deg r_0^{(0)} + \beta_1 > \deg r_1^{(0)}$, $\deg r_0^{(0)} + \beta_1 < \deg r_1^{(0)}$ and $\deg r_0^{(0)} + \beta_1 = \deg r_1^{(0)}$.

Possibility 1: $\deg r_0^{(0)} + \beta_1 > \deg r_1^{(0)}$. From (12) we have

$$\deg r_0^{(1)} = \deg \tilde{r}_0 b_1 = \deg r_0^{(0)} + \beta_1, \ \ \deg \tilde{r}_1 = \deg r_0^{(0)} + \beta_1 - B_0.$$

⊕ **If** $\beta_1 = 0$, then proceeding as in Subcase 1.2 (with $\beta_1 = 0$), we have

$$\deg r_0^{(k)} = \deg r_0^{(0)}, \ \ \deg \tilde{r}_k = \deg r_0^{(0)} - B_0 \quad (k \geq 1),$$

and so $r^{(0)}$ has a non-finite, periodic expansion.

⊕ **If** $\beta_1 < 0$, then proceeding as in Subcase 1.2 (with $\beta_1 < 0$), we arrive at the conclusion that $r^{(0)}$ has a finite expansion.

Possibility 2: $\deg r_0^{(0)} + \beta_1 < \deg r_1^{(0)}$. From (12) we have

$$\deg r_0^{(1)} = \deg r_1^{(0)}, \ \ \deg \tilde{r}_1 = \deg r_1^{(0)} - B_0.$$

⊕ **If** $\beta_1 = 0$, from (13), $r^{(2)} = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y \neq 0$, we see that $\deg \tilde{r}_1 b_1 = \deg r_1^{(0)}$, $\deg \tilde{r}_0 = \deg r_0^{(0)} - B_0$. By assumption, $\deg r_0^{(0)} + \beta_1 < \deg r_1^{(0)}$ and $\beta_1 = 0$, we get $\deg r_0^{(0)} - B_0 < \deg r_1^{(0)}$ and $\deg r_0^{(2)} = \deg \tilde{r}_1 b_1 = \deg r_1^{(0)}$, $\deg \tilde{r}_2 = \deg r_1^{(0)} - B_0$. From (14), $r^{(k)} = r_0^{(k)} + r_1^{(k)} y = (\tilde{r}_{k-1} b_1 + \tilde{r}_{k-2}) + \tilde{r}_{k-1} y$, and we similarly obtain

$$\deg r_0^{(k)} = \deg r_1^{(0)}, \ \ \deg \tilde{r}_k = \deg r_1^{(0)} - B_0 \quad (k \geq 1).$$

Using the same analysis as in Subcase 1.2 (with $\beta_1 = 0$), we come to the conclusion that $r^{(0)}$ has a non-finite, periodic expansion.

⊕ **If** $\beta_1 < 0$, then from (12), we get $\deg \tilde{r}_1 b_1 = \deg r_1^{(0)} + \beta_1$, $\deg \tilde{r}_0 = \deg r_0^{(0)} - B_0$. Treating three possible cases

$$\deg r_1^{(0)} + \beta_1 > \deg r_0^{(0)} - B_0, \deg r_1^{(0)} + \beta_1 < \deg r_0^{(0)} - B_0, \deg r_1^{(0)} + \beta_1 = \deg r_0^{(0)} - B_0$$

and using the same analysis as in Subcase 1.2 (with $\beta_1 < 0$), we conclude that $r^{(0)}$ has a finite expansion.

Possibility 3: $\deg r_0^{(0)} + \beta_1 = \deg r_1^{(0)}$. From (12), we have

$$\deg r_0^{(1)} \leq \deg r_0^{(0)} + \beta_1, \ \ \deg \tilde{r}_1 \leq \deg r_0^{(0)} + \beta_1 - B_0.$$

$\oplus$ **If** $\beta_1 = 0$, then an analysis similar to Subcase 1.2 (with $\beta_1 = 0$) shows that $\deg r_0^{(k)}$ and $\deg \tilde{r}_k$ $(k \geq 1)$ are bounded, and so either both are of constant values from certain point onward resulting in a non-finite but ultimately periodic expansion, or both are decreasing over certain subsequence resulting in a finite expansion. Example 1 and Example 2 following the statement of Theorem 1 illustrate that both of these possible conclusions do indeed exist.

