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ON THE EIGENVALUES OF CERTAIN NUMBER-THEORETIC MATRICES

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Abstract

In this paper we study the structure and give bounds for the eigenvalues of the $n \times n$ matrix, whose ij entry is $(i,j)^\alpha [i,j]^\beta$, where $\alpha, \beta \in \mathbb{R}$, (i,j) is the greatest common divisor of i and j and $[i,j]$ is the least common multiple of i and j. Currently, only O-estimates for the greatest eigenvalue of this matrix can be found in the literature, and the asymptotic behaviour of the greatest and smallest eigenvalues is known in case when $\alpha = \beta$.

1 Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. Let $(S)_f$ denote the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its ij entry. More formally, let $((S)_f)_{ij} = f((x_i, x_j))$. Analogously, let $[S]_f$ denote the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its ij entry. That is, $([S]_f)_{ij} = f([x_i, x_j])$. The matrices $(S)_f$ and $[S]_f$ are referred to as the GCD and LCM matrices on S associated with f.

The study of GCD and LCM matrices was initiated by H. J. S. Smith [19] in 1875, when he calculated $\det(S)_f$ in case when S is factor-closed and $\det[S]_f$ in a more special case. Since Smith, numerous papers have been published about GCD and LCM matrices. For general accounts, see e.g. [9, 18]. There are also various generalizations of GCD and LCM matrices to be found in the literature. The most important ones are the lattice-theoretic generalizations into meet and join matrices, see e.g. [16].

Over the years some authors have studied number-theoretic matrices that are neither GCD nor LCM matrices, but are very closely related to them. For

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example, Wintner [20] published results concerning the largest eigenvalue of the $n \times n$ matrix having

$$\left(\frac{(i,j)}{[i,j]}\right)^{\alpha}$$

as its ij entry and subsequently Lindqvist and Seip [17] investigated the asymptotic behavior of the smallest and largest eigenvalues of the same matrix. More recently Hilberdink [10] as well as Berkes and Weber [5] have studied this same topic from analytical perspective.

Also the norms of GCD, LCM and related matrices have been repeatedly studied in the literature. Altinisik et al. [2] investigated the norms of reciprocal LCM matrices, and later Altinisik [1] published a paper about the norms of GCD related matrix. Haukkanen [6, 7, 8] studied the $n \times n$ matrix having

$$\frac{(i,j)^r}{[i,j]^s}, \quad r, s \in \mathbb{R},$$

as its ij entry and, among other things, gave O-estimates for the ℓ_p and maximum row and column sum norms of this matrix. In this paper we study the same class of matrices, although we use a slightly different notation. Let $\alpha, \beta \in \mathbb{R}$. Our goal here is to find bounds for the eigenvalues of the $n \times n$ matrix having

$$(i,j)^{\alpha}[i,j]^{\beta}$$

as its ij entry. In order to do this we use similar techniques as Ilmonen et al [15] and Hong and Loewy [13]. One of the methods may be considered to originate from Hong and Loewy [12]. It should be noted that not much is known about the eigenvalues of GCD, LCM and related matrices. In addition to the articles mentioned above there are only a few publications that provide information about the eigenvalues (see e.g. [3, 11]).

2 Preliminaries

Let $A_n^{\alpha,\beta}$ denote the $n \times n$ matrix, whose ij entry is given by

$$(A_n^{\alpha,\beta})_{ij} = (i,j)^{\alpha}[i,j]^{\beta}, \tag{1}$$

where $\alpha, \beta \in \mathbb{R}$. In addition, for every $n \in \mathbb{Z}^+$ we define the $n \times n$ matrix E_n by

$$(E_n)_{ij} = \begin{cases} 1 & \text{if } j \mid i \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The matrix E_n may be referred to as the incidence matrix of the set $\{1, 2, \dots, n\}$ with respect to the divisibility relation.

