East-West J. of Mathematics: Vol.14 , No. 2 (2012) pp. 138-143

CONTINUED FRACTIONS REPRESENTING CERTAIN ANALOGUES OF EXPONENTIAL ELEMENTS IN $\mathbb{F}_q((1/x))$

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Abstract

Let $(Q_i)_{i=1}^{\infty}$ be a sequence of nonzero monic polynomials over a finite field $\mathbb{F}_q$ satisfying $Q_1 \cdots Q_i | Q_{i+1}$ $(i \in \mathbb{N})$. Let $\alpha(n) = \sum_{i=1}^{n} 1/Q_1 \cdots Q_i$ and $\alpha(\infty) = \sum_{i=1}^{\infty} 1/Q_1 \cdots Q_i$. It is shown that the continued fraction for $\alpha(\infty)$ in the function field $\mathbb{F}_q((1/x))$ can be explicitly given. As an application, by choosing suitable polynomials Q_i, explicit continued fraction expansion of a function field analogue of some exponential elements in $\mathbb{F}_q((1/x))$ $(q \geq 2)$, are derived. This gives an extension of some earlier work of Thakur in 1992.

1 Introduction

As seen in [1]-[6] and [8]-[14], there have appeared various relationships between series and continued fractions. Some of these results connect certain types of series with continued fraction expansions having symmetric patterns.

In [13], Thakur explored the question of finding "interesting patterns for interesting numbers" in the function field setting and found an interesting continued fraction for an analogue of e for $\mathbb{F}_q((1/x))$.

Let $\mathbb{F}_q$ be a finite field of q elements, $(q \geq 2)$, $\mathbf{F} := \mathbb{F}_q((1/x))$ the field of formal Laurent series over $\mathbb{F}_q$ equipped with a degree valuation $|\cdot|$ defined by $|x^{-1}| = e^{-1}$. Every element $\xi \in \mathbf{F}$ can be uniquely written as a simple continued fraction of the form

Keywords: function field, exponential element
(2010) AMS Classification: 11A55, 11F70

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} := [a_0; a_1, a_2, \ldots],$$

where $a_0 \in \mathbb{F}_q[x]$ and $a_n \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$ $(n \geq 1)$.

Let $C_n/D_n = [a_0; a_1, \ldots, a_n]$ be the nth convergent of the continued fraction for ξ. The following basic properties are easily verified by induction.

1. $D_n C_{n-1} - C_n D_{n-1} = (-1)^n \quad (n \geq 0)$, so that C_n and D_n are relatively prime;
2. $C_{n+1} = a_{n+1} C_n + C_{n-1}$ and $D_{n+1} = a_{n+1} D_n + D_{n-1} \quad (n \geq 0)$;
3. for any $n \geq 0$, we have
$$\frac{\zeta C_n + C_{n-1}}{\zeta D_n + D_{n-1}} = [a_0; a_1, a_2, \ldots, a_n, \zeta];$$
4. if $\dfrac{C_n}{D_n} = [a_0; a_1, \ldots, a_n]$, then $\dfrac{D_n}{D_{n-1}} = [a_n; a_{n-1}, \ldots, a_2, a_1]$.

An analogue of the exponential element in $\mathbf{F}$, introduced by Carlitz (see e.g. [13], [14]), is defined as follows: let $[i] = x^{q^i} - x$, $d_0 = 1$ and $d_i = [i] d_{i-1}^q$ $(i \geq 1)$. The exponential for $\mathbf{F}$ is defined by

$$e(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i}.$$

Put $e := e(1)$. Then the series expansion for e is of the form

$$e = \sum_{i=0}^{\infty} \frac{1}{d_i}.$$

In 1992, Thakur [13] proved the following result which give the continued fraction expansion for $\sum_{i=0}^{n+1} z^{q^i}/d_i$ if the continued fraction expansion for $\sum_{i=0}^{n} z^{q^i}/d_i$ is known.

