SHORT-SOLID SUPERASSOCIATIVE TYPE
(n) VARIETIES

V. Budd†, K. Denecke‡ and S. L. Wismath‡

† Institut für Mathematik,
Universität Potsdam,
Am Neuen Palais, 14415 Potsdam, Germany
kdenecke@rz.uni-potsdam.de

‡ Dept. of Mathematics and C.S.,
University of Lethbridge,
Lethbridge, Alberta, Canada T1K-3M4
wismaths@cs.uleth.ca

Abstract

Let $M$ be a monoid of hypersubstitutions and $V$ be a variety, both of a fixed type $\tau$. $V$ is said to be $M$-solid if the application of any hypersubstitution in $M$ to any identity of $V$ leads to an identity of $V$. For any monoid $M$, the collection of all $M$-solid varieties of type $\tau$ forms a sublattice of the lattice of all varieties of type $\tau$. Thus studying $M$-solid varieties gives an approach to studying the lattice of all varieties of a given type. This approach has been used most successfully in the special case of type (2) and in particular for varieties of semigroups, where the lattice of all $M$-solid varieties has been fully characterized for $M = \text{Hyp}(2)$ and various other choices of $M$. In this paper we extend this approach to type $(n)$ varieties for $n \geq 3$. Using the $n$-ary superassociative law as our analogue of associativity for the semigroup case, we investigate $M$-solid superassociative varieties of type $(n)$ for two monoids $M$ consisting of particular hypersubstitutions which we call short hypersubstitutions. For each of these monoids we find the smallest and largest $M$-solid superassociative varieties, and give complete characterizations of all such varieties. To obtain these characterizations we introduce a reduction in $M$-solidity testing based on the Green’s relations on monoids of hypersubstitutions, which we describe for our monoids.

Research of the first and third authors supported by NSERC of Canada.

Key words and phrases: M-hyperidentity, hypersubstitution, superassociativity, short-solidity.
(1991) Mathematics Subject Classification: 08A40, 08B15, 20M07.
1 Introduction

Let $\tau$ be a fixed type of algebras, with fundamental operation symbols $f_i$ of arity $n_i$, for $i \in I$. A hypersubstitution of type $\tau$ is a mapping which associates to every operation symbol $f_i$ an $n_i$-ary term $\sigma(f_i)$ of type $\tau$. Let $W_\tau(X)$ be the set of all terms of type $\tau$ on an alphabet $X = \{x_1, x_2, x_3, \ldots\}$. Any hypersubstitution $\sigma$ can be uniquely extended to a map $\hat{\sigma}$ on $W_\tau(X)$ inductively as follows:

(i) if $t = x_i$ for some $i \geq 1$, then $\hat{\sigma}[t] = x_i$;
(ii) if $t = f_i(t_1, \ldots, t_{n_i})$ for some $n_i$-ary operation symbol $f_i$ and some terms $t_1, \ldots, t_{n_i}$, then $\hat{\sigma}[t] = \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

Here the left side of (ii) means the composition of the term $\sigma(f_i)$ and the terms $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]$. A binary operation $\circ_h$ can be defined on the set $Hyp(\tau)$ of all hypersubstitutions of type $\tau$, by letting $\sigma_1 \circ_h \sigma_2$ be the hypersubstitution which maps each fundamental operation symbol $f_i$ to the term $\hat{\sigma_1}(\sigma_2(f_i))$. The set $Hyp(\tau)$ of all hypersubstitutions of type $\tau$ is closed under this associative binary operation. It also contains an identity element for $\circ_h$, namely the identity hypersubstitution $\sigma_{id}$ which maps every $f_i$ to $f_i(x_1, \ldots, x_{n_i})$. Thus $Hyp(\tau)$ is a monoid.

Now let $M$ be any submonoid of $Hyp(\tau)$. An identity $u \approx v$ of a variety $V$ is called an $M$-hyperidentity of $V$ if for every hypersubstitution $\sigma \in M$, the identity $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ holds in $V$. A variety $V$ is called $M$-solid if every identity of $V$ is an $M$-hyperidentity of $V$. When $M$ is the whole monoid $Hyp(\tau)$, an $M$-hyperidentity is called a hyperidentity, and an $M$-solid variety is called a solid variety.

The motivation for studying $M$-solid varieties comes from the following result of Denecke and Reichel in [4]. For each submonoid $M$ of $Hyp(\tau)$, the collection of all $M$-solid varieties of type $\tau$ forms a complete lattice, which is a complete sublattice of the lattice $L(\tau)$ of all varieties of type $\tau$. This lattice $L(\tau)$ is in general large and complicated, and difficult to study, and the $M$-solid-sublattices give us a way to study at least some of its sublattices. Thus it may be useful to study the monoid $Hyp(\tau)$ and its submonoids $M$ and the corresponding $M$-solid varieties, both in general and for specific types $\tau$, and the intersection of the lattice of all $M$-solid varieties with a fixed variety of type $\tau$. For specific types, much work has been done for type (2), and in particular for varieties of semigroups. Polák ([9]) has given an equational description of the largest solid variety of semigroups and a characterization of the lattice of solid semigroup varieties, and various authors have studied $M$-solid semigroup varieties for various choices of $M$ (see for instance [1], [2]).

