

SOME RELATIONSHIPS BETWEEN POLY-CAUCHY TYPE NUMBERS AND POLY-BERNOULLI TYPE NUMBERS

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Abstract

In this paper, we show some relationships between poly-Cauchy type numbers and poly-Bernoulli type numbers. Poly-Cauchy type numbers include poly-Cauchy numbers introduced by T. Komatsu, and poly-Bernoulli type numbers include poly-Bernoulli numbers introduced by M. Kaneko. Both numbers are generalizations of classical Cauchy numbers and classical Bernoulli numbers, respectively.

1 Introduction

Let n and k be integers with $n \geq 0$. Let $f_k(m)$ be a function independent of n . Define *poly-Bernoulli type numbers* $\mathcal{B}_{k,n}(f)$ by

$$\mathcal{B}_{k,n}(f) := (-1)^n \sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} m! f_k(m), \quad (1)$$

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where $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} = \frac{1}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^n$$

(see e.g. [3]). Define *poly-Cauchy type numbers of the first kind* $\mathcal{C}_{k,n}(f)$ by

$$\mathcal{C}_{k,n}(f) := (-1)^n \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] f_k(m) \quad (2)$$

and *poly-Cauchy type numbers of the second kind* $\hat{\mathcal{C}}_{k,n}(f)$ by

$$\hat{\mathcal{C}}_{k,n}(f) := (-1)^n \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] (-1)^m f_k(m), \quad (3)$$

where $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$ are the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1) \dots (x+n-1) = \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] x^m$$

(see e.g. [3]). Hence, if

$$f = f_k(m) = (-1)^m \sum_{i=0}^m \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}, \quad (4)$$

then

$$\mathcal{B}_{k,n}(f) = B_n^{(k)}(z) = (-1)^n \sum_{m=0}^n \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} m! (-1)^m \sum_{i=0}^m \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}$$

is one of the poly-Bernoulli polynomials defined in [8], and

$$\mathcal{C}_{k,n}(f) = c_n^{(k)}(z) = (-1)^n \sum_{m=0}^n \left[\begin{smallmatrix} m \\ n \end{smallmatrix} \right] (-1)^{n-m} \sum_{i=0}^m \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}$$

is the poly-Cauchy polynomial of the first kind defined by

$$c_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_k (x_1 x_2 \dots x_k - z)(x_1 x_2 \dots x_k - 1 - z) \dots (x_1 x_2 \dots x_k - (n-1) - z) dx_1 dx_2 \dots dx_k$$

([4, Theorem 1]). In addition,

$$\hat{\mathcal{C}}_{k,n}(f) = \hat{c}_n^{(k)}(z) = (-1)^n \sum_{m=0}^n \begin{bmatrix} m \\ n \end{bmatrix} (-1)^{n-m} \sum_{i=0}^m \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}$$

is the poly-Cauchy polynomial of the second kind defined by

$$\hat{c}_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1 x_2 \dots x_k + z)(-x_1 x_2 \dots x_k - 1 + z) \dots (-x_1 x_2 \dots x_k - (n-1) + z) dx_1 dx_2 \dots dx_k$$

([4, Theorem 4]). Furthermore, if $z = 0$ in (4), namely,

$$f = f_k(m) = \frac{(-1)^m}{(m+1)^k}, \quad (5)$$

then

$$\mathcal{B}_{k,n}(f) = B_n^{(k)} = (-1)^n \sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} m! (-1)^m \frac{(-1)^m m!}{(m+1)^k}$$

is the poly-Bernoulli number defined in [5],

$$\mathcal{C}_{k,n}(f) = c_n^{(k)} = (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{(m+1)^k}$$

is the poly-Cauchy number of the first kind defined in [6], and

$$\hat{\mathcal{C}}_{k,n}(f) = \hat{c}_n^{(k)} = (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k}$$

is the poly-Cauchy number of the second kind, also defined in [6].

In this paper, we shall show the following relationships between poly-Cauchy type numbers and poly-Bernoulli type numbers.