Next we define some important arithmetical functions that we need. First of all, let $N^{\alpha-\beta}$ be the function such that $N^{\alpha-\beta}(k) = k^{\alpha-\beta}$ for all $k \in \mathbb{Z}^+$. In addition, let $J_{\alpha-\beta}$ denote the arithmetical function with

$$J_{\alpha-\beta}(k) = k^{\alpha-\beta} \prod_{p \mid k} \left(1 - \frac{1}{p^{\alpha-\beta}}\right) \tag{3}$$

for all $k \in \mathbb{Z}_+$. This function may be seen as a generalization of the Jordan totient function, and it is easy to see that the function $J_{\alpha-\beta}$ can be written as

$$J_{\alpha-\beta} = N^{\alpha-\beta} * \mu, \tag{4}$$

the Dirichlet convolution of $N^{\alpha-\beta}$ and the number-theoretic Möbius function.

Remark 1. *If $\alpha - \beta > 0$, then clearly $J_{\alpha-\beta}(k) > 0$ for all $k \in \mathbb{Z}^+$.*

Before we begin to analyze the eigenvalues of the matrix $A_n^{\alpha,\beta}$ we first need to obtain suitable factorizations for it.

Proposition 1. *Let $F_n = \operatorname{diag}(1, 2, \ldots, n)$ and $D_n = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, where*

$$d_i = J_{\alpha-\beta}(i) = (N^{\alpha-\beta} * \mu)(i). \tag{5}$$

Then the matrix $A_n^{\alpha,\beta}$ can be written as

$$A_n^{\alpha,\beta} = F_n^{\beta} E_n D_n E_n^T F_n^{\beta}. \tag{6}$$

Proof. Since the ij element of the matrix $E_n D_n E_n^T$ is

$$\begin{aligned} \sum_{k \mid (i,j)} J_{\alpha-\beta}(k) &= \sum_{k \mid (i,j)} (N^{\alpha-\beta} * \mu)(k) = [(N^{\alpha-\beta} * \mu) * \zeta]((i,j)) \\ &= N^{\alpha-\beta}((i,j)) = (i,j)^{\alpha-\beta}, \end{aligned} \tag{7}$$

the ij element of the matrix $F_n^{\beta} E_n D_n E_n^T F_n^{\beta}$ is

$$i^{\beta}(i,j)^{\alpha-\beta} j^{\beta} = (i,j)^{\alpha}[i,j]^{\beta}, \tag{8}$$

which is also the ij element of $A_n^{\alpha,\beta}$. □

Remark 2. *By applying Proposition 1, it is easy to see that*

$$\det A_n^{\alpha,\beta} = (n!)^{2\beta} \prod_{k=1}^{n} J_{\alpha-\beta}(k) = (n!)^{2\beta} \prod_{k=1}^{n} \left(N^{\alpha-\beta} * \mu\right)(k). \tag{9}$$

In case when $\alpha > \beta$, we are able to use a different factorization presented in the following proposition.

Proposition 2. *Suppose that $\alpha > \beta$. Let $J_{\alpha-\beta}$, D_n and F_n be as in Proposition 1, and let B_n denote the real $n \times n$ matrix with*

$$(B_n)_{ij} = \begin{cases} \sqrt{J_{\alpha-\beta}(j)} & \text{if } j \mid i \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Then the matrix $A_n^{\alpha,\beta}$ can be written as

$$A_n^{\alpha,\beta} = (F_n^\beta B_n)(F_n^\beta B_n)^T = (F_n^\beta E_n D_n^{\frac{1}{2}})(F_n^\beta E_n D_n^{\frac{1}{2}})^T. \tag{11}$$

Proof. First we observe that the ij element of $B_n B_n^T$ is equal to

$$\begin{aligned} \sum_{k \mid (i,j)} J_{\alpha-\beta}(k) &= \sum_{k \mid (i,j)} (N^{\alpha-\beta} * \mu)(k) = [(N^{\alpha-\beta} * \mu) * \zeta]((i,j)) \\ &= N^{\alpha-\beta}((i,j)) = (i,j)^{\alpha-\beta}. \end{aligned} \tag{12}$$

Thus the ij element of $(F_n^\beta B_n)(F_n^\beta B_n)^T = F_n^\beta (B_n B_n^T) F_n^\beta$ is

$$i^\beta (i,j)^{\alpha-\beta} j^\beta = (i,j)^\alpha [i,j]^\beta, \tag{13}$$

which is also the ij element of the matrix $A_n^{\alpha,\beta}$. Thus, we have proven the first equality. The second equality follows from the fact that the matrix B_n can be written as $B_n = E_n D_n^{\frac{1}{2}}$. □