Our goal in this paper is to begin a similar investigation for type (n), for $n \geq 3$. Only a few solid varieties of type (n) have been known, although recently Denecke, Koppitz and Wismath ([3]) constructed some infinite chains of such solid varieties, and little is known about $M$-solidity for any choices of $M$. Our strategy is to use our knowledge about type (2), and semigroup varieties,
as a model. In particular, even in type (2) the basic problem of finding all solid varieties of the type has not been solved; instead, work has focussed on generalizations of this problem, in two essential ways. First, instead of looking at all type (2) varieties, we began with semigroup varieties only, so that we had the associative law as an identity to work with. Secondly, instead of looking only for all solid varieties, we also looked for $M$-solid varieties, for various choices of $M$. To use a similar approach for type ($n$) varieties, we need some identity to work with, analogous to associativity in the semigroup case. We will use an identity called superassociativity, which is a natural extension of associativity to higher types and occurs in the axiomatization of the concept of an abstract clone in clone theory. We also introduce two submonoids of $Hyp(n)$, called $Mon$ and $Short$, and study their properties. We give a characterization of all $Mon$-solid superassociative varieties of type ($n$), analogous to a similar theorem of Denecke and Koppitz for type (2). We also use the Green’s relations on these two monoids, to show how testing for $M$-solidity may be made easier. Using this result, we produce the largest $Short$-solid superassociative variety of type ($n$), and characterize all such varieties.

2 The Monoid $Short$

In this section we present some background information about hypersubstitutions and varieties of type ($n$), and introduce the special monoids we shall be studying. We assume throughout a fixed type ($n$), with $n \geq 2$, so we have one $n$-ary operation symbol which we shall denote by $f$. For $\Sigma$ any set of identities of type ($n$), we will denote by $Mod(\Sigma)$ the variety determined by the set $\Sigma$. For convenience, we list here some identities and varieties of type ($n$) that we shall discuss later:

- $x \approx y$, the trivial identity,
- $f(x, \ldots, x) \approx x$, the idempotent identity $Id$,
- $f(f(x_1, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nn})) \approx f(x_{11}, x_{22}, \ldots, x_{nn})$, the diagonal identity $D$,
- $f(f(x_1, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nn})) \approx f(f(x_1, \ldots, x_{n1}), \ldots, f(x_{1n}, \ldots, x_{nn}))$, the medial identity $M$,
- $f(f(x, \ldots, x), \ldots, f(x, \ldots, x)) \approx f(x, \ldots, x)$, the normalized-idempotent identity $NId$,
- $f(x_1, \ldots, x_n) \approx f(y_1, \ldots, y_n)$, the zero identity $Z$.

- $TR = Mod(x \approx y)$, the trivial variety,
- $RA = Mod(Id, D)$, the rectangular variety,
- $Med = Mod(M)$, the medial variety,
- $Idem = Mod(Id)$, the idempotent variety,
- $NIdem = Mod(NId)$, the normalized-idempotent variety,
- $C = Mod(Z)$, the zero variety.

A hypersubstitution $\sigma$ is called a pre-hypersubstitution if $\sigma(f)$ is not a vari-
Short-Solid Superassociative Type (n) Varieties

The set $Pre$ of all pre-hypersubstitutions forms a submonoid of $Hyp(n)$, and a variety is called presolid if it is $M$-solid for $M = Pre$. Note that any solid variety is also presolid.

An identity $u \approx v$ is said to be non-normal if one of the terms $u$ and $v$ is a variable and the other is not, otherwise the identity is called normal. A variety is said to be normal if all of its identities are normal, and non-normal otherwise. If $V$ is a non-normal variety, then there is a unique smallest normal variety containing $V$, called the normalization of $V$.

The following Lemma lists some well-known facts about solid and presolid varieties and normalizations of varieties; see for instance [6].

Lemma 2.1
i) The smallest non-trivial solid variety of type $(n)$ is $RA$.
ii) $C$ is the smallest presolid but not solid variety of type $(n)$.
iii) The normalization of a solid (or presolid) variety is solid (or presolid), and is obtained by taking the join with $C$. (Note that this is only true for type $(n)$, but not for arbitrary type.)
iv) If $RA \leq V$, then $V$ is presolid if $V$ is solid.

v) The varieties $Med$, $Idem$ and $NIdem$ are all solid.

Since any hypersubstitution $\sigma$ in $Hyp(n)$ is completely determined by what it does to $f$, we will denote by $\sigma_t$ the hypersubstitution which maps $f$ to the $n$-ary term $t$. For any $n$-ary term, we can define two parameters called length and content.

Definition 2.2 Let $t$ be an $n$-ary term of type $(n)$. Then
a) the content $c(t)$ of $t$ is defined to be the set of variables used in $t$;
b) the length $|t|$ of $t$ is defined inductively by $|t| = 1$ when $t$ is a variable, and $|f(t_1, \ldots, t_n)| = \Sigma_{i=1}^n |t_i|$.
c) the content and length of a hypersubstitution $\sigma_t$ are defined to be the content and length of the term $t$.

We will be particularly interested in hypersubstitutions which have length of $n$ or less. We will use the following notation for sets of hypersubstitutions:

$Pr = \{\sigma_t : |t| = 1\}$, the subsemigroup of projections; this set is a RZ band;
$Perm = \{\sigma_t : |t| = n, |c(t)| = n\}$, the submonoid (actually a group) of all permutations;
$SpE = \{\sigma_t : |t| = n, |c(t)| = 1\}$, the $n$ special idempotents; this set is a right zero band;
$Mon = Pr \cup Perm \cup SpE$,
$Short = \{\sigma_t : |t| \leq n\}$.