$\oplus$ **If** $\beta_1 < 0$, then (13) gives $\deg \tilde{r}_1 b_1 \leq \deg r_0^{(0)} + 2\beta_1$, $\deg \tilde{r}_0 = \deg r_0^{(0)} - B_0$. Treating the three possible cases

$$\deg r_0^{(0)} + 2\beta_1 > \deg r_0^{(0)} - B_0,$$

$$\deg r_0^{(0)} + 2\beta_1 < \deg r_0^{(0)} - B_0,$$

$$\deg r_0^{(0)} + 2\beta_1 = \deg r_0^{(0)} - B_0,$$

and using an analysis similar to that of Subcase 1.2 (with $\beta_1 < 0$), we conclude that $r^{(0)}$ has a finite expansion.

Summarizing the findings in Case 2, we have

(a) if $\tilde{r}_0 = 0$, $\beta_1 = 0$, or $\tilde{r}_0 \neq 0$, $\deg r_0^{(0)} + \beta_1 > \deg r_1^{(0)}$, $\beta_1 = 0$, or $\tilde{r}_0 \neq 0$, $\deg r_0^{(0)} + \beta_1 < \deg r_1^{(0)}$, $\beta_1 = 0$, then $r^{(0)}$ has a non-finite and ultimately periodic expansion;

(b) if $\tilde{r}_0 \neq 0$, $\deg r_0^{(0)} + \beta_1 = \deg r_1^{(0)}$, $\beta_1 = 0$, then $r^{(0)}$ can have either a finite or a non-finite but periodic expansion;

(c) in all other cases $r^{(0)}$ has a finite expansion.

Collecting all the results found, we have:

Theorem 1. *Let* $p(x,y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x,y]$ *with*

$$B_0 := \deg b_0 > 0, \quad B_1 := \deg b_1 = \beta_1 + B_0 \quad \textit{with} \quad \beta_1 \leq 0,$$

and let $r := r^{(0)} \in \mathcal{R} \setminus \{0\} := \{c_0 + c_1 y \ ; \ c_i \in \mathbb{F}_q[x]\} \setminus \{0\}$.

(i) If $\beta_1 < 0$, *then* $r^{(0)}$ *always has a finite expansion.*

(ii) If $\beta_1 = 0$, *then* $r^{(0)}$ *always has a non-finite, ultimately periodic expansion except only when* $\tilde{r}_0 \neq 0$, $\deg r_0^{(0)} + \beta_1 = \deg r_1^{(0)}$, $\beta_1 = 0$ *where both finite and non-finite expansions can occur.*

We end this section by giving two examples exemplifying the point made in (b) above.

Example 1. Let $p(x,y) = y^2 + xy - x \in \mathbb{F}_3[x,y]$ with $b_1 = b_0 = x$ and $B_0 = 1 = 0+1 = B_1, \beta_1 = 0$. Take

$$r^{(0)} := r_0^{(0)} + r_1^{(0)} y = (x^2 + x) + (2x^2 + x)y.$$

From algorithm steps (11) - (14), we have

$$\begin{aligned} &r^{(0)} = (x+1)x + (2x^2+x)y \ \ ; \ \tilde{r}_0 \neq 0, \ \deg r_0^{(0)} + \beta_1 = \deg r_1^{(0)} = 2 \\ &r^{(1)} = 2x + (x+1)y, \ r^{(2)} = 1 + 2y, \ r^{(3)} = 2, \ r^{(4)} = 0, \end{aligned}$$

i.e., $r^{(0)}$ has a finite expansion.

