East-West J. of Mathematics: Vol. 14 , No 2 (2012) pp. 114-120

SOME RELATIONSHIPS BETWEEN POLY-CAUCHY TYPE NUMBERS AND POLY-BERNOULLI TYPE NUMBERS

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Abstract

In this paper, we show some relationships between poly-Cauchy type numbers and poly-Bernoulli type numbers. Poly-Cauchy type numbers include poly-Cauchy numbers introduced by T. Komatsu, and poly-Bernoulli type numbers include poly-Bernoulli numbers introduced by M. Kaneko. Both numbers are generalizations of classical Cauchy numbers and classical Bernoulli numbers, respectively.

1 Introduction

Let n and k be integers with $n \geq 0$. Let $f_k(m)$ be a function independent of n. Define *poly-Bernoulli type numbers* $\mathcal{B}_{k,n}(f)$ by

$$\mathcal{B}_{k,n}(f) := (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} m! f_k(m) \,, \tag{1}$$

Keywords: poly-Cauchy numbers, poly-Bernoulli numbers, poly-Cauchy type numbers, poly-Bernoulli type numbers, Cauchy numbers, Bernoulli numbers
(2010) AMS Classification: 05A15, 11B75
Research supported in part by the Grant-in-Aid for Scientific research (C) (No.22540005), the Japan Society for the Promotion of Science.

where $\left\{ {n \atop m} \right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (m-j)^n$$

(see e.g. [3]). Define *poly-Cauchy type numbers of the first kind* $\mathcal{C}_{k,n}(f)$ by

$$\mathcal{C}_{k,n}(f) := (-1)^n \sum_{m=0}^{n} \left[{n \atop m} \right] f_k(m) \tag{2}$$

and *poly-Cauchy type numbers of the second kind* $\hat{\mathcal{C}}_{k,n}(f)$ by

$$\hat{\mathcal{C}}_{k,n}(f) := (-1)^n \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^m f_k(m)\,, \tag{3}$$

where $\left[{n \atop m} \right]$ are the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{m=0}^{n} \left[{n \atop m} \right] x^m$$

(see e.g. [3]). Hence, if

$$f = f_k(m) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}\,, \tag{4}$$

then

$$\mathcal{B}_{k,n}(f) = B_n^{(k)}(z) = (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} m!(-1)^m \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}$$

is one of the poly-Bernoulli polynomials defined in [8], and

$$\mathcal{C}_{k,n}(f) = c_n^{(k)}(z) = (-1)^n \sum_{m=0}^{n} \left[{m \atop n} \right] (-1)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}$$

is the poly-Cauchy polynomial of the first kind defined by

$$c_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1 x_2 \dots x_k - z)(x_1 x_2 \dots x_k - 1 - z) \cdots (x_1 x_2 \dots x_k - (n-1) - z) dx_1 dx_2 \dots dx_k$$

([4, Theorem 1]). In addition,

$$\hat{\mathcal{C}}_{k,n}(f)=\hat{c}_n^{(k)}(z)=(-1)^n\sum_{m=0}^n\begin{bmatrix}m\\n\end{bmatrix}(-1)^{n-m}\sum_{i=0}^m\binom{m}{i}\frac{(-z)^i}{(m-i+1)^k}$$

is the poly-Cauchy polynomial of the second kind defined by

$$\hat{c}_n^{(k)}(z)=n!\underbrace{\int_0^1\dots\int_0^1}_{k}(-x_1x_2\dots x_k+z)(-x_1x_2\dots x_k-1+z)$$
$$\cdots(-x_1x_2\dots x_k-(n-1)+z)dx_1dx_2\dots dx_k$$

([4, Theorem 4]). Furthermore, if $z=0$ in (4), namely,

$$f=f_k(m)=\frac{(-1)^m}{(m+1)^k}\,, \tag{5}$$

then

$$\mathcal{B}_{k,n}(f)=B_n^{(k)}=(-1)^n\sum_{m=0}^n\left\{ n \atop m\right\}m!(-1)^m\frac{(-1)^m m!}{(m+1)^k}$$

is the poly-Bernoulli number defined in [5],

$$\mathcal{C}_{k,n}(f)=c_n^{(k)}=(-1)^n\sum_{m=0}^n\begin{bmatrix}n\\m\end{bmatrix}\frac{(-1)^m}{(m+1)^k}$$

is the poly-Cauchy number of the first kind defined in [6], and

$$\hat{\mathcal{C}}_{k,n}(f)=\hat{c}_n^{(k)}=(-1)^n\sum_{m=0}^n\begin{bmatrix}n\\m\end{bmatrix}\frac{1}{(m+1)^k}$$

is the poly-Cauchy number of the second kind, also defined in [6].