$ShE = \{\sigma_t : |t| = n, |c(t)| > 1 \text{ and } \sigma_t \text{ is an idempotent, } \sigma_t \neq \sigma_{id}\}$, the set of short, non-special idempotents;
$Oth = Short \setminus (Mon \cup ShE)$, the set of other short elements.

The following Lemma is easy to verify, from the definition of the composition operation $\circ_h$ in $Hyp(n)$.
Lemma 2.3 The sets $\text{Perm}$, $\text{Mon}$ and $\text{Short}$ are submonoids of $\text{Hyp}(n)$.

Example 2.4 For $n = 2$, we will abbreviate a short binary term $f(x_i, x_j)$ by $x_ix_j$ only, for $i, j \in \{1, 2\}$. Here the monoid $\text{Short}$ actually equals $\text{Mon} = \text{Pr} \cup \text{Perm} \cup \text{SpE}$, and the remaining two sets $\text{ShE}$ and $\text{Oth}$ are empty. We have $\text{Mon} = \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}, \sigma_{x_2x_1}, \sigma_{x_1x_1}, \sigma_{x_2x_2}\}$.

Note that content size is limited for binary terms, since only content sizes 1 or 2 are possible. Short hypersubstitutions with content size 2 are permutations, while those with content size 1 can only be projections or special idempotents. So in a sense type (2) is limited, and we should expect more interesting behaviour for type $(n)$ for $n \geq 3$. Denecke and Koppitz described all the submonoids of the monoid Short when $n = 2$, and characterized all Short-solid semigroup varieties. We will call a variety of type $(n)$ short-solid if it is Short-solid, and perm-solid if it is $\text{M}$-solid for the monoid $\text{M} = \text{Perm}$.

In the case $n = 2$, perm-solid is also known as dual-solid.

Theorem 2.5 ([DK]) Let $V$ be a type (2) variety of semigroups. Then $V$ is Short-solid iff $V$ is dual-solid and $\text{RA} \subseteq V \subseteq \text{Mod}(\text{NId})$.

To describe the sets $\text{ShE}$ and $\text{Oth}$ for $n \geq 3$, we need the following characterization of idempotents in $\text{Hyp}(n)$.

Lemma 2.6 ([10]) Let $t$ be a non-variable $n$-ary term, $t = f(t_1, \ldots, t_n)$. The hypersubstitution $\sigma_t$ is an idempotent element of $\text{Hyp}(n)$ iff for every index $1 \leq i \leq n$ such that the variable $x_i$ is in the content of $t$, the entry $t_i$ is the variable $x_i$.

Example 2.7 For $n = 3$, the monoid Short consists of thirty short hypersubstitutions: there are 3 projections, 6 permutations, 3 special idempotents, 6 more short idempotents, and 12 Others. To list these elements, we use the convenient notation of representing $\sigma_{f(x_i, x_j, x_k)}$ by the triple $ijk$, and similarly for $\sigma_{x_i}$. Then the elements of Short are:

- Pr: 1,2,3
- Perm: 123, 132, 231, 213, 321, 312
- SpE: 111, 222, 333
- ShE: 121, 122, 113, 133, 223, 323
- Oth: 112, 131, 211, 212, 221, 232, 233, 311, 313, 322, 331, 332

3 Mon-Solid Superassociative Varieties

To have an identity analogous to associativity in type (2) to work with in our search for $\text{M}$-solid varieties, we consider for any $n \geq 2$ the following superassociative identity
\[ f(f(x_1, x_2, \ldots, x_n), y_2, \ldots, y_n) \approx f(x_1, f(f(x_2, y_2, \ldots, y_n), f(x_3, y_2, \ldots, y_n), \ldots, f(x_n, y_2, \ldots, y_n)) \]  

\((SA^n_n)\)

Note that the superscript \(n\) in \(SA^n_n\) tells us the arity of the terms, while the subscript 1 refers to association from the first component to the others. This identity occurs as an axiom for clones (see for instance [6]), and for \(n = 2\) reduces to the usual associative law. We will call a type \((n)\) variety superassociative if it satisfies this identity \(SA^n_n\). We begin the investigation of solid superassociative varieties by looking for \(M\)-solid superassociative varieties, for \(M\) equal to \(Mon\) and then \(Short\).