In order to obtain bounds for the eigenvalues of the matrix $A_n^{\alpha,\beta}$ we find out the eigenvalues of the matrix $E_n^T E_n$ for different $n \in \mathbb{Z}^+$. The smallest eigenvalue of this matrix is denoted by t_n and the largest by T_n. Table 1 shows the values of the constants t_n and T_n for small values of n. The ij element of the matrix $E_n^T E_n$ is in fact equal to

$$\left|\{k \in \mathbb{Z}^+ \;\middle|\; k \le n,\ i \mid k \text{ and } j \mid k\}\right| = \left\lfloor \frac{n}{[i,j]} \right\rfloor, \tag{14}$$

the greatest integer that is less than or equal to $\frac{n}{[i,j]}$. This same matrix is also studied by Bege [4] when he considers it as an example.

As can be seen from Table 1, the sequences $(t_n)_{n=1}^\infty$ and $(T_n)_{n=1}^\infty$ seem to possess certain monotonic behavior. This encourages us to present the following conjecture.

Conjecture 1. *For every $n \in \mathbb{Z}^+$ we have*

$$t_{n+1} \le t_n \quad \text{and} \quad T_n \le T_{n+1}. \tag{15}$$

Calculations show that this conjecture holds for $n = 2, \ldots, 100$.

n	t_n	T_n	n	t_n	T_n	n	t_n	T_n
2	0.381966	2.61803	15	0.0616080	23.6243	28	0.0411874	47.1773
3	0.267949	3.73205	16	0.0616079	26.1117	29	0.0401315	47.7330
4	0.252762	5.78339	17	0.0591935	26.70841	30	0.0343360	51.4915
5	0.204371	6.60665	18	0.0584344	29.8007	31	0.0336797	52.0305
6	0.129425	9.21230	19	0.0562263	30.3787	32	0.0336797	54.6056
7	0.118823	9.92035	20	0.0550575	33.2123	33	0.0322295	55.7392
8	0.118764	12.2892	21	0.0505600	34, 4522	34	0.0306762	57.2482
9	0.116597	13.4520	22	0.0466545	36.0618	35	0.0295618	58.2226
10	0.0930874	15.4428	23	0.0452547	36.6470	36	0.0295298	62.7258
11	0.087262	16.113	24	0.0452214	41.0878	37	0.0289990	63.2500
12	0.087262	16.113	25	0.0451569	41.8465	38	0.0277260	64.7226
13	0.0791480	20.4160	26	0.0419049	43.3920	39	0.0267584	65.8548
14	0.0681283	22.1909	27	0.0419033	44.6343	40	0.0267526	69.2188

Table 1: The constants t_n and T_n for $n \leq 40$.

3 Estimations for the eigenvalues

First we assume that $\alpha > \beta$. From Proposition 2, it follows that in this case the matrix $A_n^{\alpha,\beta}$ is positive definite, and thus we are able to give a lower bound for the smallest eigenvalue of $A_n^{\alpha,\beta}$.

Theorem 1. *Let $\alpha > \beta$ and let $\lambda_1^{n,\alpha,\beta}$ denote the smallest eigenvalue of the matrix $A_n^{\alpha,\beta}$. Then*

$$\lambda_1^{n,\alpha,\beta} \geq t_n \cdot \min_{1 \leq i \leq n} J_{\alpha-\beta}(i) \cdot \min\{1, n^{2\beta}\} > 0. \tag{16}$$

Proof. By applying Proposition 2, we have

$$A_n^{\alpha,\beta} = (F_n^\beta E_n D_n^{\frac{1}{2}})(F_n^\beta E_n D_n^{\frac{1}{2}})^T. \tag{17}$$

