

# CONTINUED FRACTIONS REPRESENTING CERTAIN ANALOGUES OF EXPONENTIAL ELEMENTS IN $\mathbb{F}_q((1/x))$

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## Abstract

Let  $(Q_i)_{i=1}^{\infty}$  be a sequence of nonzero monic polynomials over a finite field  $\mathbb{F}_q$  satisfying  $Q_1 \cdots Q_i \mid Q_{i+1}$  ( $i \in \mathbb{N}$ ). Let  $\alpha(n) = \sum_{i=1}^n 1/Q_1 \cdots Q_i$  and  $\alpha(\infty) = \sum_{i=1}^{\infty} 1/Q_1 \cdots Q_i$ . It is shown that the continued fraction for  $\alpha(\infty)$  in the function field  $\mathbb{F}_q((1/x))$  can be explicitly given. As an application, by choosing suitable polynomials  $Q_i$ , explicit continued fraction expansion of a function field analogue of some exponential elements in  $\mathbb{F}_q((1/x))$  ( $q \geq 2$ ), are derived. This gives an extension of some earlier work of Thakur in 1992.

## 1 Introduction

As seen in [1]-[6] and [8]-[14], there have appeared various relationships between series and continued fractions. Some of these results connect certain types of series with continued fraction expansions having symmetric patterns.

In [13], Thakur explored the question of finding “interesting patterns for interesting numbers” in the function field setting and found an interesting continued fraction for an analogue of  $e$  for  $\mathbb{F}_q((1/x))$ .

Let  $\mathbb{F}_q$  be a finite field of  $q$  elements, ( $q \geq 2$ ),  $\mathbf{F} := \mathbb{F}_q((1/x))$  the field of formal Laurent series over  $\mathbb{F}_q$  equipped with a degree valuation  $|\cdot|$  defined by  $|x^{-1}| = e^{-1}$ . Every element  $\xi \in \mathbf{F}$  can be uniquely written as a simple continued fraction of the form

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$$\xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots}} := [a_0; a_1, a_2, \dots],$$

where  $a_0 \in \mathbb{F}_q[x]$  and  $a_n \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$  ( $n \geq 1$ ).

Let  $C_n/D_n = [a_0; a_1, \dots, a_n]$  be the  $n$ th convergent of the continued fraction for  $\xi$ . The following basic properties are easily verified by induction.

1.  $D_n C_{n-1} - C_n D_{n-1} = (-1)^n$  ( $n \geq 0$ ), so that  $C_n$  and  $D_n$  are relatively prime;
2.  $C_{n+1} = a_{n+1} C_n + C_{n-1}$  and  $D_{n+1} = a_{n+1} D_n + D_{n-1}$  ( $n \geq 0$ );
3. for any  $n \geq 0$ , we have

$$\frac{\zeta C_n + C_{n-1}}{\zeta D_n + D_{n-1}} = [a_0; a_1, a_2, \dots, a_n, \zeta];$$

4. if  $\frac{C_n}{D_n} = [a_0; a_1, \dots, a_n]$ , then  $\frac{D_n}{D_{n-1}} = [a_n; a_{n-1}, \dots, a_2, a_1]$ .

An analogue of the exponential element in  $\mathbf{F}$ , introduced by Carlitz (see e.g. [13], [14]), is defined as follows: let  $[i] = x^{q^i} - x$ ,  $d_0 = 1$  and  $d_i = [i]d_{i-1}^q$  ( $i \geq 1$ ). The exponential for  $\mathbf{F}$  is defined by

$$e(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i}.$$

Put  $e := e(1)$ . Then the series expansion for  $e$  is of the form

$$e = \sum_{i=0}^{\infty} \frac{1}{d_i}.$$

In 1992, Thakur [13] proved the following result which give the continued fraction expansion for  $\sum_{i=0}^{n+1} z^{q^i}/d_i$  if the continued fraction expansion for  $\sum_{i=0}^n z^{q^i}/d_i$  is known.