Example 2. Let $p(x,y) = y^2 + xy - x \in \mathbb{F}_3[x,y]$ with $B_0 = 1 = 0+1 = B_1$ and $\beta_1 = 0$. Take

$$r^{(0)} := r_0^{(0)} + r_1^{(0)} y = x + xy.$$

From algorithm steps (11) - (14), we have

$$\begin{aligned} &r^{(0)} = x + xy \ \ ; \ \tilde{r}_0 = 1 \neq 0, \ \deg r_0^{(0)} + \beta_1 = \deg r_1^{(0)} = 1 \\ &r^{(1)} = 2x + y, \ r^{(2)} = (2x+1) + 2y, \ r^{(3)} = (2x+2) + 2y = r^{(4)}, \end{aligned}$$

and so $r^{(0)}$ has a non-finite, ultimately periodic expansion.

3 The case $B_1 > B_0$

This is the case investigated in [1]. The analysis in [1] yields

Theorem 2. *Let*

$$p(x,y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x,y], B_1 := \deg b_1, B_0 := \deg b_0.$$

If $B_1 > B_0 > 0$, then each $r \in \mathcal{R} \setminus \{0\} := \{c_0 + c_1 y; c_i \in \mathbb{F}_q[x]\} \setminus \{0\}$ either has a finite or an infinite but non-periodic Scheicher-Thuswaldner representations.

More precisely, $r \in \mathcal{R} \setminus \{0\}$ has a finite expansion if and only if there exists a non-negative integer k such that

$$\max\left(\deg r_0^{(k)}, \deg r_1^{(k)}\right) < B_0,$$

where $r^{(k)} := r_0^{(k)} + r_1^{(k)} y$; moreover, the sequence $\mathcal{U}_r$ is finite of length $k+1$ if $r_1^{(k)} = 0$ and of length $k+2$ if $r_1^{(k)} \neq 0$.

From Theorem 2, there are only finite and infinite, non-periodic expansions possible. We push here a little further by imposing a condition about a unique maximal element, which enables us to determine all possible shapes of expansions.

Theorem 3. *Let*

$$\begin{aligned}
&p(x,y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x,y] \\
&\deg b_0 := B_0 \in \mathbb{N},\ \deg b_1 := \beta_1 + B_0 \quad (\beta_1 \in \mathbb{Z}) \\
&r^{(0)} := r_0^{(0)} + r_1^{(0)} y = (\tilde{r}_0 b_0 + d_0) + r_1^{(0)} y \in \mathcal{R} \setminus \{0\}
\end{aligned}$$

and generally (from the Scheicher-Thuswaldner algorithm) for $k \geq 1$, let

$$\begin{aligned}
r^{(k)} &:= r_0^{(k)} + r_1^{(k)} y \\
&= \left(\tilde{r}_{k-1} b_1 + \tilde{r}_{k-2} b_2 + \cdots + \tilde{r}_0 b_k + r_k^{(0)}\right) + r_1^{(k)} y \\
&\qquad \textit{where } \tilde{r}_j = 0 \textit{ if } j < 0,\ b_j = 0 \textit{ if } j > 2,\ b_2 = 1,\ r_j^{(0)} = 0 \textit{ if } j > 1 \\
&= (\tilde{r}_k b_0 + d_k) + r_1^{(k)} y.
\end{aligned}$$

For $1 \leq j \leq 2$, let

$$S_j = \left\{ \deg(\tilde{r}_{j-1} b_1) - B_0,\ \deg(\tilde{r}_{j-2} b_2) - B_0,\ \deg r_j^{(0)} - B_0 \right\}.$$

For $j \in \mathbb{N}$, let

$$m_j = \max(S_j) := \max\{x\ ;\ x \in S_j\}.$$

For $j \geq 3$, let

$$S_j = \{m_{j-1} + \beta_1,\ m_{j-2} - B_0\}.$$

To avoid the trivial case, assume that at least one of $\deg r_0^{(0)}$ and $\deg r_1^{(0)}$ is $\geq B_0$. For all $j \geq 1$, assume that

$$m_j > \max(S_j - \{m_j\}).$$

Then $r^{(0)}$ has an infinite, non-periodic expansion if and only if $\beta_1 > 0$.