By applying Remarks 1 and 2 we deduce that $\det A_n^{\alpha,\beta} \neq 0$ and furthermore that $A_n^{\alpha,\beta}$ is invertible. Thus, the matrices $A_n^{\alpha,\beta}$ and $(A_n^{\alpha,\beta})^{-1}$ are real symmetric and positive definite and therefore the greatest eigenvalue of $(A_n^{\alpha,\beta})^{-1}$ is also the inverse of the smallest eigenvalue of $A_n^{\alpha,\beta}$. In addition, the greatest eigenvalue of $(A_n^{\alpha,\beta})^{-1}$ is equal to $\left|\left|\left|(A_n^{\alpha,\beta})^{-1}\right|\right|\right|_S$, the spectral norm of the matrix $(A_n^{\alpha,\beta})^{-1}$. Thus

$$\lambda_1^{n,\alpha,\beta} = \frac{1}{|||(A_n^{\alpha,\beta})^{-1}|||_S} = \frac{1}{\left|\left|\left|[(F_n^\beta E_n D_n^{\frac{1}{2}})(F_n^\beta E_n D_n^{\frac{1}{2}})^T]^{-1}\right|\right|\right|_S}. \tag{18}$$

By applying the submultiplicativity of the spectral norm we obtain

$$\left|\!\left|\!\left|[(F_n^\beta E_n D_n^{\frac{1}{2}})(F_n^\beta E_n D_n^{\frac{1}{2}})^T]^{-1}\right|\!\right|\!\right|_S = |||(F_n^\beta)^{-1}(E_n^T)^{-1}D_n^{-1}E_n^{-1}(F_n^\beta)^{-1}|||_S \tag{19}$$
$$\leq |||(F_n^\beta)^{-1}|||_S^2 \cdot (|||E_n^{-1}|||_S \cdot |||(E_n^{-1})^T|||_S) \cdot |||D_n^{-1}|||_S$$
$$= |||(F_n^\beta)^{-1}|||_S^2 \cdot |||(E_n^T E_n)^{-1}|||_S \cdot |||D_n^{-1}|||_S .$$

Since $J_{\alpha-\beta}(i) > 0$ for all $i = 1, \ldots, n$ we have

$$|||D_n^{-1}|||_S = \left|\!\left|\!\left|\operatorname{diag}\left(\frac{1}{(J_{\alpha-\beta}(1)}, \frac{1}{J_{\alpha-\beta}(2)}, \ldots, \frac{1}{J_{\alpha-\beta}(n)}\right)\right|\!\right|\!\right|_S$$
$$= \max_{1\leq i\leq n} \frac{1}{J_{\alpha-\beta}(i)} = \frac{1}{\min_{1\leq i\leq n} J_{\alpha-\beta}(i)}, \tag{20}$$

and similarly

$$|||(F_n^\beta)^{-1}|||_S^2 = \left|\!\left|\!\left|\operatorname{diag}\left(\frac{1}{1^\beta}, \frac{1}{2^\beta}, \ldots, \frac{1}{n^\beta}\right)\right|\!\right|\!\right|_S^2 = \max_{1\leq i\leq n} \frac{1}{i^{2\beta}}$$
$$= \frac{1}{\min_{1\leq i\leq n} i^{2\beta}} = \frac{1}{\min\{1, n^{2\beta}\}}. \tag{21}$$

For the spectral norm of the matrix $(E_n^T E_n)^{-1}$, we have

$$|||(E_n^T E_n)^{-1}|||_S = \frac{1}{t_n}. \tag{22}$$

Now by combining equations (20), (21) and (22) with (19), we obtain

$$\lambda_1^{n,\alpha,\beta} = \frac{1}{\left|\!\left|\!\left|[(F_n^\beta E_n D_n^{\frac{1}{2}})(F_n^\beta E_n D_n^{\frac{1}{2}})^T]^{-1}\right|\!\right|\!\right|_S}$$
$$\geq \frac{1}{\left|\!\left|\!\left|(F_n^\beta)^{-1}\right|\!\right|\!\right|_S^2 \cdot |||(E_n^T E_n)^{-1}|||_S \cdot |||D_n^{-1}|||_S}$$
$$= t_n \cdot \min_{1\leq i\leq n} J_{\alpha-\beta}(i) \cdot \min\{1, n^{2\beta}\}, \tag{23}$$

which completes the proof. □

Remark 3. *For* $\alpha - \beta \geq 1$ *we have* $\min_{1\leq i\leq n} J_{\alpha-\beta}(i) = 1$. *In addition, if* $\beta \geq 0$, *then* $\min\{1, n^{2\beta}\} = 1$ *and we simply have*

$$\lambda_1^{n,\alpha,\beta} \geq t_n. \tag{24}$$

In particular, this holds for the so called power GCD matrix $A_n^{\alpha,\beta}$ *in which* $\beta = 0$ *and* $\alpha > 1$ *and for the matrix* $A_n^{1,0}$, *which is the usual GCD matrix of the set* $\{1, 2, \ldots, n\}$.