**Proposition 1.** *Let  $x_1 = [0; z^{-q}[1]]$ , and if  $x_n = [a_0; a_1, \dots, a_{2^n-1}]$ , then set*

$$x_{n+1} = [a_0; a_1, \dots, a_{2^n-1}, \frac{-z^{-q^n(q-2)}d_{n+1}}{d_n^2}, -a_{2^n-1}, \dots, -a_1].$$

Then

$$x_n = \sum_{i=1}^n \frac{z^{q^i}}{d_i}.$$

In particular,  $e(z) = z + \lim_{n \rightarrow \infty} x_n$  and for  $q = 2$ ,

$$e = [1; \underbrace{[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5], \dots]}_{\dots}]$$

(More explicitly, for  $n > 0$  the  $n$ -th partial quotient is  $x^{2^{u_n}} - x$  with  $u_n$  being the exponent of the highest power of 2 dividing  $2n$ ).

Our objective here is to extend this proposition of Thakur by proving

**Theorem 1.** Let  $(Q_i)_{i=1}^{\infty}$  be a sequence of nonzero monic polynomials over a finite field  $\mathbb{F}_q$ , let

$$\alpha(n) = \sum_{i=1}^n \frac{1}{Q_1 \cdots Q_i} \quad (n \geq 1) \quad \text{and let} \quad \alpha(\infty) = \sum_{i=1}^{\infty} \frac{1}{Q_1 \cdots Q_i}.$$

Assume that  $Q_1 \cdots Q_i | Q_{i+1}$  ( $i \in \mathbb{N}$ ). Then  $\alpha(1) = [0; Q_1]$  and if  $\alpha(n) = [0; a_1, \dots, a_{k_n}]$ , then

$$\alpha(n+1) = [0; a_1, \dots, a_{k_n}, \frac{-Q_{n+1}}{Q_1 Q_2 \cdots Q_n}, -a_{k_n}, \dots, -a_1].$$

Moreover,

$$\alpha(\infty) = [0; \underbrace{Q_1, \frac{-Q_2}{Q_1}, -Q_1, \frac{-Q_3}{Q_1 Q_2}, Q_1, \frac{Q_2}{Q_1}, -Q_1, \frac{-Q_4}{Q_1 Q_2 Q_3}, \dots}_{\dots}].$$

## 2 Proofs

Our work makes use of the following lemma, see e.g. [7] and [15].

**Lemma 1 (Folding Lemma).** Let  $y \in \mathbf{F}$  and  $\frac{C_n}{D_n} = [a_0; a_1, a_2, \dots, a_n]$ . Then

$$[a_0; a_1, \dots, a_n, y, -a_n, \dots, -a_1] = \frac{C_n}{D_n} + \frac{(-1)^n}{y D_n^2}.$$

**Proof of Theorem 1.** It is clear that  $\alpha(1) = \frac{1}{Q_1} = [0; Q_1] := \frac{C_{k_1}}{D_{k_1}}$ . Since  $\gcd(1, Q_1) = 1$  and  $Q_1$  is monic, we have  $1 = C_{k_1}$  and  $D_{k_1} = Q_1$ . By Folding Lemma, we get

$$\alpha(2) = \frac{1}{Q_1} + \frac{1}{Q_1 Q_2} = \frac{1}{Q_1} + \frac{-1}{\frac{-Q_2}{Q_1} Q_1^2} = [0; Q_1, \frac{-Q_2}{Q_1}, -Q_1] := \frac{C_{k_2}}{D_{k_2}}.$$

For  $n \geq 2$ , let  $[0; a_1, \dots, a_{k_n}]$  be a continued fraction expansion for  $\alpha(n)$  whose  $k_n$ th convergent is  $C_{k_n}/D_{k_n}$ ; observe that  $k_n$  is an odd integer. By Folding Lemma, we get

$$\begin{aligned} [0; a_1, \dots, a_{k_n}, \frac{-Q_{n+1}}{Q_1 \cdots Q_n}, -a_{k_n}, \dots, -a_1] &= \frac{C_{k_n}}{D_{k_n}} + \frac{(-1)^{k_n}}{\frac{-Q_{n+1}}{Q_1 \cdots Q_n} D_{k_n}^2} \\ &= \alpha(n) + \frac{(-1)^{k_n}}{\frac{-Q_{n+1}}{Q_1 \cdots Q_n} D_{k_n}^2}. \end{aligned}$$

On the other hand, we have

$$\frac{1}{Q_1} + \dots + \frac{1}{Q_1 \cdots Q_n} = \frac{(Q_2 \cdots Q_n) + (Q_3 \cdots Q_n) + \dots + Q_n + 1}{Q_1 \cdots Q_n}.$$