Theorem 1. For $n \geq 1$ we have

$$\begin{aligned} \mathcal{B}_{k,n}(f) &= \sum_{l=1}^n \sum_{m=1}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \left\{ \begin{matrix} m-1 \\ l-1 \end{matrix} \right\} \mathcal{C}_{k,l}(f), \\ &= (-1)^n \sum_{l=1}^n \sum_{m=1}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} \hat{\mathcal{C}}_{k,l}(f), \\ \mathcal{C}_{k,n}(f) &= (-1)^n \sum_{l=1}^n \sum_{m=1}^n \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} \mathcal{B}_{k,l}(f) \\ \hat{\mathcal{C}}_{k,n}(f) &= (-1)^n \sum_{l=1}^n \sum_{m=1}^n \frac{1}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} \mathcal{B}_{k,l}(f). \end{aligned}$$

Remark. If

$$f = f_k(m) = (-1)^m \sum_{i=0}^m \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k},$$

then this theorem is reduced to [8, Theorem 4.1].

2 Proof

Poly-Bernoulli type numbers can be expressed by poly-Cauchy type numbers. Notice that

$$\sum_{l=i}^m (-1)^l \left\{ \begin{matrix} m-1 \\ l-1 \end{matrix} \right\} \left[\begin{matrix} l \\ i \end{matrix} \right] = (-1)^m \binom{m-1}{i-1}$$

(see, e.g. [3, (6.26)]) and

$$\sum_{m=i}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} (-1)^m \binom{m-1}{i-1} = (-1)^n n! \left\{ \begin{matrix} n \\ i \end{matrix} \right\}$$

(see, e.g. [3, (6.19),(6.21)]). Then by the identity (2) we have

$$\begin{aligned} & \sum_{l=1}^n \sum_{m=1}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \left\{ \begin{matrix} m-1 \\ l-1 \end{matrix} \right\} \mathcal{C}_{k,l}(f) \\ &= \sum_{l=1}^n \sum_{m=l}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \left\{ \begin{matrix} m-1 \\ l-1 \end{matrix} \right\} (-1)^l \sum_{i=0}^l \left[\begin{matrix} l \\ i \end{matrix} \right] f_k(i) \\ &= \sum_{i=1}^n f_k(i) \sum_{l=i}^n \sum_{m=l}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \left\{ \begin{matrix} m-1 \\ l-1 \end{matrix} \right\} (-1)^l \left[\begin{matrix} l \\ i \end{matrix} \right] \\ &= \sum_{i=1}^n f_k(i) \sum_{m=i}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \sum_{l=i}^m (-1)^l \left\{ \begin{matrix} m-1 \\ l-1 \end{matrix} \right\} \left[\begin{matrix} l \\ i \end{matrix} \right] \\ &= \sum_{i=1}^n f_k(i) \sum_{m=i}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} (-1)^m \binom{m-1}{i-1} \\ &= \sum_{i=1}^n f_k(i) (-1)^n n! \left\{ \begin{matrix} n \\ i \end{matrix} \right\} = \mathcal{B}_{k,n}(f). \end{aligned}$$

Hence, the first identity holds. Next, by the identity (3) we have

$$\begin{aligned}
& (-1)^n \sum_{l=1}^n \sum_{m=1}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} \hat{\mathcal{C}}_{k,l}(f) \\
&= (-1)^n \sum_{l=1}^n \sum_{m=1}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (-1)^l \sum_{i=0}^l \begin{bmatrix} l \\ i \end{bmatrix} (-1)^i f_k(i) \\
&= (-1)^n \sum_{i=1}^n (-1)^i f_k(i) \sum_{l=i}^n \sum_{m=l}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \left\{ \begin{matrix} m \\ l \end{matrix} \right\} (-1)^l \begin{bmatrix} l \\ i \end{bmatrix} \\
&= (-1)^n \sum_{i=1}^n (-1)^i f_k(i) \sum_{m=i}^n m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \sum_{l=i}^m (-1)^l \left\{ \begin{matrix} m \\ l \end{matrix} \right\} \begin{bmatrix} l \\ i \end{bmatrix} \\
&= (-1)^n \sum_{i=1}^n (-1)^i f_k(i) i! \left\{ \begin{matrix} n \\ i \end{matrix} \right\} (-1)^i \\
&= (-1)^n \sum_{i=1}^n f_k(i) i! \left\{ \begin{matrix} n \\ i \end{matrix} \right\} = \mathcal{B}_{k,n}(f).
\end{aligned}$$