Proof. Under the stated hypotheses, we get $m_k = \deg \tilde{r}_k \ \ (k \geq 1)$.
To prove the sufficiency part, we assume that $\beta_1 > 0$. From

$$r^{(k)} = r_0^{(k)} + r_1^{(k)} y = (\tilde{r}_k b_0 + d_k) + r_1^{(k)} y \qquad (k \geq 3),$$

we observe that $m_k = \deg \tilde{r}_k = \deg r_0^{(k)} - B_0$. Our strategy is to show that the sequence $(m_k)_{k \geq 1}$ is strictly increasing, which implies that $r^{(k)}$ can neither vanish nor become periodic, yielding an infinite, non-periodic expansion for $r = r^{(0)}$.

Take a nontrivial $r^{(0)} := r_0^{(0)} + r_1^{(0)} y \in \mathcal{R} \setminus \{0\}$ with $r_0^{(0)} \neq 0, \deg r_0^{(0)} \geq B_0$

or $\deg r_1^{(0)} \geq B_0$. Recall that

$$\begin{aligned} r^{(0)} &= r_0^{(0)} + r_1^{(0)} y = (\tilde{r}_0 b_0 + d_0) + r_1^{(0)} y \quad (\deg \tilde{r}_0 = \deg r_0^{(0)} - B_0) \\ r^{(1)} &= r_0^{(1)} + r_1^{(1)} y = \left(\tilde{r}_0 b_1 + r_1^{(0)}\right) + r_1^{(1)} y \\ &= (\tilde{r}_1 b_0 + d_1) + r_1^{(1)} y \\ \deg \tilde{r}_0 b_1 &= \left(\deg r_0^{(0)} - B_0\right) + (\beta_1 + B_0) = \deg r_0^{(0)} + \beta_1. \end{aligned}$$

That m_1 is non-trivial, i.e. $\neq -\infty$, follows by noting that since

$$\deg \tilde{r}_1 = m_1 = \max\left\{\deg(\tilde{r}_0 b_1) - B_0,\ \deg r_1^{(0)} - B_0\right\},\ \text{and } r_0^{(0)} \neq 0,$$

we get

$$\deg \tilde{r}_1 = m_1 \geq \deg(\tilde{r}_0 b_1) - B_0 = \deg r_0^{(0)} + \beta_1 - B_0 \geq 0.$$

From

$$\begin{aligned} r^{(2)} &= r_0^{(2)} + r_1^{(2)} y = (\tilde{r}_1 b_1 + \tilde{r}_0) + r_1^{(2)} y \\ &= (\tilde{r}_2 b_0 + d_2) + r_1^{(2)} y \\ \deg \tilde{r}_1 b_1 &= \deg \tilde{r}_1 + \deg b_1 = m_1 + \beta_1 + B_0, \end{aligned}$$

since

$$\deg \tilde{r}_2 = m_2 = \max\{\deg(\tilde{r}_1 b_1) - B_0,\ \deg(\tilde{r}_0) - B_0\},$$

we get

$$m_2 \geq \deg(\tilde{r}_1 b_1) - B_0 = m_1 + \beta_1 > m_1.$$

Next, from

$$\begin{aligned} r^{(3)} &= r_0^{(3)} + r_1^{(3)} y = (\tilde{r}_2 b_1 + \tilde{r}_1) + r_1^{(3)} y \\ &= (\tilde{r}_3 b_0 + d_3) + r_1^{(3)} y \\ \deg \tilde{r}_2 b_1 &= \deg \tilde{r}_2 + \deg b_1 = m_2 + \beta_1 + B_0, \end{aligned}$$

since

$$\begin{aligned} \deg \tilde{r}_3 = m_3 &= \max\{\deg(\tilde{r}_2 b_1) - B_0,\ \deg(\tilde{r}_1) - B_0\} \\ &= \max\{m_2 + \beta_1,\ m_1 - B_0\}, \end{aligned} \tag{21}$$

we get

$$m_3 \geq \deg(\tilde{r}_2 b_1) - B_0 = m_2 + \beta_1 > m_2.$$

Proceeding one more step further, from

$$\begin{aligned} r^{(4)} &= r_0^{(4)} + r_1^{(4)} y = (\tilde{r}_3 b_1 + \tilde{r}_2) + r_1^{(4)} y \\ &= (\tilde{r}_4 b_0 + d_4) + r_1^{(4)} y \\ \deg \tilde{r}_3 b_1 &= \deg \tilde{r}_3 + \deg b_1 = m_3 + \beta_1 + B_0, \end{aligned}$$