We assert that  $\gcd((Q_2 \cdots Q_n) + (Q_3 \cdots Q_n) + \dots + Q_n + 1, Q_1 Q_2 \cdots Q_n) = 1$  for  $n \geq 2$ . Suppose there exists a prime  $p \in \mathbb{F}_q[x]$  such that

$$p | ((Q_2 \cdots Q_n) + (Q_3 \cdots Q_n) + \dots + Q_n + 1) \quad \text{and} \quad p | (Q_1 Q_2 \cdots Q_n).$$

Since  $p | (Q_1 Q_2 \cdots Q_n)$ ,  $p | Q_k$  for some  $1 \leq k \leq n$ , then  $p | Q_j Q_{j+1} \cdots Q_n$  for all  $2 \leq j \leq k$ . Since  $Q_1 \cdots Q_k | Q_{k+l}$  for all  $1 \leq l \leq n - k$ , we have  $Q_k | Q_{k+l} \cdots Q_n$  for all  $1 \leq l \leq n - k$  and so  $p | Q_{k+l} \cdots Q_n$  for all  $1 \leq l \leq n - k$ . Since  $p | ((Q_2 \cdots Q_n) + (Q_3 \cdots Q_n) + \dots + Q_n + 1)$ , then we get  $p | 1$ , which is a contradiction. Thus  $\gcd((Q_2 \cdots Q_n) + (Q_3 \cdots Q_n) + \dots + Q_n + 1, Q_1 Q_2 \cdots Q_n) = 1$ . Since all  $Q_i$  are monic, then  $D_{k_n} = Q_1 Q_2 \cdots Q_n$ . Hence

$$\begin{aligned} \alpha(n) + \frac{(-1)^{k_n}}{\frac{-Q_{n+1}}{Q_1 \cdots Q_n} D_{k_n}^2} &= \alpha(n) + \frac{-1}{\frac{-Q_{n+1}}{Q_1 \cdots Q_n} (Q_1 Q_2 \cdots Q_n)^2} \\ &= \alpha(n) + \frac{1}{Q_1 \cdots Q_n Q_{n+1}} = \alpha(n+1). \end{aligned}$$

Consequently, we have

$$\begin{aligned} \alpha(1) &= [0; Q_1] \\ \alpha(2) &= [0; \underbrace{Q_1}, \frac{-Q_2}{Q_1}, \underbrace{-Q_1}] \\ \alpha(3) &= [0; \underbrace{Q_1}, \frac{-Q_2}{Q_1}, \underbrace{-Q_1}, \frac{-Q_3}{Q_1 Q_2}, Q_1, \underbrace{\frac{Q_2}{Q_1}, -Q_1}] \\ &\vdots \\ \alpha(\infty) &= [0; \underbrace{Q_1}, \frac{-Q_2}{Q_1}, \underbrace{-Q_1}, \frac{-Q_3}{Q_1 Q_2}, Q_1, \underbrace{\frac{Q_2}{Q_1}, -Q_1}, \frac{-Q_4}{Q_1 Q_2 Q_3}, \dots]. \end{aligned}$$

Specializing the parameters in Theorem 1, we now derive several continued fraction expansion of analogous of exponential in  $\mathbf{F}$ .

**Corollary 1.** Let  $[i] := x^{q^i} - x$ ,  $d_0 := 1$ , and  $d_i := [i]d_{i-1}^q$  ( $i \geq 1$ ). Then

$$\begin{aligned} I) \quad e - 1 &= \sum_{i=1}^{\infty} \frac{1}{d_i} = [0; \underbrace{d_1, \frac{-d_2}{d_1^2}, -d_1}_{d_1}, \frac{-d_3}{d_2^2}, d_1, \frac{d_2}{d_1^2}, -d_1, \frac{-d_4}{d_3^2}, \dots] \\ &= [0; \underbrace{[1], -[2]d_1^{q-2}, -[1]}_{d_1}, -[3]d_2^{q-2}, \underbrace{[1], [2]d_1^{q-2}, -[1]}_{d_1}, -[4]d_3^{q-2}, \dots] \end{aligned}$$

$$II) \quad \sum_{i=1}^{\infty} \frac{1}{d_1 \cdots d_i} = [0; \underbrace{d_1, \frac{-d_2}{d_1}, -d_1}_{d_1 d_2}, \frac{-d_3}{d_1 d_2}, d_1, \frac{d_2}{d_1}, -d_1, \frac{-d_4}{d_1 d_2 d_3}, \dots]$$