On the other hand, if $\beta < 0$, then $\min\{1, n^{2\beta}\} = n^{2\beta}$ and

$$\lambda_1^{n,\alpha,\beta} \geq t_n \cdot n^{2\beta}. \tag{25}$$

For example, when considering the so called reciprocal matrix $A_n^{1,-1}$, Theorem 1 yields this bound.

Example 1. *Let $n = 6$, $\alpha = 2$ and $\beta = \frac{1}{2}$. Then we have*

$$A_6^{2,\frac{1}{2}} = \begin{bmatrix} 1 & \sqrt{2} & \sqrt{3} & 2 & \sqrt{5} & \sqrt{6} \\ \sqrt{2} & 4\sqrt{2} & \sqrt{6} & 8 & \sqrt{10} & 4\sqrt{6} \\ \sqrt{3} & \sqrt{6} & 9\sqrt{3} & 2\sqrt{3} & \sqrt{15} & 9\sqrt{6} \\ 2 & 8 & 2\sqrt{3} & 32 & 2\sqrt{5} & 8\sqrt{3} \\ \sqrt{5} & \sqrt{10} & \sqrt{15} & 2\sqrt{5} & 25\sqrt{5} & \sqrt{30} \\ \sqrt{6} & 4\sqrt{6} & 9\sqrt{6} & 8\sqrt{3} & \sqrt{30} & 36\sqrt{6} \end{bmatrix}, \tag{26}$$

and by Theorem 1 and Remark 3 we have $\lambda_1^{6,2,\frac{1}{2}} \geq t_6 \approx 0.129425$. Direct calculation shows that in fact $\lambda_1^{6,2,\frac{1}{2}} \approx 0.459959$.

Example 2. *Let $n = 5$, $\alpha = -2$ and $\beta = -3$. This time we have*

$$A_5^{-2,-3} = \begin{bmatrix} 1 & \frac{1}{8} & \frac{1}{27} & \frac{1}{64} & \frac{1}{125} \\ \frac{1}{8} & \frac{1}{32} & \frac{1}{216} & \frac{1}{256} & \frac{1}{1000} \\ \frac{1}{27} & \frac{1}{216} & \frac{1}{243} & \frac{1}{1728} & \frac{1}{3375} \\ \frac{1}{64} & \frac{1}{256} & \frac{1}{1728} & \frac{1}{1024} & \frac{1}{8000} \\ \frac{1}{125} & \frac{1}{1000} & \frac{1}{3375} & \frac{1}{8000} & \frac{1}{7776} \end{bmatrix}, \tag{27}$$

$\min_{1 \leq i \leq n} J_1(i) = 1$ *and* $\min\{1, 5^{2\cdot(-3)}\} = \frac{1}{15625}$. *Thus, by Theorem 1 we have*

$$\lambda_1^{5,-2,-3} \geq t_5 \cdot 1 \cdot \frac{1}{15625} \approx 1.30797 \cdot 10^{-5},$$

although a direct calculation gives $\lambda_1^{5,-2,-3} \approx 6.45967 \cdot 10^{-5}$.

In Theorem 1 we assume that $\alpha > \beta$. Next we are going to prove a more robust theorem which can be used in any circumstances, but as a downside it also gives a bit more broad bounds for the eigenvalues of the matrix $A_n^{\alpha,\beta}$.