Suppose that a type \((n)\) variety \(V\) is superassociative and at least \(perm\)-solid. Then \(V\) must satisfy the identities obtained from \(SA^n_n\) by applying \(\sigma\) for each \(\sigma\) in \(Perm\). In particular, for each \(2 \leq i \leq n\), we can obtain the following superassociative identity \(SA^n_n\) by applying the permutation \(\sigma_i\) with \(t = f(x_2, \ldots, x_i, x_1, x_{i+1}, \ldots, x_n)\):

\[ f(y_2, \ldots, y_i, f(x_2, \ldots, x_i, x_1, x_{i+1}, \ldots, x_n), y_{i+1}, \ldots, y_n) \approx f(f(y_2, \ldots, y_i, x_2, y_{i+1}, \ldots, y_n), \ldots, f(y_2, \ldots, y_i, x_i, y_{i+1}, \ldots, y_n), x_1, \ldots, f(y_2, \ldots, y_i, x_n, y_{i+1}, \ldots, y_n)) \]  

\((SA^n_n)\)

For \(V\) to be \(Mon\)-solid, we also have to consider the result of applying \(\hat{\sigma}\) to any of these identities, for \(\sigma\) any element of \(SpE\). It is easily verified that applying \(\sigma_i\) for \(t = f(x_1, \ldots, x_1)\) to \(SA^n_n\) gives the normalized idempotent identity \(NIId\), and moreover that any identity produced from application of \(\sigma\) in \(SpE\) to any of the \(SA^n_n\) is a consequence of \(NIId\). This leads us to define a variety \(V_n\) which we shall show is the largest \(Mon\)-solid superassociative variety of type \((n)\).

**Lemma 3.1** Let \(n \geq 3\). The variety \(V_n = Mod(SA^n_1, \ldots, SA^n_n, NIId)\) is the largest \(mon\)-solid superassociative type \((n)\) variety.

**Proof** We saw above that any superassociative mon-solid variety must be contained in \(V_n\), so it will suffice to show that \(V_n\) is mon-solid. We have to show that for any \(u \approx v\) from the set \(\Sigma = \{SA^n_1, \ldots, SA^n_n, NIId\}\) and any \(\sigma\) in \(Mon\), the identity \(\hat{\sigma}[u] \approx \hat{\sigma}[v]\) also holds in \(V_n\). We consider several cases:

a) Let \(\sigma \in Pr\). Then since all the identities in \(\Sigma\) hold in \(RA\), any projection gives only a trivial identity, which also holds in \(V_n\).

b) Let \(\sigma \in SpE\), so \(\sigma\) is one of \(\sigma f(x_i, \ldots, x_i)\) for \(1 \leq i \leq n\). As remarked above, any of the identities \(\hat{\sigma}[u] \approx \hat{\sigma}[v]\) is a consequence of the \(NIId\) identity, and so holds in \(V_n\).

c) Let \(\sigma \in Perm\). It is clear that if \(u \approx v\) is the identity \(NIId\), then \(\hat{\sigma}[u] \approx \hat{\sigma}[v]\) is also the identity \(NIId\), so we need consider only the identities \(SA^n_n\). For each \(1 \leq i \leq n\), let \(u_i\) be the term from the left side of \(SA^n_n\) and let \(v_i\) be the term from the right side.

Since \(\sigma \in Perm\), we can write it as \(\sigma_i\) for a term \(t = f(x_{j_1}, \ldots, x_{j_n})\) for some set of indices \(\{j_1, \ldots, j_n\}\) of size \(n\). The variable \(x_i\) occurs as one of
the inputs in $t$, say in position $k$, so $x_{jk} = x_i$. We denote by $s_i$ the term $f(x_2, \ldots, x_i, x_{i+1}, \ldots, x_n)$ which occurs in $u_i$. Then

$$\hat{\sigma}[s_i] = \sigma(f)(x_2, \ldots, x_i, x_{i+1}, \ldots, x_n),$$

and this is a term which has $x_1$ in the $k$-th position and the other variables $x_p$ permuted among themselves according to $t$. Therefore

$$\hat{\sigma}[u_i] = \sigma(f)(y_2, \ldots, y_i, \hat{\sigma}[s_i], y_{i+1}, \ldots, y_n),$$

and this is a permutation of the term $f(y_2, \ldots, y_i, \hat{\sigma}[s_i], y_{i+1}, \ldots, y_n)$, with the term $\hat{\sigma}[s_i]$ in the $k$-th position and the $y_p$ variables permuted among themselves according to $t$. This means that $\hat{\sigma}[u_i]$ is just $u_k$, possibly with the variables relabeled. To do a similar analysis for $v_i$, we set $w_q = f(y_2, \ldots, y_i, x_q, y_{i+1}, \ldots, y_n)$, for each $2 \leq q \leq n$. Then $\hat{\sigma}[w_q]$ is a permutation of the term $w_q$, with the variable $x_q$ moved to the $k$-th position and the other variables $y_p$ permuted among themselves according to $t$. Then we have

$$\hat{\sigma}[v_i] = \hat{\sigma}[f(w_2, \ldots, w_i, x_1, w_{i+1}, \ldots, w_n)]$$

$$= \sigma(f)(\hat{\sigma}[w_2], \ldots, \hat{\sigma}[w_i], x_1, \hat{\sigma}[w_{i+1}], \ldots, \hat{\sigma}[w_n]).$$

This is a permutation of the term $f(\hat{\sigma}[w_2], \ldots, \hat{\sigma}[w_i], x_1, \hat{\sigma}[w_{i+1}], \ldots, \hat{\sigma}[w_n])$, with the entry $x_1$ in the $k$-th position and the remaining entries $\hat{\sigma}[w_q]$ permuted among themselves according to $t$. This means that $\hat{\sigma}[v_i] = v_k$, possibly with the variables relabeled. Altogether, we see that $\hat{\sigma}[u_i] \approx \hat{\sigma}[v_i]$ is equivalent to the identity $u_k \approx v_k$, or $SA^n_k$. □

Next we find the smallest non-trivial $Mon$-solid superassociative variety, and characterize all such varieties.

