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ARITHMETIC PROGRESSION OF SQUARES AND SOLVABILITY OF THE DIOPHANTINE EQUATION $8x^4 + 1 = y^2$

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Abstract

There is no arithmetic progression consisting of square terms and with a square common difference. Alternatively, the diophantine equation $1 + x^4 = 2y^2$ has no solution in positive integers. Consequently, the diophantine equation $8x^4 + 1 = y^2$ has no positive integral solution other than $x = 1, y = 3$, a clear indication that no balancing number other that 1 is a perfect square.

1 Introduction

Balancing numbers and balancers, originally introduced by Behera and Panda [1], are respectively natural numbers n and r satisfying the equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

This equation when simplified yields

$$n^2 = \frac{(n+r)(n+r+1)}{2},$$

leading to the conclusion that a natural number n is a balancing number if and only if n^2 is a triangular number, or, equivalently, $8n^2 + 1$ is a perfect square. Since $8 \cdot 1^2 + 1 = 9$, Panda [4] and Panda and Ray [5] accepted as 1 as the first balancing number, though the definition for the balancing numbers suggests that every balancing number must be greater than 1. The

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n^{th} balancing number is denoted by B_n and is well known that the balancing numbers satisfy the recurrence relation (also see [1])

$$B_{n+1} = 6B_n - B_{n-1}, \quad B_1 = 1, \quad B_2 = 6.$$

The first fifteen balancing numbers are 1, 6, 35, 204, 1189, 6930, 40391, 1372105, 7997214, 46611179, 271669860, 1583407981, 9228778026, 53789260175 and except the first one, none is a square.This leads to a natural question "Is it possible for any other balancing number to be a perfect square ?" The answer to this question lies in the solvability of the diophantine equation $8x^4 + 1 = y^2$ in natural number with $x > 1$, which, in turn, depends on the solvability of a problem on arithmetic progressions, consist of finding three distinct natural numbers P, Q and R in arithmetic progression such that each of P, Q and R and the common difference $d = Q - P = R - Q$ are perfect squares.

2 Three perfect squares in arithmetic progression

Let x, y and z be distinct natural numbers such that x^2, y^2 and z^2 are in arithmetic progression. Then x, y and z satisfy the Pythagorean-type equation

$$x^2 + z^2 = 2y^2.$$

Observe that both x and z must be of the same parity. If both x and z are even, then y must be even and hence, we can cancel the highest power of 2 that is common to each of x^2, y^2 and z^2. We can assume, both x and z are odd and hence y is also odd; else,

$$x^2 + z^2 \equiv 2(mod\,4)$$

and

$$2y^2 \equiv 0(mod\,4)$$

contradict

$$x^2 + z^2 = 2y^2.$$

Since x, y and z are distinct, we may assume without loss of generality that $x < z$. If d is the greatest common divisor of x and z then certainly, d divides y and we can cancel d^2 from both sides of the equation. Thus, we also assume without loss of generality that x, y and z are pairwise coprime. Since both x and z are odd, $\frac{z+x}{2}$ and $\frac{z-x}{2}$ are natural numbers and the diophantine equation $x^2 + z^2 = 2y^2$ reduces to the Pythagorean equation

$$\left(\frac{z + x}{2}\right)^2 + \left(\frac{z - x}{2}\right)^2 = y^2.$$

Since x and z are coprime, $z+x$ and $z-x$ are either coprime or their greatest common divisor is 2. Thus, in the present case, $\frac{z+x}{2}$ and $\frac{z-x}{2}$ are coprime. Let (A,B,C) be any arbitrary primitive Pythagorean triple, that is A,B and C are pairwise coprime and $A^2+B^2=C^2$. Assume that $A>B$. Then we have $\frac{z+x}{2}=A$, $\frac{z-x}{2}=B$ and $y=C$, implying that $z=A+B, x=A-B$ and $y=C$ leading to

$$(A-B)^2+(A+B)^2=2C^2.$$

We also note here that if x^2, y^2 and z^2 are in arithmetic progression then, $(kx)^2, (ky)^2$ and $(kz)^2$ are also in arithmetic progression for each natural number k.