Note that

$$\sum_{l=i}^m (-1)^{m-l} \begin{bmatrix} m \\ l \end{bmatrix} \left\{ \begin{matrix} l \\ i \end{matrix} \right\} = \begin{cases} 1 & (i = m); \\ 0 & (i \neq m). \end{cases}$$

On the other hand, poly-Cauchy type numbers $\mathcal{C}_{k,n}(f)$ can be expressed by using poly-Bernoulli numbers $\mathcal{B}_{k,n}(f)$. By (1) we have

$$\begin{aligned}
& (-1)^n \sum_{l=1}^n \sum_{m=1}^n \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} \mathcal{B}_{k,l}(f) \\
&= (-1)^n \sum_{l=1}^n \sum_{m=1}^n \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} (-1)^l \sum_{i=0}^l \left\{ \begin{matrix} l \\ i \end{matrix} \right\} i! f_k(i) \\
&= (-1)^n \sum_{m=1}^n \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \sum_{l=0}^n \begin{bmatrix} m \\ l \end{bmatrix} (-1)^l \sum_{i=0}^l \left\{ \begin{matrix} l \\ i \end{matrix} \right\} i! f_k(i) \\
&= (-1)^n \sum_{m=1}^n \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \sum_{i=0}^n i! f_k(i) \sum_{l=i}^n (-1)^l \begin{bmatrix} m \\ l \end{bmatrix} \left\{ \begin{matrix} l \\ i \end{matrix} \right\} \\
&= (-1)^n \sum_{m=0}^n \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} m! f_k(m) (-1)^m \\
&= (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} f_k(m) = \mathcal{C}_{k,n}(f).
\end{aligned}$$

Note that $\begin{bmatrix} m \\ 0 \end{bmatrix} = 0$ ($m \geq 1$) and $\begin{bmatrix} m \\ l \end{bmatrix} = 0$ ($l > m$). Hence, the third identity holds.

Finally, the fourth identity holds because

$$\begin{aligned}
& (-1)^n \sum_{l=1}^n \sum_{m=1}^n \frac{1}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} \mathcal{B}_{k,l}(f) \\
&= (-1)^n \sum_{l=1}^n \sum_{m=1}^n \frac{1}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} (-1)^l \sum_{i=0}^l \begin{Bmatrix} l \\ i \end{Bmatrix} i! f_k(i) \\
&= (-1)^n \sum_{m=1}^n \frac{1}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \sum_{l=0}^n \begin{bmatrix} m \\ l \end{bmatrix} (-1)^l \sum_{i=0}^l \begin{Bmatrix} l \\ i \end{Bmatrix} i! f_k(i) \\
&= (-1)^n \sum_{m=1}^n \frac{1}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \sum_{i=0}^n i! f_k(i) \sum_{l=i}^n (-1)^l \begin{bmatrix} m \\ l \end{bmatrix} \begin{Bmatrix} l \\ i \end{Bmatrix} \\
&= (-1)^n \sum_{m=0}^n \frac{1}{m!} \begin{bmatrix} n \\ m \end{bmatrix} m! f_k(m) (-1)^m \\
&= (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} (-1)^m f_k(m) = \hat{\mathcal{C}}_{k,n}(f).
\end{aligned}$$

3 Comments

For a real number $q \neq 0$, poly-Cauchy numbers, with q parameter, of the first kind are defined by

$$c_{n,q}^{(k)} = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-q)^{n-m}}{(m+1)^k} \quad (n \geq 0, k \geq 1)$$

and the poly-Cauchy numbers, with q parameter, of the second kind by

$$\hat{c}_{n,q}^{(k)} = (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{q^{n-m}}{(m+1)^k}$$

([7]). Though the corresponding poly-Bernoulli numbers with q parameter seem to be defined by

$$B_{n,q}^{(k)} = \sum_{m=0}^n \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{(-q)^{n-m} m!}{(m+1)^k},$$

but these numbers do not satisfy similar identities as in Theorem 1 unless $q = \pm 1$. For, in these cases

$$f_k(m) = \frac{(-1)^m q^{n-m}}{(m+1)^k},$$

which is not independent of n , unless $q = \pm 1$. In other words, $c_{n,q}^{(k)}$, $\hat{c}_{n,q}^{(k)}$ and $B_{n,q}^{(k)}$ ($q \neq \pm 1$) are not involved in the poly-Cauchy type numbers and poly-Bernoulli type numbers defined in this paper.

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