III) for  $n \in \{1, \dots, q\}$ , we have

$$\sum_{i=1}^{\infty} \frac{1}{x^n d_1 \cdots d_i} = [0; \underbrace{x^n d_1, \frac{-d_2}{x^n d_1}, -x^n d_1}_{x^n d_1 d_2}, \frac{-d_3}{x^n d_1 d_2}, x^n d_1, \frac{d_2}{x^n d_1}, -x^n d_1, \frac{-d_4}{x^n d_1 d_2 d_3}, \dots].$$

*Proof.* I) Let  $Q_1 = d_1$  and  $Q_i = [i]d_{i-1}^{q-1}$  ( $i \geq 2$ ). Since

$$\begin{aligned} Q_1 Q_2 \cdots Q_i &= d_1 \left( [2]d_1^{q-1} \right) \left( [3]d_2^{q-1} \right) \cdots \left( [i]d_{i-1}^{q-1} \right) \\ &= d_2 \left( [3]d_2^{q-1} \right) \cdots \left( [i]d_{i-1}^{q-1} \right) = \cdots = d_{i-1} \left( [i]d_{i-1}^{q-1} \right) = d_i \end{aligned}$$

and  $Q_{i+1} = [i+1]d_i^{q-1}$ , we get  $Q_1 Q_2 \cdots Q_i | Q_{i+1}$  ( $i \geq 1$ ). For  $i \geq 1$ , we have

$$\frac{-Q_{i+1}}{Q_1 Q_2 \cdots Q_i} = \frac{-Q_1 Q_2 \cdots Q_i Q_{i+1}}{(Q_1 Q_2 \cdots Q_i)^2} = \frac{-d_{i+1}}{d_i^2}.$$

By Theorem 1, we get

$$\sum_{i=1}^{\infty} \frac{1}{d_i} = [0; \underbrace{d_1, \frac{-d_2}{d_1^2}, -d_1}_{d_1}, \frac{-d_3}{d_2^2}, d_1, \frac{d_2}{d_1^2}, -d_1, \frac{-d_4}{d_3^2}, \dots].$$

II) Let  $Q_i = d_i$  ( $i \geq 1$ ). Consider

$$d_{i+1} = [i+1]d_i^q = [i+1]d_i^{q-1}d_i = [i+1]d_i^{q-1}[i]d_{i-1}^q = \cdots = [i+1] \cdots [2]d_i^{q-1} \cdots d_1^{q-1}.$$

Since  $q \geq 2$ ,  $d_1 d_2 \cdots d_i | d_{i+1}$ . By Theorem 1, we get

$$\sum_{i=1}^{\infty} \frac{1}{d_1 \cdots d_i} = [0; \underbrace{d_1, \frac{-d_2}{d_1}, -d_1}_{d_1 d_2}, \frac{-d_3}{d_1 d_2}, d_1, \frac{d_2}{d_1}, -d_1, \frac{-d_4}{d_1 d_2 d_3}, \dots].$$

III) Let  $Q_1 = x^n d_1$  and  $Q_i = d_i$  ( $i \geq 2$ ). Consider

$$d_{i+1} = [i+1]d_i^q = [i+1]d_i^{q-1}d_i = [i+1]d_i^{q-1}[i]d_{i-1}^q = \cdots = [i+1] \cdots [2]d_i^{q-1} \cdots d_1^{q-1}.$$

Since  $x \mid [i]$  for all  $i \geq 1$  and  $q \geq 2$ ,  $x^n d_1 d_2 \cdots d_i \mid d_{i+1}$ . By Theorem 1, we get

$$\sum_{i=1}^{\infty} \frac{1}{x^n d_1 \cdots d_i} = [0; \underbrace{x^n d_1, \frac{-d_2}{x^n d_1}, -x^n d_1}_{x^n d_1 d_2}, \frac{-d_3}{x^n d_1 d_2}, \underbrace{x^n d_1, \frac{d_2}{x^n d_1}, -x^n d_1}_{x^n d_1 d_2 d_3}, \frac{-d_4}{x^n d_1 d_2 d_3}, \dots].$$

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