From the above discussion, it is clear that using primitive Pythagorean triples we can always generate three pairwise coprime perfect squares to form arithmetic progressions. Conversely, if three perfect squares are in arithmetic progression, we can always find a Pythagorean triple from these squares. Thus if x^2, y^2 and z^2 are in arithmetic progression, then x, y and z must be of the same parity and taking

$$A=\frac{z+x}{2}, B=\frac{z-x}{2}, C=y$$

we have

$$A^2+B^2=C^2.$$

3 Arithmetic progressions with square terms and square common difference

Let x, y and z be distinct natural numbers such that x^2, y^2 and z^2 are in arithmetic progression with a perfect square common difference. We can assume without loss of generality that x, y and z are pairwise coprime. This leads us to consider the diophantine equation $x^2+z^2=2y^2$ such that the common difference $y^2-x^2=z^2-y^2$ is a perfect square. Let y be the smallest number to satisfy this property. In view of the results of the last section, we have

$$x=A-B, y=C, z=A+B$$

where A, B and C are natural numbers forming a primitive Pythagorean triple. It is well known that each primitive solution of the Pythagorean equation $A^2+B^2=C^2$ is of the form

$$A=a^2-b^2, B=2ab, C=a^2+b^2$$

where a and b are two coprime natural numbers of opposite parity, see [2] and [3, p.584]. We may assume without loss of generality that $a>b$. The common difference of the arithmetic progression takes the form

$$y^2 - x^2 = z^2 - y^2 = 2AB = 4ab(a^2 - b^2) = 4ab(a+b)(a-b).$$

Since a and b are coprime and of opposite parity, the numbers $a, b, a+b$ and $a-b$ are also pairwise coprime and hence, all must be perfect squares. Thus, there exist natural numbers u, v, p and q such that

$$a = u^2, b = v^2, a+b = p^2, a-b = q^2,$$

leading to the simultaneous diophantine equation

$$u^2 + v^2 = p^2, u^2 - v^2 = q^2,$$

a clear indication that $u^2 - v^2, u^2, u^2 + v^2$ are in arithmetic progression with common difference v^2. But in view of

$$u^2 = a \leq a^2 < a^2 + b^2 = C < C^2 = y^2,$$

it follows that $u < y$, contradicting the assumption that y is the smallest number such that x^2, y^2 and z^2 are in arithmetic progression with a perfect square common difference. Hence, three squares cannot be in arithmetic progression if their common difference is a square.

4 The diophantine equation $x^4 + 1 = 2y^2$

Our next objective is to show that no balancing number other than 1 is a perfect square, or, equivalently, the diophantine equation $8x^4 + 1 = y^2$ has no positive integral solution except $x = 1, y = 3$. But before we do so, we need to study the diophantine equation $x^4 + 1 = 2y^2$.

The diophantine equation $x^4 + 1 = 2y^2$ has the trivial solution in positive integers $x = y = 1$. To take care of other solutions, we assume that $x \neq y$. Further, we observe that if $x \geq 2$, then we have

$$4x^2 \leq x^4 < x^4 + 1 = 2y^2$$

leading to $x < y$. We next convert $x^4 + 1 = 2y^2$ to a Pythagorean equation. Adding $2x^2$ to both sides we get

$$(x^2 + 1)^2 = 2(x^2 + y^2) = (x+y)^2 + (y-x)^2.$$

Observe that if $x^4 + 1 = 2y^2$, then x is odd and hence y is also odd. Further, if d is the greatest common divisor of x and y, then d^2 divides $2y^2 - x^4 = 1$, implying that $d = 1$ and hence that x and y are coprime. Thus, we have the equation

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 = \left(\frac{x^2+1}{2}\right)^2.$$

It is also well known that if a and b are coprime, then $a+b$ and $a-b$ are either coprime or their greatest common divisor is 2. Since for the case at hand, x and y are odd $\frac{x+y}{2}$ and $\frac{y-x}{2}$ are coprime. We claim that $\frac{x+y}{2}$, $\frac{y-x}{2}$ and $\frac{x^2+1}{2}$ are pairwise coprime. Observe that if d divides $\frac{x+y}{2}$ and $\frac{x^2+1}{2}$ then d^2 must divide

$$\left(\frac{x^2+1}{2}\right)^2 - \left(\frac{x+y}{2}\right)^2 = \left(\frac{y-x}{2}\right)^2$$

and then $\frac{x+y}{2}$ and $\frac{y-x}{2}$ cannot be coprime. Thus, there exist coprime natural numbers a and b of opposite parity with $a > b$ such that the solution of the equation

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{y-x}{2}\right)^2 = \left(\frac{x^2+1}{2}\right)^2$$

is given by

$$\frac{x+y}{2} = 2ab, \quad \frac{y-x}{2} = a^2-b^2, \quad \frac{x^2+1}{2} = a^2+b^2$$

or

$$\frac{x+y}{2} = a^2-b^2, \quad \frac{y-x}{2} = 2ab, \quad \frac{x^2+1}{2} = a^2+b^2.$$