since

$$\begin{aligned} \deg \tilde{r}_4 = m_4 &= \max \{\deg (\tilde{r}_3 b_1) - B_0,\ \deg (\tilde{r}_2) - B_0\} \\ &= \max \{m_3 + \beta_1,\ m_2 - B_0\}, \end{aligned} \tag{22}$$

we get

$$m_4 \geq \deg (\tilde{r}_3 b_1) - B_0 = m_3 + \beta_1 > m_3.$$

In general, using induction, for $k \geq 3$, from

$$\begin{aligned} r^{(k)} &= r_0^{(k)} + r_1^{(k)} y = (\tilde{r}_{k-1} b_1 + \tilde{r}_{k-2}) + r_1^{(k)} y \\ &= (\tilde{r}_k b_0 + d_k) + r_1^{(k)} y \\ \deg \tilde{r}_{k-1} b_1 &= \deg \tilde{r}_{k-1} + \deg b_1 = m_{k-1} + \beta_1 + B_0, \end{aligned}$$

since

$$\deg \tilde{r}_k = m_k = \max \{m_{k-1} + \beta_1,\ m_{k-2} - B_0\}, \tag{23}$$

we get

$$m_k \geq m_{k-1} + \beta_1 > m_{k-1},$$

as desired.

To prove the necessity part, assume that $r^{(0)}$ has an infinite, non-periodic expansion, i.e., the sequence $\left(r^{(0)}, r^{(1)}, r^{(2)}, \ldots\right)$ is infinite and non-periodic. By the result II of Scheicher-Thuswaldner mentioned in the introduction, $\beta_1 + B_0 := \deg b_1 > \deg b_0 := B_0$, which in turn yields that $\beta_1 > 0$. □

Our final analysis shows that even in the simplest case of second degree polynomial $p(x, y)$ to determine all elements having finite expansions can be tedious.

From

$$r^{(0)} := r_0^{(0)} + r_1^{(0)} y = (\tilde{r}_0 b_0 + d_0) + r_1^{(0)} y \in \mathcal{R} \setminus \{0\}$$

and

$$r^{(1)} = \left(\tilde{r}_0 b_1 + r_1^{(0)}\right) + \tilde{r}_0 y,$$

there are two possible cases.
Case 1: $\tilde{r}_0 b_1 + r_1^{(0)} = 0$. Thus,

$$\begin{aligned} r^{(2)} &= \tilde{r}_0 = \tilde{r}_2 b_0 + d_2, \quad \deg(\tilde{r}_0) = \deg(\tilde{r}_2) + B_0 \text{ (provided } \tilde{r}_2 \neq 0) \\ r^{(3)} &= \tilde{r}_2 b_1 + \tilde{r}_2 y = (\tilde{r}_3 b_0 + d_3) + \tilde{r}_2 y, \\ &\quad \deg(\tilde{r}_3 b_0) = \deg(\tilde{r}_2 b_1) = \deg(\tilde{r}_0) + \beta_1 \text{ (provided } \tilde{r}_2 \neq 0) \\ r^{(4)} &= (\tilde{r}_3 b_1 + \tilde{r}_2) + \tilde{r}_3 y, \quad \deg(\tilde{r}_3 b_1) = \deg(\tilde{r}_0) + 2\beta_1. \end{aligned}$$

If $\tilde{r}_0 = 0$, then $r_1^{(0)} = 0 = r^{(1)}$ and so $r^{(0)} = d_0$ has a **finite expansion of length 1(A)**.