**Lemma 3.2** The rectangular variety $RA$ of type $(n)$ is a solid (and hence mon-solid and short-solid) superassociative variety, and is the smallest non-trivial solid, mon-solid or short-solid superassociative variety.

**Proof** It is well-known that the rectangular variety of any type is solid, and is the smallest non-trivial solid variety of its type. Moreover it is straightforward to show that the $V_n$ identities are consequences of the two identities $Id$ and $D$ which define $RA$, so that $RA \subseteq V_n$. Finally, any non-trivial $Mon$- or short-solid variety $V$ must be $M$-solid for the monoid consisting of the projections and $\sigma_{id}$, which is equivalent to $RA \subseteq V$. □

The following theorem generalizes the Denecke-Koppitz theorem, Theorem 2.5, for $Mon$-solid semigroup varieties of type $(2)$.

**Lemma 3.3** Let $n \geq 3$. A type $(n)$ superassociative variety $V$ is $Mon$-solid iff $V$ is perm-solid and $RA \leq V \leq V_n$. 

V. BUDD, K. DENECKE AND S. L. WISMATH

135
Short-Solid Superassociative Type (n) Varieties

Proof Let \( V \) be \( \text{Mon} \)-solid and superassociative. Since the projections and the permutations are all in \( \text{Mon} \), the identities of \( V \) must be closed under projections, which means that \( RA \leq V \), and \( V \) must also be perm-solid. Finally, we have already seen that \( V \) must be a subvariety of \( V_n \).

Conversely, let \( V \) be perm-solid with \( RA \leq V \leq V_n \). We will show that for any \( \sigma \in \text{Mon} \) and any identity \( u \approx v \) of \( V \), we have \( \hat{\sigma}[u] \approx \hat{\sigma}[v] \) also an identity of \( V \). From \( RA \leq V \) we know that this claim is true for any \( \sigma \in \text{Pr} \). Since \( V \) is also perm-solid, it is true for \( \sigma \in \text{Perm} \) as well. Finally, application of any special idempotent element to any identity of \( V \) always gives a consequence of the normalized idempotent law which holds in \( V \). Thus \( V \) is \( \text{Mon} \)-solid.

Lemma 3.4 Let \( n \geq 3 \). The normalization of any \( \text{Mon} \)-solid type \((n)\) superassociative variety is also \( \text{Mon} \)-solid.

Proof Let \( V \) be \( \text{Mon} \)-solid. Then by the previous theorem, we know that \( V \) is perm-solid and \( RA \leq V \leq V_n \). Since \( V_n \) is normal, its join with the zero variety \( C \) is itself, and we get \( RA \leq V \vee C \leq V_n \). Since \( V \) and \( C \) are perm-solid, so is \( V \vee C \). Therefore \( V \vee C \), which is the normalization of \( V \), is also \( \text{Mon} \)-solid.

Example 3.5 The variety \( \text{Mod}(D) \) is a solid superassociative variety. We have \( \text{Mod}(D) = RA \vee C \), the normalization of \( RA \).

Lemma 3.6 The variety \( \text{Mod}(SA_1^n, \ldots, SA_n^n, NId, M) \) is the largest \( \text{Mon} \)-solid superassociative medial variety. Moreover, any solid superassociative medial variety is contained in \( V \).

Proof Since this variety is the intersection of \( V_n \) with the medial variety \( \text{Med} \), the claim follows from the previous proof for \( V_n \) and the fact that the medial variety \( \text{Med} \) is solid.

Our monoid \( \text{Mon} \) contains all the projection hypersubstitutions \( \sigma_{x_i} \). If we want to exclude these, we can consider the monoid \( \text{Mon} - \text{Pr} \), which we shall call \( \text{PreMon} \). Then we have the following result, analogous to what happens in type (2).

Lemma 3.7 Let \( V = \text{Mod}(SA_1^n, \ldots, SA_3^n, M, f(x_1, \ldots, x_1) \approx f(x_2, \ldots, x_2)) \). Then \( V \) is the largest \( \text{PreMon} \)-solid but not \( \text{Mon} \)-solid superassociative medial variety.

Proof It is clear that this variety is not solid or \( \text{Mon} \)-solid, since it satisfies an identity which does not hold in \( RA \). To prove the claim we have to prove that application of any of the non-projection hypersubstitutions in \( \text{Mon} \) to any of the basis identities results in an identity of \( V \). For the \( n \) superassociative laws and \( NId \), this has already been done, and we considered \( M \) in the previous
proof. It is easy to check that our new identity $f(x_1, \ldots, x_1) \approx f(x_2, \ldots, x_2)$ also works.

Then we also have to verify that $V$ is the largest such variety. Let $U$ be any superassociative, medial variety which is $PreMon$-solid but not $Mon$-solid. Then there is an identity $u \approx v$ satisfied by this variety for which some projection hypersubstitution $\sigma_{x_j}$ gives us an identity $x_i \approx x_k$ for some indices $i \neq k$. Applying the hypersubstitution using $f(x_j, \ldots, x_j)$ to $u \approx v$ and reducing using the normalized idempotent law gives us $f(x_i, \ldots, x_i) \approx f(x_j, \ldots, x_j)$, as required.