Theorem 2. *Every eigenvalue of the matrix $A_n^{\alpha,\beta}$ lies in the union of the real intervals*

$$\bigcup_{k=1}^{n} \Big[2k^{\alpha+\beta} - T_n \cdot \max\{1, n^{2\beta}\} \cdot \max_{1 \leq i \leq n} |J_{\alpha-\beta}(i)|\ ,\ T_n \cdot \max\{1, n^{2\beta}\} \cdot \max_{1 \leq i \leq n} |J_{\alpha-\beta}(i)|\Big]. \tag{28}$$

Proof. Let the matrices E_n, D_n and F_n be as above. In addition, we denote

$$\Lambda_n = |D_n|^{\frac{1}{2}} = \mathrm{diag}\left(\sqrt{|J_{\alpha-\beta}(1)|}, \sqrt{|J_{\alpha-\beta}(2)|}, \ldots, \sqrt{|J_{\alpha-\beta}(n)|}\right). \tag{29}$$

By applying Proposition 1, we obtain

$$A_n^{\alpha,\beta} = F_n^{\beta} E_n D_n E_n^T F_n^{\beta}, \tag{30}$$

and next we observe that

$$0_{n\times n} \leq A_n^{\alpha,\beta} \leq F_n^{\beta} E_n |D_n| E_n^T F_n^{\beta} = F_n^{\beta} E_n \Lambda_n \Lambda_n^T E_n^T (F_n^{\beta})^T = (F_n^{\beta} E_n \Lambda_n)(F_n^{\beta} E_n \Lambda_n)^T, \tag{31}$$

where $\leq$ is understood componentwise. By Theorem 8.2.12 in [14], we know that now every eigenvalue of $A_n^{\alpha,\beta}$ lies in the region

$$\bigcup_{k=1}^{n} \left\{ z \in \mathbb{C} \;\middle|\; |z - k^{\alpha+\beta}| \leq \rho((F_n^{\beta} E_n \Lambda_n)(F_n^{\beta} E_n \Lambda_n)^T) - k^{\alpha+\beta} \right\}, \tag{32}$$

where $\rho((F_n^{\beta} E_n \Lambda_n)(F_n^{\beta} E_n \Lambda_n)^T)$ is the spectral radius of the matrix $(F_n^{\beta} E_n \Lambda_n)(F_n^{\beta} E_n \Lambda_n)^T$. Since the matrix $(F_n^{\beta} E_n \Lambda_n)(F_n^{\beta} E_n \Lambda_n)^T$ is clearly positive semidefinite, we have

$$\begin{aligned}
\rho((F_n^{\beta} E_n \Lambda_n)(F_n^{\beta} E_n \Lambda_n)^T) &= \left\|\left|(F_n^{\beta} E_n \Lambda_n)(F_n^{\beta} E_n \Lambda_n)^T\right|\right\|_S \\
&\leq \left\|\left|F^{\beta}\right|\right\|_S^2 \cdot \left\|\left|E_n^T E_n\right|\right\|_S \cdot \left\|\left|\Lambda_n \Lambda_n^T\right|\right\|_S \qquad (33)\\
&= T_n \cdot \max_{1\leq i\leq n} i^{2\beta} \cdot \max_{1\leq i\leq n} |J_{\alpha-\beta}(i)| \\
&= T_n \cdot \max\{1, n^{2\beta}\} \cdot \max_{1\leq i\leq n} |J_{\alpha-\beta}(i)|. \qquad (34)
\end{aligned}$$

Finally, the matrix $A_n^{\alpha,\beta}$ is real and symmetric, which means that all its eigenvalues are real. So we have proven that every eigenvalue of $A_n^{\alpha,\beta}$ lies in the region

$$\bigcup_{k=1}^{n} \left\{ z \in \mathbb{R} \;\middle|\; |z - k^{\alpha+\beta}| \leq T_n \cdot \max\{1, n^{2\beta}\} \cdot \max_{1\leq i\leq n} |J_{\alpha-\beta}(i)| - k^{\alpha+\beta} \right\}. \tag{35}$$

The claim now follows easily by removing the absolute value function. □

Remark 4. *Theorem 2 is not very useful when $\beta > 0$, since in this case the term $\max\{1, n^{2\beta}\}$ often becomes large.*

Example 3. *Let $n = 4$, $\alpha = -1$ and $\beta = -1$. Then we obtain*

$$A_4^{-1,-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{6} & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{9} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{12} & \frac{1}{16} \end{bmatrix}. \tag{36}$$