For $\tilde{r}_0 \neq 0$, if $\deg(\tilde{r}_0) < B_0$, then $\tilde{r}_2 = 0$ and so $r^{(3)} = 0$ showing that $r^{(0)} = (d_2 b_0 + d_0) + (-d_2 b_1)y$ has a **finite expansion of length 3 (B)**, while if $\deg(\tilde{r}_0) \geq B_0$, then the degrees of the constant terms in $r^{(k)}$ are strictly increasing nonnegative integers, and so $r^{(0)}$ has an infinite non-periodic expansion.
Case 2: $\tilde{r}_0 b_1 + r_1^{(0)} \neq 0$. Thus,

$$\begin{aligned} r^{(1)} &= (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = (\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y, \\ &\quad \deg(\tilde{r}_1 b_0) = \deg(\tilde{r}_0 b_1 + r_1^{(0)}) \text{ (provided } \tilde{r}_1 \neq 0) \qquad (24) \\ r^{(2)} &= (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y, \\ &\quad \deg(\tilde{r}_2 b_0) = \deg(\tilde{r}_1 b_1 + \tilde{r}_0) \text{ (provided } \tilde{r}_2 \neq 0) \qquad (25) \\ r^{(3)} &= (\tilde{r}_2 b_1 + \tilde{r}_1) + \tilde{r}_2 y = (\tilde{r}_3 b_0 + d_3) + \tilde{r}_2 y, \\ &\quad \deg(\tilde{r}_3 b_0) = \deg(\tilde{r}_2 b_1 + \tilde{r}_1) \text{ (provided } \tilde{r}_3 \neq 0) \qquad (26) \\ r^{(4)} &= (\tilde{r}_3 b_1 + \tilde{r}_2) + \tilde{r}_3 y = (\tilde{r}_4 b_0 + d_4) + \tilde{r}_3 y, \\ &\quad \deg(\tilde{r}_4 b_0) = \deg(\tilde{r}_3 b_1 + \tilde{r}_2) \text{ (provided } \tilde{r}_4 \neq 0). \qquad (27) \end{aligned}$$

• If $\deg r_1^{(0)} > \deg(\tilde{r}_0 b_1)$, then (24) gives

$$\deg(\tilde{r}_1 b_0) = \deg(r_1^{(0)}) > \deg(\tilde{r}_0 b_1), \text{ i.e., } \deg(\tilde{r}_1) > \deg(\tilde{r}_0) + \beta_1,$$

which in turn, (25) gives

$$\deg(\tilde{r}_2 b_0) = \deg(\tilde{r}_1 b_1), \text{ i.e., } \deg(\tilde{r}_2) > \deg(\tilde{r}_0) + 2\beta_1.$$

Continuing in the same manner, we see that the degrees of the constant terms in $r^{(k)}$ are strictly increasing integers, and so $r^{(0)}$ has an infinite non-periodic expansion.

• If $\deg r_1^{(0)} < \deg(\tilde{r}_0 b_1)$, then (24) gives

$$\deg(\tilde{r}_1 b_0) = \deg(\tilde{r}_0 b_1) > \deg r_1^{(0)}, \text{ i.e., } \deg(\tilde{r}_1) = \deg(\tilde{r}_0) + \beta_1,$$

which in turn, (25) gives

$$\deg(\tilde{r}_2 b_0) = \deg(\tilde{r}_1 b_1), \text{ i.e., } \deg(\tilde{r}_2) = \deg(\tilde{r}_0) + 2\beta_1.$$

Continuing in the same manner, we see that the degrees of the constant terms in $r^{(k)}$ are strictly increasing integers, and so $r^{(0)}$ has an infinite non-periodic expansion.

The preceding discussion shows that for $r^{(0)}$ to have a finite expansion, we must have

$$\deg r_1^{(0)} = \deg(\tilde{r}_0 b_1) = \deg(\tilde{r}_0) + B_1. \tag{28}$$

Note that if $\tilde{r}_0 = 0$, then the case defining condition indicates that $r_1^{(0)} \neq 0$, and so $\deg r_1^{(0)} > \deg(\tilde{r}_0 b_1)$, and the above analysis shows that $r^{(0)}$ has an infinite non-periodic expansion. Henceforth, we assume that $\tilde{r}_0 \neq 0$. Letting $D := \deg(\tilde{r}_0 b_1 + r_1^{(0)}) \geq 0$.