We can prove similar results using the idempotent law $Idem$ instead of the Medial law, since the variety $Idem$ defined by the idempotent law is also solid.

4 A Reduction in Testing

Before we extend our results on $Mon$-solid superassociative varieties to short-solid, we consider a method to reduce the amount of testing necessary. We know that to show that a variety $V = Mod(\Sigma)$ is $M$-solid, for some monoid $M$, it suffices to show that for each identity $u \approx v$ in $\Sigma$ and each $\sigma \in M$, we have $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ an identity of $V$. For $V = Mod(\Sigma)$ and $\sigma$ any hypersubstitution, we will say that $V$ is $\sigma$-closed if for any identity $u \approx v$ in $\Sigma$, we have $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ an identity of $V$. We will show in this section that for some monoids $M$, including the monoid $Short$, it is not necessary to test every $\sigma$ in $M$; that is, we can find a subset $S$ of $M$ such that $V$ is $M$-solid iff for every $u \approx v$ in $\Sigma$ and every $\sigma \in S$, we have $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ an identity of $V$. Such reductions in testing were considered in [7], where such sets $S$ were called $M$-solidity testing systems. The reduction we give here is a new one, based on the so-called Green’s relations on any monoid.

The Green’s relations are relations defined on any monoid $M$. Two elements $\sigma$ and $\rho$ in a monoid $(M, \circ)$ are said to be $R$-related to each other if there exist elements $\lambda$ and $\delta$ in $M$ such that

$$\sigma \circ \lambda = \rho \quad \text{and} \quad \rho \circ \delta = \sigma.$$  

The relation $L$ is defined dually, using left multiplication. The relation $D$ is defined as $R \circ L$. Two elements $\sigma$ and $\rho$ are said to be $J$-related if there exist elements $\lambda$, $\delta$, $\kappa$ and $\iota$ such that

$$\lambda \circ \sigma \circ \delta = \rho \quad \text{and} \quad \kappa \circ \rho \circ \iota = \sigma.$$  

These four relations are all equivalence relations on the monoid, although not usually congruences. For more information on these relations, we refer the reader to [8].

The Green’s relations have been investigated for the monoid of hypersubstitutions, for all elements of $Hyp(2)$ in [5] and for projections, permutations.
and idempotents in $Hyp(n)$ in [10]. We will summarize these results here, then extend them to include the Green’s relations classes for all the elements of our monoid Short. We also show how these relations, particularly $R$ and $L$, can be used to shorten the task of testing whether a given variety is $M$-solid, for certain monoids $M$.

We need the following notation. We let $J = \{1, 2, \ldots, n\}$. For any permutation $\pi$ on the set $J$, we denote by $\sigma_\pi$ the corresponding hypersubstitution taking $f$ to $f(x_{\pi(1)}, \ldots, x_{\pi(n)})$. For any $n$-ary term $t$, we let $C_\pi(t)$ be the term obtained from $t$ by replacing each variable $x_i$ in $t$ by the variable $x_{\pi(i)}$. We also define $\pi[t]$ inductively, by $\pi[x_i] := x_{\pi(i)}$ for any variable $x_i$ and $\pi[f(u_1, \ldots, u_n)] := f(\pi[u_{\pi(1)}], \ldots, \pi[u_{\pi(n)}])$.

**Lemma 4.1** ([10]) Let $\sigma_s$ and $\sigma_t$ be elements of $Hyp(n)$.  
(i) Then $\sigma_s R \sigma_t$ iff there is a permutation $\pi$ on $J$ such that $\sigma_s \circ_h \sigma_\pi = \sigma_t$ and $\sigma_t \circ_h \sigma_\pi^{-1} = \sigma_s$; and in this case, $\sigma_t = \sigma_{C_\pi(s)}$.

(ii) If $\sigma_s$ is an idempotent in $Hyp(n)$, then $\sigma_s L \sigma_t$ iff there exist an idempotent $\sigma_w$ with the same content as $\sigma_s$ and a permutation $\pi$ on $J$ such that $\sigma_w \circ_h \sigma_\pi = \sigma_t$ and $\sigma_t \circ_h \sigma_w = \sigma_t$; and in this case, $\sigma_w = \sigma_\pi(t)$.

In particular, two elements are $R$-related iff each of them can be written as the other right-multiplied by a permutation. The description for $L$ is a bit more complicated. But for any idempotents $\sigma$ and $\rho$, these elements are $L$-related to each other iff we have $\sigma \circ_h \rho = \sigma$ and $\rho \circ_h \sigma = \rho$. This means that any element which is $L$-related to an idempotent is related by left multiplication by either a permutation or an idempotent. This motivates the following definition: we will say that two elements $\sigma$ and $\rho$ are $L^P$-related if there exists a permutation $\sigma_\pi$ such that $\sigma_\pi \circ_h \sigma = \rho$ and $\sigma_\pi^{-1} \circ_h \rho = \sigma$. Clearly $L^P$ is an equivalence relation and is contained in $L$.

**Lemma 4.2** Let $V = Mod(\Sigma)$ be a perm-solid variety. Let $\sigma L^P \rho$. Then $V$ is $\sigma$-closed iff it is $\rho$-closed.