In this paper, we shall show the following relationships between poly-Cauchy type numbers and poly-Bernoulli type numbers.

Theorem 1. *For $n\ge 1$ we have*

$$\mathcal{B}_{k,n}(f)=\sum_{l=1}^n\sum_{m=1}^n m!\left\{ n \atop m\right\}\left\{ m-1 \atop l-1\right\}\mathcal{C}_{k,l}(f)\,,$$
$$=(-1)^n\sum_{l=1}^n\sum_{m=1}^n m!\left\{ n \atop m\right\}\left\{ m \atop l\right\}\hat{\mathcal{C}}_{k,l}(f)\,,$$
$$\mathcal{C}_{k,n}(f)=(-1)^n\sum_{l=1}^n\sum_{m=1}^n\frac{(-1)^m}{m!}\begin{bmatrix}n\\m\end{bmatrix}\begin{bmatrix}m\\l\end{bmatrix}\mathcal{B}_{k,l}(f)$$
$$\hat{\mathcal{C}}_{k,n}(f)=(-1)^n\sum_{l=1}^n\sum_{m=1}^n\frac{1}{m!}\begin{bmatrix}n\\m\end{bmatrix}\begin{bmatrix}m\\l\end{bmatrix}\mathcal{B}_{k,l}(f)\,.$$

Remark. If

$$f = f_k(m) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}\,,$$

then this theorem is reduced to [8, Theorem 4.1].

2 Proof

Poly-Bernoulli type numbers can be expressed by poly-Cauchy type numbers. Notice that

$$\sum_{l=i}^{m} (-1)^l \left\{ {m-1 \atop l-1} \right\} \left[{l \atop i} \right] = (-1)^m \binom{m-1}{i-1}$$

(see, e.g. [3, (6.26)]) and

$$\sum_{m=i}^{n} m! \left\{ {n \atop m} \right\} (-1)^m \binom{m-1}{i-1} = (-1)^n i! \left\{ {n \atop i} \right\}$$

(see, e.g. [3, (6.19),(6.21)]). Then by the identity (2) we have

$$\begin{aligned}
&\sum_{l=1}^{n} \sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} \mathcal{C}_{k,l}(f) \\
&= \sum_{l=1}^{n} \sum_{m=l}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} (-1)^l \sum_{i=0}^{l} \left[{l \atop i} \right] f_k(i) \\
&= \sum_{i=1}^{n} f_k(i) \sum_{l=i}^{n} \sum_{m=l}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} (-1)^l \left[{l \atop i} \right] \\
&= \sum_{i=1}^{n} f_k(i) \sum_{m=i}^{n} m! \left\{ {n \atop m} \right\} \sum_{l=i}^{m} (-1)^l \left\{ {m-1 \atop l-1} \right\} \left[{l \atop i} \right] \\
&= \sum_{i=1}^{n} f_k(i) \sum_{m=i}^{n} m! \left\{ {n \atop m} \right\} (-1)^m \binom{m-1}{i-1} \\
&= \sum_{i=1}^{n} f_k(i) (-1)^n i! \left\{ {n \atop i} \right\} = \mathcal{B}_{k,n}(f)\,.
\end{aligned}$$

Hence, the first identity holds. Next, by the identity (3) we have

$$
\begin{aligned}
&(-1)^n \sum_{l=1}^n \sum_{m=1}^n m! \left\{ n \atop m \right\} \left\{ m \atop l \right\} \hat{\mathcal{C}}_{k,l}(f) \\
&= (-1)^n \sum_{l=1}^n \sum_{m=1}^n m! \left\{ n \atop m \right\} \left\{ m \atop l \right\} (-1)^l \sum_{i=0}^l \left[l \atop i \right] (-1)^i f_k(i) \\
&= (-1)^n \sum_{i=1}^n (-1)^i f_k(i) \sum_{l=i}^n \sum_{m=l}^n m! \left\{ n \atop m \right\} \left\{ m \atop l \right\} (-1)^l \left[l \atop i \right] \\
&= (-1)^n \sum_{i=1}^n (-1)^i f_k(i) \sum_{m=i}^n m! \left\{ n \atop m \right\} \sum_{l=i}^m (-1)^l \left\{ m \atop l \right\} \left[l \atop i \right] \\
&= (-1)^n \sum_{i=1}^n (-1)^i f_k(i) i! \left\{ n \atop i \right\} (-1)^i \\
&= (-1)^n \sum_{i=1}^n f_k(i) i! \left\{ n \atop i \right\} = \mathcal{B}_{k,n}(f)\,.
\end{aligned}
$$

Note that

$$
\sum_{l=i}^m (-1)^{m-l} \left[m \atop l \right] \left\{ l \atop i \right\} = \begin{cases} 1 & (i = m); \\ 0 & (i \neq m). \end{cases}
$$