∘ If $D < B_0$, then (24) gives $\tilde{r}_1 = 0$ and so

$$\begin{aligned}
r^{(1)} &= (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = d_1 + \tilde{r}_0 y, \ \ \deg d_1 = \deg(\tilde{r}_0 b_1 + r_1^{(0)}) = D \\
r^{(2)} &= \tilde{r}_0 = \tilde{r}_2 b_0 + d_2, \quad \deg(\tilde{r}_2 b_0) = \deg \tilde{r}_0 \text{ (provided } \tilde{r}_2 \neq 0) \\
r^{(3)} &= \tilde{r}_2 b_1 + \tilde{r}_2 y = (\tilde{r}_3 b_0 + d_3) + \tilde{r}_2 y, \ \ \deg(\tilde{r}_3 b_0) = \deg(\tilde{r}_2 b_1) \text{ (provided } \tilde{r}_3 \neq 0) \\
r^{(4)} &= (\tilde{r}_3 b_1 + \tilde{r}_2) + \tilde{r}_3 y = (\tilde{r}_4 b_0 + d_4) + \tilde{r}_3 y, \ \ \deg(\tilde{r}_4 b_0) = \deg(\tilde{r}_3 b_1 + \tilde{r}_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(provided } \tilde{r}_4 \neq 0).
\end{aligned}$$

If $\deg \tilde{r}_0 < B_0$, then $\tilde{r}_2 = 0$ and so $r^{(3)} = 0$, i.e., $r^{(0)} = (d_2 b_0 + d_0) + (d_1 - d_2 b_1) y$ has a **finite expansion of length 3 (C)**. If $\deg \tilde{r}_0 \geq B_0$, then $\tilde{r}_2 \neq 0$ and so $\deg \tilde{r}_3 = \deg(\tilde{r}_2) + \beta_1$. Continuing in the same manner, we get $\deg \tilde{r}_4 = \deg(\tilde{r}_3) + \beta_1 = \deg(\tilde{r}_2) + 2\beta_1$, i.e., the degrees of the constant terms in $r^{(k)}$ are strictly increasing integers, and so $r^{(0)}$ has an infinite non-periodic expansion.

∘ If $D \geq B_0$, then (24) gives $\tilde{r}_1 \neq 0$ and so

$$r^{(1)} = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = (\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y, \ \deg(\tilde{r}_1 b_0) = \deg(\tilde{r}_0 b_1 + r_1^{(0)}) = D.$$

Proceeding to the next step, from

$$r^{(2)} = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y,$$

using the same arguments as above, we see that for a finite expansion, we must have $\deg(\tilde{r}_1 b_1) = \deg \tilde{r}_0$, and together with (28), we must have

$$\deg r_1^{(0)} = \deg(\tilde{r}_0) + B_1 = \deg(\tilde{r}_1) + 2B_1,$$

and if the procedure can be continued, then $\deg r_1^{(0)} = \deg(\tilde{r}_k) + (k+1)B_1$ for each $k \in \mathbb{N}$. Since $\deg r_1^{(0)}$ is fixed, for a finite expansion to occur this cannot be continued indefinitely, and so one of the possibilities (A), (B), (C) must occur for some index k, i.e., we have:

Theorem 4. *Let* $p(x,y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x,y]$. *Then* $r^{(0)} := r_0^{(0)} + r_1^{(0)} y \in \mathcal{R} \setminus \{0\}$ *has a finite expansion if and only if there is* $k \in \mathbb{N}$ *for which* $r^{(k)}$ *takes one of the following three forms*

$$r^{(k)} = d_k, r^{(k)} = (d_{k+2}b_0 + d_k) + (-d_{k+2}b_1)y, r^{(k)} = (d_{k+2}b_0 + d_k) + (d_{k+1} - d_{k+2}b_1)y,$$

where $d_k, d_{k+1}, d_{k+2} \in \mathcal{N}$.

In particular, for the first cycle as elaborated above, finite expansions occur if and only if the starting element takes one of the following three forms (corresponding to the forms stated in (A), (B), (C), respectively)

$$r^{(0)} = d_0, r^{(0)} = (d_2 b_0 + d_0) + (-d_2 b_1)y, r^{(0)} = (d_2 b_0 + d_0) + (d_1 - d_2 b_1)y,$$

where $d_0, d_1, d_2 \in \mathcal{N}$.

References

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