**Proof** Since $\sigma L^P \rho$, there is a permutation $\sigma_\pi$ such that $\sigma_\pi \circ_h \sigma = \rho$ and $\sigma_\pi^{-1} \circ_h \rho = \sigma$. Now let $u \approx v$ be any identity in $\Sigma$. To test whether $\hat{\rho}[u] \approx \hat{\rho}[v]$ is an identity of $V$, we ask whether $\hat{\sigma}_\pi[\hat{\sigma}[u]] \approx \hat{\sigma}_\pi[\hat{\sigma}[v]]$ is an identity of $V$. If $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ is an identity of $V$, then the fact that $V$ is perm-solid and $\sigma_\pi$ is a permutation means that $\hat{\sigma}_\pi[\hat{\sigma}[u]] \approx \hat{\sigma}_\pi[\hat{\sigma}[v]]$ and hence $\hat{\rho}[u] \approx \hat{\rho}[v]$ is an identity of $V$. Thus if $V$ is $\sigma$-closed it is also $\rho$-closed. A dual argument using $\sigma_\pi^{-1} \circ_h \rho = \sigma$ gives us the converse as well. \(\square\)

A similar claim, but weakened slightly to deal with multiplication on the right instead of the left by a permutation, holds for $R$-related elements.

**Lemma 4.3** Let $V = Mod(\Sigma)$ be a perm-solid variety. Let $\sigma$ be an idempotent in $Hyp(n)$ and let $\sigma R \rho$. If $V$ is $\sigma$-closed, then $V$ is $\rho$-closed.
Proof  Let V be σ-closed. Since σRρ, there exists a permutation σπ such that σ ◦ h ◦ σπ = ρ. Let u ≃ v be any identity from Σ; we want to show that \( \tilde{\rho}_h[u] \approx \tilde{\rho}_h[v] \) is an identity of V. This is the same as \( \tilde{\sigma}[\tilde{\sigma}_\pi[u]] \approx \tilde{\sigma}[\tilde{\sigma}_\pi[v]] \). By perm-solidity of V, we know that \( \tilde{\sigma}[\tilde{\sigma}_\pi[u]] \approx \tilde{\sigma}[\tilde{\sigma}_\pi[v]] \) is an identity of V. Moreover, since σ is an idempotent, the monoid it generates is just \( N = \{ \sigma_{id}, \sigma \} \). So closure of V under σ, along with automatic closure of V under σid, means that V is N-solid. Therefore, application of σ to any identity of V gives an identity of V. In particular, we have \( \tilde{\sigma}[\tilde{\sigma}_\pi[u]] \approx \tilde{\sigma}[\tilde{\sigma}_\pi[v]] \) an identity of V. This makes V ρ-closed, as required. \( \square \)

To use this result in testing for Short-solidity in the next section, we need the following information about \( R \) and \( L^\rho \) on elements of Short.

Lemma 4.4 In the monoid Short \( \leq H_{yp}(n) \), any element of the set Oth is \( L^\rho \)-related to an element of ShE and \( R \)-related to an element of ShE.

Proof  Let \( \sigma_t \) be an element of the set Oth. This means that the term \( t \) has content size \( c \) with \( 2 \leq c \leq n - 1 \), since elements of content size one or \( n \) belong to Pr ∪ SpE or Perm, respectively. We write \( t = f(t_1, \ldots, t_n) \), where each \( t_j \) is a variable from the set \( c(t) \). We define a permutation \( \pi \) on the set \( J = \{1, 2, \ldots, n\} \), as follows. For each index \( i \) such that \( x_i \) occurs in \( t \), we set \( \pi(i) \) to be the first index \( j \) for which \( x_i \) occurs in position \( j \) in \( t \). This defines a partial function on \( J \) which is injective on its domain, and so can be extended to a bijection on \( J \).

Now by Lemma 4.1, it will be enough to show that both the compositions \( \sigma_\pi \circ h \circ \sigma_t \) and \( \sigma_t \circ h \circ \sigma_\pi \) are idempotents. From Lemma 4.1, we know that \( \sigma_\pi \circ h \circ \sigma_t = \sigma_{\pi[t]} \), and by the inductive definition of \( \pi[t] \) we have \( \pi[t] = f(t_{\pi(1)}, \ldots, t_{\pi(n)}) \). Any variable \( x_i \) in this latter term is a variable in \( t \), and by construction \( \pi(t) \) is an index \( j \) for which \( t_j = x_i \), so in \( \pi[t] \) the \( i \)-th entry is the variable \( x_i \). This makes \( \sigma_{\pi[t]} \) an idempotent, by Lemma 2.6, and so an element of ShE.

From Lemma 4.1, we also have \( \sigma_t \circ h \circ \sigma_\pi = \sigma_{C_\pi[t]} \). The term \( C_\pi[t] \) is formed from \( t \) by replacing each variable \( x_i \) in \( t \) by the variable \( x_{\pi(i)} \). But \( \pi(i) \) is the index \( j \) of the position where the variable \( x_i \) first occurs in \( t \). Thus for each variable \( x_i \) which occurs in \( C_\pi[t] \), that variable occurs in the \( i \)-th position. Again, this makes \( \sigma_{C_\pi[t]} \) an idempotent element, and hence in ShE. \( \square \)

Corollary 4.5 Let \( V = Mod(\Sigma) \) be a rectangular and perm-solid superassociative type \( (n) \) variety. Then to test if V is short-solid, it suffices to show that for any identity \( u \approx v \) in \( \Sigma \), the identity \( \tilde{\sigma}[u] \approx \tilde{\sigma}[v] \) holds in V for every \( \sigma \) in SpE ∪ ShE. We can also replace ShE in the previous sentence with S, for any set S of representatives, one from each R-class on the set ShE.