On the other hand, poly-Cauchy type numbers $\mathcal{C}_{k,n}(f)$ can be expressed by using poly-Bernoulli numbers $\mathcal{B}_{k,n}(f)$. By (1) we have

$$
\begin{aligned}
&(-1)^n \sum_{l=1}^n \sum_{m=1}^n \frac{(-1)^m}{m!} \left[n \atop m \right] \left[m \atop l \right] \mathcal{B}_{k,l}(f) \\
&= (-1)^n \sum_{l=1}^n \sum_{m=1}^n \frac{(-1)^m}{m!} \left[n \atop m \right] \left[m \atop l \right] (-1)^l \sum_{i=0}^l \left\{ l \atop i \right\} i! f_k(i) \\
&= (-1)^n \sum_{m=1}^n \frac{(-1)^m}{m!} \left[n \atop m \right] \sum_{l=0}^n \left[m \atop l \right] (-1)^l \sum_{i=0}^l \left\{ l \atop i \right\} i! f_k(i) \\
&= (-1)^n \sum_{m=1}^n \frac{(-1)^m}{m!} \left[n \atop m \right] \sum_{i=0}^n i! f_k(i) \sum_{l=i}^n (-1)^l \left[m \atop l \right] \left\{ l \atop i \right\} \\
&= (-1)^n \sum_{m=0}^n \frac{(-1)^m}{m!} \left[n \atop m \right] m! f_k(m) (-1)^m \\
&= (-1)^n \sum_{m=0}^n \left[n \atop m \right] f_k(m) = \mathcal{C}_{k,n}(f)\,.
\end{aligned}
$$

Note that $\left[{m \atop 0}\right] = 0$ $(m \geq 1)$ and $\left[{m \atop l}\right] = 0$ $(l > m)$. Hence, the third identity holds.

Finally, the fourth identiy holds because

$$
\begin{aligned}
&(-1)^n \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{1}{m!} \left[{n \atop m}\right] \left[{m \atop l}\right] \mathcal{B}_{k,l}(f) \\
&= (-1)^n \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{1}{m!} \left[{n \atop m}\right] \left[{m \atop l}\right] (-1)^l \sum_{i=0}^{l} \left\{{l \atop i}\right\} i! f_k(i) \\
&= (-1)^n \sum_{m=1}^{n} \frac{1}{m!} \left[{n \atop m}\right] \sum_{l=0}^{n} \left[{m \atop l}\right] (-1)^l \sum_{i=0}^{l} \left\{{l \atop i}\right\} i! f_k(i) \\
&= (-1)^n \sum_{m=1}^{n} \frac{1}{m!} \left[{n \atop m}\right] \sum_{i=0}^{n} i! f_k(i) \sum_{l=i}^{n} (-1)^l \left[{m \atop l}\right] \left\{{l \atop i}\right\} \\
&= (-1)^n \sum_{m=0}^{n} \frac{1}{m!} \left[{n \atop m}\right] m! f_k(m) (-1)^m \\
&= (-1)^n \sum_{m=0}^{n} \left[{n \atop m}\right] (-1)^m f_k(m) = \hat{\mathcal{C}}_{k,n}(f)\,.
\end{aligned}
$$

3 Comments

For a real number $q \neq 0$, poly-Cauchy numbers, with q parameter, of the first kind are defined by

$$
c_{n,q}^{(k)} = \sum_{m=0}^{n} \left[{n \atop m}\right] \frac{(-q)^{n-m}}{(m+1)^k} \quad (n \geq 0,\, k \geq 1)
$$

and the poly-Cauchy numbers, with q parameter, of the second kind by

$$
\hat{c}_{n,q}^{(k)} = (-1)^n \sum_{m=0}^{n} \left[{n \atop m}\right] \frac{q^{n-m}}{(m+1)^k}
$$

([7]). Though the corresponding poly-Bernoulli numbers with q parameter seem to be defined by

$$
B_{n,q}^{(k)} = \sum_{m=0}^{n} \left\{{n \atop m}\right\} \frac{(-q)^{n-m} m!}{(m+1)^k}\,,
$$

but these numbers do not satisfy similar identities as in Theorem 1 unless $q = \pm 1$. For, in these cases

$$
f_k(m) = \frac{(-1)^m q^{n-m}}{(m+1)^k}\,,
$$

which is not independent of n, unless $q = \pm 1$. In other words, $c_{n,q}^{(k)}$, $\hat{c}_{n,q}^{(k)}$ and $B_{n,q}^{(k)}$ ($q \neq \pm 1$) are not involved in the poly-Cauchy type numbers and poly-Bernoulli type numbers defined in this paper.

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