Now $\max\{1, 4^{2\cdot(-1)}\} = 1$, $\max_{1\le i\le 4} |J_0(i)| = |J_0(1)| = 1$ *and thus by Theorem 2 we know that the eigenvalues of* $A_4^{-1,-1}$ *lie in the union*

$$[-3.78, 5.78] \cup [-5.28, 5.78] \cup [-5.56, 5.78] \cup [-5.65, 5.78] = [-5.65, 5.78]. \quad (37)$$

Direct calculation shows that this really is the case, since $A_4^{-1,-1}$ *has* 0 *as an eigenvalue of multiplicity* 3 *and the only nonzero eigenvalue is* 1.42361.

The following corollary is a direct consequece of Theorem 2.

Corollary 1. *If* λ *is an eigenvalue of the matrix* $A_n^{\alpha,\beta}$, *then*

$$|\lambda| \le T_n \cdot \max\{1, n^{2\beta}\} \cdot \max_{1\le i\le n} |J_{\alpha-\beta}(i)|. \quad (38)$$

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References

[1] E. Altinisik, *On the matrix norms of a GCD related matrix*, Math. Inequal. Appl., **11** (2008), 635–646.

[2] E. Altinisik, N. Tuglu and P. Haukkanen, *A note on bounds for norms of the reciprocal LCM matrix*, Math. Inequal. Appl., **7** (2004), 491–496.

[3] F. Balatoni, *On the eigenvalues of the matrix of the Smith determinant*, Mat. Lapok, **20** (1969), 397–403 (in Hungarian).

[4] A. Bege, *Generalized LCM matrices*, Publ. Math. Debrecen, **79** (2011), 3-4 (5).

[5] I. Berkes, M. Weber, *On series of dilated functions*, arXiv:1205.1326v1 [math.CA]

[6] P. Haukkanen, *On the ℓ_p norm of GCD and related matrices*, J. Inequal. Pure Appl. Math., **5** (2004) no.3, Art. 61.

[7] P. Haukkanen, *An upper bound for the ℓ_p norm of a GCD-related matrix*, J. Inequal. Appl., (2006) Art. ID 25020, 6 pp.

[8] P. Haukkanen, *On the maximum row and column sum norm of GCD and related matrices*, J. Inequal. Pure Appl. Math., **8** (2007) no. 4, Art. 97.

[9] P. Haukkanen, J. Wang, J. Sillanpää, *On Smiths determinant*, Linear Algebra Appl., **258** (1997), 251-269.

[10] T. Hilberdink, *An arithmetical mapping and applications to Ω-results for the Riemann zeta function*, Acta Arith., **139** (2009) no. 4, 341-367.

[11] S. Hong and K. S. Enoch Lee, *Asymptotic behavior of eigenvalues of reciprocal power LCM matrices*, Glasg. Math. J., **50** (2008), 163–174.

[12] S. Hong and R. Loewy, *Asymptotic behavior of eigenvalues of greatest common divisor matrices*, Glasg. Math. J., **46** (2004), 551–569.

[13] S. Hong and R. Loewy, *Asymptotic behavior of the smallest eigenvalue of matrices associated with completely even functions (mod r)*, Int. J. Number Theory, **7** (2011), 1681-1704.

[14] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[15] P. Ilmonen, P. Haukkanen and J. K. Merikoski, *On eigenvalues of meet and join matrices associated with incidence functions*, Linear Algebra Appl., **429** (2008), 859–874.

[16] I. Korkee, P. Haukkanen, *On meet and join matrices associated with incidence functions*, Linear Algebra Appl., **372** (2003), 127153.

[17] P. Lindqvist and K. Seip, *Note on some greatest common divisor matrices*, Acta Arith., **84** (1998), 149–154.

[18] J. Sándor and B. Crstici, *Handbook of Number Theory II*, Kluwer Academic Publishers, 2004.

[19] H. J. S. Smith, *On the value of certain arithmetical determinant*, Proc. London Math. Soc., **7** (1875/76), 208–212.

[20] A. Wintner, *Diophantine approximations and Hilbert's space*, Amer. J. Math., **66** (1944), 564–578.