Proposition 1. *Let* $x_1 = [0; z^{-q}[1]]$, *and if* $x_n = [a_0; a_1, \ldots, a_{2^n-1}]$, *then set*

$$x_{n+1} = [a_0; a_1, \ldots, a_{2^n-1}, \frac{-z^{-q^n(q-2)} d_{n+1}}{d_n^2}, -a_{2^n-1}, \ldots, -a_1].$$

Then

$$x_n = \sum_{i=1}^{n} \frac{z^{q^i}}{d_i}.$$

In particular, $e(z) = z + \lim_{n\to\infty} x_n$ *and for* $q = 2$,

$$e = [1; \underbrace{\underbrace{\underbrace{[1]}, [2], \underbrace{[1]}}, [3], \underbrace{[1], [2], [1]}}, [4], \underbrace{[1], [2], [1], [3], [1], [2], [1]}, [5], \ldots]$$

(More explicitly, for $n > 0$ *the* n*-th partial quotient is* $x^{2^{u_n}} - x$ *with* u_n *being the exponent of the highest power of* 2 *dividing* $2n$*).*

Our objective here is to extend this proposition of Thakur by proving

Theorem 1. *Let* $(Q_i)_{i=1}^{\infty}$ *be a sequence of nonzero monic polynomials over a finite field* $\mathbb{F}_q$*, let*

$$\alpha(n) = \sum_{i=1}^{n} \frac{1}{Q_1 \cdots Q_i} \quad (n \geq 1) \quad \textit{and let} \quad \alpha(\infty) = \sum_{i=1}^{\infty} \frac{1}{Q_1 \cdots Q_i}.$$

Assume that $Q_1 \cdots Q_i | Q_{i+1}$ $(i \in \mathbb{N})$*. Then* $\alpha(1) = [0; Q_1]$ *and if* $\alpha(n) = [0; a_1, \ldots, a_{k_n}]$*, then*

$$\alpha(n+1) = [0; a_1, \ldots, a_{k_n}, \frac{-Q_{n+1}}{Q_1 Q_2 \cdots Q_n}, -a_{k_n}, \ldots, -a_1].$$

Moreover,

$$\alpha(\infty) = [0; \underbrace{\underbrace{Q_1}, \frac{-Q_2}{Q_1}, \underbrace{-Q_1}}, \frac{-Q_3}{Q_1 Q_2}, \underbrace{Q_1, \frac{Q_2}{Q_1}, -Q_1}, \frac{-Q_4}{Q_1 Q_2 Q_3}, \ldots].$$

2 Proofs

Our work makes use of the following lemma, see e.g. [7] and [15].

Lemma 1 (Folding Lemma). *Let* $y \in \mathbf{F}$ *and* $\frac{C_n}{D_n} = [a_0; a_1, a_2, \ldots, a_n]$*. Then*

$$[a_0; a_1, \ldots a_n, y, -a_n, \ldots, -a_1] = \frac{C_n}{D_n} + \frac{(-1)^n}{y D_n^2}.$$

Proof of Theorem 1. It is clear that $\alpha(1) = \frac{1}{Q_1} = [0; Q_1] := \frac{C_{k_1}}{D_{k_1}}$. Since $\gcd(1, Q_1) = 1$ and Q_1 is monic, we have $1 = C_{k_1}$ and $D_{k_1} = Q_1$. By Folding Lemma, we get

$$\alpha(2) = \frac{1}{Q_1} + \frac{1}{Q_1 Q_2} = \frac{1}{Q_1} + \frac{-1}{\frac{-Q_2}{Q_1} Q_1^2} = [0; Q_1, \frac{-Q_2}{Q_1}, -Q_1] := \frac{C_{k_2}}{D_{k_2}}.$$

For $n \geq 2$, let $[0; a_1, \ldots, a_{k_n}]$ be a continued fraction expansion for $\alpha(n)$ whose k_nth convergent is C_{k_n}/D_{k_n}; observe that k_n is an odd integer. By Folding Lemma, we get

$$[0; a_1, \ldots, a_{k_n}, \frac{-Q_{n+1}}{Q_1 \cdots Q_n}, -a_{k_n}, \ldots, -a_1] = \frac{C_{k_n}}{D_{k_n}} + \frac{(-1)^{k_n}}{\frac{-Q_{n+1}}{Q_1 \cdots Q_n} D_{k_n}^2}$$
$$= \alpha(n) + \frac{(-1)^{k_n}}{\frac{-Q_{n+1}}{Q_1 \cdots Q_n} D_{k_n}^2}.$$

On the other hand, we have

$$\frac{1}{Q_1} + \cdots + \frac{1}{Q_1 \cdots Q_n} = \frac{(Q_2 \cdots Q_n) + (Q_3 \cdots Q_n) + \cdots + Q_n + 1}{Q_1 \cdots Q_n}.$$