In either case, we have

$$y = a^2-b^2+2ab,\ x = |a^2-b^2-2ab|,\ x^2 = 2(a^2+b^2)-1.$$

This gives

$$(a^2-b^2-2ab)^2 = 2(a^2+b^2)-1,$$

which, on simplification yields

$$(a^2+b^2-1)^2 = 4ab(a+b)(a-b).$$

Since a and b are coprime and of opposite parity, $a+b$ and $a-b$ are also coprime, so that each of $a, b, a+b$ and $a-b$ are perfect squares. Thus, there exist natural numbers u, v, p and q such that

$$a = u^2,\ b = v^2,\ a+b = p^2,\ a-b = q^2$$

leading to

$$u^2+v^2 = p^2,\ u^2-v^2 = q^2,$$

implying that q^2, u^2 and p^2 are in arithmetic progression with common difference v^2. By virtue of the results of last section, this is impossible. Hence, the diophantine equation $x^4+1 = 2y^2$ has no solution in positive integers other than $x = y = 1$.

5 The diophantine equation $8x^2+y^2=z^2$

To prove the impossibility of solution of the diophantine equation $8x^4+1=y^2$ other than $x=1$ and $y=3$, we further need to study a similar type of diophantine equation $8x^2+y^2=z^2$ when both y and z are odd natural numbers. The equation $8x^2+y^2=z^2$ generates balancing and Lucas-balancing numbers [4, 5, 6] when $y=1$.

The diophantine equation $8x^2+y^2=z^2$ is also a variant of the Pythagorean equation $x^2+y^2=z^2$; however, the method of solution we present here does not use Pythagorean triples. We assume without loss of generality that y and z are coprime; else if d is the greatest common divisor of y and z then d^2 divides z^2-y^2 and hence d^2 divides x^2; so d^2 can be canceled from each of x^2, y^2 and z^2 and the equation $8x^2+y^2=z^2$ is reduced to another equation of the same type.

We rewrite the equation $8x^2+y^2=z^2$ as $\frac{z+y}{2}\cdot\frac{z-y}{2}=2x^2$. Since y and z are both odd and coprime, $\frac{z+y}{2}$ and $\frac{z-y}{2}$ are also coprime. Hence, there exist natural numbers u and v such that

$$\frac{z+y}{2}=u^2, \frac{z-y}{2}=2v^2$$

or

$$\frac{z+y}{2}=2v^2, \frac{z-y}{2}=u^2$$

leading to

$$z=u^2+2v^2, y=|u^2-2v^2|, x=uv$$

and we finally have

$$8u^2v^2+(u^2-2v^2)^2=(u^2+2v^2)^2.$$

Note that if (x,y,z) is a solution of $8x^2+y^2=z^2$ and k is any natural number, then (kx,ky,kz) is a solution.

6 Perfect square balancing numbers and the diophantine equation $8x^4+1=y^2$

We know that the first balancing number 1 is a square. As remarked in Section 1, there is no square balancing number in a visible distance. It is also well known that a natural number x is a balancing number if and only if $8x^2+1$ is a perfect square and hence, for any natural number x, x^2 is a balancing number if and only if $8x^4+1$ is a perfect square. This leads us to consider the diophantine equation $8x^4+1=y^2$, which has the trivial solution $x=1, y=3$. We try for solutions with $x > 1$. In view of the results of last section, the solutions of $8x^4 + 1 = y^2$ can be written as

$$x^2 = uv, y = u^2 + 2v^2$$

such that $|u^2 - 2v^2| = 1$, where the numbers u and v are coprime. Thus, there exist natural numbers a and b such that

$$u = a^2,\ v = b^2,\ |a^4 - 2b^4| = 1.$$

The equation $a^4 - 2b^4 = 1$ is equivalent to

$$(b^4 + 1)^2 = a^4 + b^8$$

which, clearly has no solutions in natural numbers [3, p.591] and if $2b^4 - a^4 = 1$ then we have $1 + a^4 = 2b^4$ and by virtue of results of Section 4, this equation has also no solution in natural numbers other than $a = b = 1$. But $a = b = 1$ corresponds to $u = v = 1$ and finally $x = 1, y = 3$. Thus the diophantine equation $8x^4 + 1 = y^2$ has the only solution $x = 1$ and $y = 3$. This suggests that other than 1, no balancing number is a perfect square.

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