We assert that $\gcd\left((Q_2 \cdots Q_n) + (Q_3 \cdots Q_n) + \cdots + Q_n + 1, Q_1 Q_2 \cdots Q_n\right) = 1$ for $n \geq 2$. Suppose there exists a prime $p \in \mathbb{F}_q[x]$ such that

$$p | \left((Q_2 \cdots Q_n) + (Q_3 \cdots Q_n) + \cdots + Q_n + 1\right) \quad \text{and} \quad p | (Q_1 Q_2 \cdots Q_n).$$

Since $p|(Q_1 Q_2 \cdots Q_n)$, $p|Q_k$ for some $1 \leq k \leq n$, then $p|Q_j Q_{j+1} \cdots Q_n$ for all $2 \leq j \leq k$. Since $Q_1 \cdots Q_k | Q_{k+l}$ for all $1 \leq l \leq n-k$, we have $Q_k | Q_{k+l} \cdots Q_n$ for all $1 \leq l \leq n-k$ and so $p|Q_{k+l} \cdots Q_n$ for all $1 \leq l \leq n-k$. Since $p| \left((Q_2 \cdots Q_n) + (Q_3 \cdots Q_n) + \cdots + Q_n + 1\right)$, then we get $p|1$, which is a contradiction. Thus $\gcd\left((Q_2 \cdots Q_n) + (Q_3 \cdots Q_n) + \cdots + Q_n + 1, Q_1 Q_2 \cdots Q_n\right) = 1$. Since all Q_i are monic, then $D_{k_n} = Q_1 Q_2 \cdots Q_n$. Hence

$$\alpha(n) + \frac{(-1)^{k_n}}{\frac{-Q_{n+1}}{Q_1 \cdots Q_n} D_{k_n}^2} = \alpha(n) + \frac{-1}{\frac{-Q_{n+1}}{Q_1 \cdots Q_n} (Q_1 Q_2 \cdots Q_n)^2}$$
$$= \alpha(n) + \frac{1}{Q_1 \cdots Q_n Q_{n+1}} = \alpha(n+1).$$

Consequently, we have

$$\alpha(1) = [0; Q_1]$$
$$\alpha(2) = [0; \underbrace{Q_1}, \frac{-Q_2}{Q_1}, \underbrace{-Q_1}]$$
$$\alpha(3) = [0; \underbrace{\underbrace{Q_1}, \frac{-Q_2}{Q_1}, \underbrace{-Q_1}}, \frac{-Q_3}{Q_1 Q_2}, \underbrace{Q_1, \frac{Q_2}{Q_1}, -Q_1}]$$
$$\vdots$$
$$\alpha(\infty) = [0; \underbrace{\underbrace{Q_1}, \frac{-Q_2}{Q_1}, \underbrace{-Q_1}}, \frac{-Q_3}{Q_1 Q_2}, \underbrace{Q_1, \frac{Q_2}{Q_1}, -Q_1}, \frac{-Q_4}{Q_1 Q_2 Q_3}, \ldots].$$

Specializing the parameters in Theorem 1, we now derive several continued fraction expansion of analogous of exponential in **F**.

Corollary 1. *Let* $[i] := x^{q^i} - x$, $d_0 := 1$, *and* $d_i := [i]d_{i-1}^q$ $(i \geq 1)$. *Then*

I)
$$e - 1 = \sum_{i=1}^{\infty} \frac{1}{d_i} = [0; \underbrace{\underbrace{d_1}_{}, \frac{-d_2}{d_1^2}, \underbrace{-d_1}_{}}_{}, \frac{-d_3}{d_2^2}, \underbrace{d_1, \frac{d_2}{d_1^2}, -d_1}_{}, \frac{-d_4}{d_3^2}, \ldots]$$
$$= [0; \underbrace{\underbrace{[1]}_{}, -[2]d_1^{q-2}, \underbrace{-[1]}_{}}_{}, -[3]d_2^{q-2}, \underbrace{[1], [2]d_1^{q-2}, -[1]}_{}, -[4]d_3^{q-2}, \ldots]$$

II)
$$\sum_{i=1}^{\infty} \frac{1}{d_1 \cdots d_i} = [0; \underbrace{\underbrace{d_1}_{}, \frac{-d_2}{d_1}, \underbrace{-d_1}_{}}_{}, \frac{-d_3}{d_1 d_2}, \underbrace{d_1, \frac{d_2}{d_1}, -d_1}_{}, \frac{-d_4}{d_1 d_2 d_3}, \ldots]$$

III) for $n \in \{1, \ldots, q\}$, *we have*
$$\sum_{i=1}^{\infty} \frac{1}{x^n d_1 \cdots d_i} = [0; \underbrace{\underbrace{x^n d_1}_{}, \frac{-d_2}{x^n d_1}, \underbrace{-x^n d_1}_{}}_{}, \frac{-d_3}{x^n d_1 d_2}, \underbrace{x^n d_1, \frac{d_2}{x^n d_1}, -x^n d_1}_{}, \frac{-d_4}{x^n d_1 d_2 d_3}, \ldots].$$

Proof. *I*) Let $Q_1 = d_1$ and $Q_i = [i]d_{i-1}^{q-1}$ $(i \geq 2)$. Since
$$Q_1 Q_2 \cdots Q_i = d_1 \left([2]d_1^{q-1}\right)\left([3]d_2^{q-1}\right) \cdots \left([i]d_{i-1}^{q-1}\right)$$
$$= d_2 \left([3]d_2^{q-1}\right) \cdots \left([i]d_{i-1}^{q-1}\right) = \cdots = d_{i-1}\left([i]d_{i-1}^{q-1}\right) = d_i$$
and $Q_{i+1} = [i+1]d_i^{q-1}$, we get $Q_1 Q_2 \cdots Q_i | Q_{i+1}$ $(i \geq 1)$. For $i \geq 1$, we have
$$\frac{-Q_{i+1}}{Q_1 Q_2 \cdots Q_i} = \frac{-Q_1 Q_2 \cdots Q_i Q_{i+1}}{(Q_1 Q_2 \cdots Q_i)^2} = \frac{-d_{i+1}}{d_i^2}.$$
By Theorem 1, we get
$$\sum_{i=1}^{\infty} \frac{1}{d_i} = [0; \underbrace{\underbrace{d_1}_{}, \frac{-d_2}{d_1^2}, \underbrace{-d_1}_{}}_{}, \frac{-d_3}{d_2^2}, \underbrace{d_1, \frac{d_2}{d_1^2}, -d_1}_{}, \frac{-d_4}{d_3^2}, \ldots].$$

II) Let $Q_i = d_i$ $(i \geq 1)$. Consider
$$d_{i+1} = [i+1]d_i^q = [i+1]d_i^{q-1}d_i = [i+1]d_i^{q-1}[i]d_{i-1}^q = \cdots = [i+1] \cdots [2]d_i^{q-1} \cdots d_1^{q-1}.$$
Since $q \geq 2$, $d_1 d_2 \cdots d_i | d_{i+1}$. By Theorem 1, we get
$$\sum_{i=1}^{\infty} \frac{1}{d_1 \cdots d_i} = [0; \underbrace{\underbrace{d_1}_{}, \frac{-d_2}{d_1}, \underbrace{-d_1}_{}}_{}, \frac{-d_3}{d_1 d_2}, \underbrace{d_1, \frac{d_2}{d_1}, -d_1}_{}, \frac{-d_4}{d_1 d_2 d_3}, \ldots].$$

III) Let $Q_1 = x^n d_1$ and $Q_i = d_i$ $(i \geq 2)$. Consider
$$d_{i+1} = [i+1]d_i^q = [i+1]d_i^{q-1}d_i = [i+1]d_i^{q-1}[i]d_{i-1}^q = \cdots = [i+1] \cdots [2]d_i^{q-1} \cdots d_1^{q-1}.$$

Since $x|[i]$ for all $i \geq 1$ and $q \geq 2$, $x^n d_1 d_2 \cdots d_i | d_{i+1}$. By Theorem 1, we get

$$\sum_{i=1}^{\infty} \frac{1}{x^n d_1 \cdots d_i} = [0; \underbrace{\underbrace{x^n d_1}, \frac{-d_2}{x^n d_1}, \underbrace{-x^n d_1}}, \frac{-d_3}{x^n d_1 d_2}, \underbrace{x^n d_1, \frac{d_2}{x^n d_1}, -x^n d_1}, \frac{-d_4}{x^n d_1 d_2 d_3}, \ldots].$$

□

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