This Corollary represents a significant reduction in testing for Short-solidity. For \( n = 3 \), for instance, the 18 elements of the set Oth need not be tested, reducing our testing to 12 hypersubstitutions. For \( n = 4 \), a total of 192 of the
260 Short hypersubstitutions are in $Oth$, again giving a significant reduction. It may also be possible to use this idea on varieties other than perm-solid ones, using the following generalization of our technique.

**Definition 4.6** Let $M$ be a monoid of hypersubstitutions of a fixed type $\tau$. Define a relation $\sim_M$ on $Hyp(\tau)$, by setting $\sigma_1 \sim_M \sigma_2$ iff for any $M$-solid variety $V$, $V$ is $\sigma_1$-closed iff $V$ is $\sigma_2$-closed.

This is clearly an equivalence relation on $Hyp(\tau)$; we do not know if it is a congruence.

We conclude this section by describing the equivalence classes of the Green's relations $R$, $L$, $D$ and $J$ for all elements from our monoid $Short$, as well as some related elements. First, from [10], we know that for any $\sigma \in Pr$, the $L$-class of $\sigma$ is just $\{\sigma\}$, while the $R$-, $D$- and $J$-classes are all equal to $Pr$ itself. For any $\sigma$ in $Perm$, all four classes for $\sigma$ are equal to $Perm$. It follows from Lemma 4.1 above that any special idempotent $\sigma_f(x_1, \ldots, x_i)$ is $L$-related only to itself, but $R$-related to all elements of $SpE$. Notice that all of these classes are the same, whether we are talking about the Green's relations only on the monoid $Short$, or on the whole monoid $Hyp(n)$; elements of $Mon$ are only related to other elements of $Mon$. The same is true for $R$-classes of elements of $Short$, but no longer true for $L$-classes of elements in $ShE \cup Oth$.

**Lemma 4.7** Let $\sigma_t \in ShE \cup Oth$. Then

(i) the $R$-class of $\sigma_t$ in both $Short$ and $Hyp(n)$, consists of those $\sigma_s$ for which $s$ has the same size content and the same variable pattern as $t$; that is, $s = c_\pi(t)$ for some permutation $\pi$ of $J$.

(ii) the $L$-class of $\sigma_t$ for the monoid $Short$ consists of those $\sigma_s$ in $ShE \cup Oth$ such that $s$ and $t$ have the same content.

(iii) the $D$-class and the $J$-class of $\sigma_t$ consist of all $\sigma_s \in ShE \cup Oth$ such that $s$ and $t$ have the same size content.

(iv) On the monoid $Short$, the relations $D$ and $J$ are equal.

**Proof** (i) This follows from Lemma 4.1 above.

(ii) It was shown in [10] that any two elements which are $L$-related must have the same content. Conversely, we know from the proof of Lemma 4.4 that any element of $Oth$ is $L$-related to an idempotent in $ShE$ with the same content. Finally, any two idempotents $\sigma_p$ and $\sigma_q$ with the same content are $L$-related, since $\sigma_p \circ h \sigma_q = \sigma_p$ and $\sigma_q \circ h \sigma_p = \sigma_q$.

(iii) The claim for $D$ follows from the definition of $D = R \circ h L$ and parts (i) and (ii). The claim for $J$ follows from Lemma 4.4 and the fact that any idempotent element $\sigma$ has equal $D$- and $J$-class (see [10]).

(iv) This follows from the previous parts.

We can extend our results to some elements beyond the monoid $Short$. As remarked above, the relation $R$ is fully known on all of $Hyp(n)$ (see Lemma...
4.1) and elements of Short cannot be $\mathcal{R}$-related to elements outside of Short. But for the relation $\mathcal{L}$ we have the following argument. Let $t = f(t_1, \ldots, t_n)$ be a term with content size $k$, for some $1 \leq k \leq n$, for which each variable $x_j$ in the content of $t$ occurs as one of the entries $t_p$, for $1 \leq p \leq n$. Then there is a permutation $\pi$ which permutes the entries $t_i$ in such a way that each variable $x_j$ occuring at the top level is moved into position $j$. This means exactly that $\sigma_t \circ \sigma_\pi$ is an idempotent, with the same content as $t$. Thus each such $\sigma_t$ is $\mathcal{L}^P$-related to an idempotent element with the same content. We can use this fact, along with Lemma 4.2, to make new reductions in testing. Moreover, each such idempotent is also $\mathcal{L}$-related to any other idempotent with the same content, including an idempotent in the set $\text{ShE}$. This means that any such hypersubstitution $\sigma_t$ is $\mathcal{L}$-related to an element of Short.

5 Short-Solid Superassociative Varieties

In this section we want to extend our results on Mon-solid varieties, from Section 3, to the short-solid case. We notice that for type (2), in fact $\text{Mon} = \text{Short}$, so that all the results from Section 3 remain true if we replace Mon by Short in this case.

For $n = 3$, we know that Mon is a proper subset of Short. However, it turns out that in this case, our largest Mon-solid variety $V_3$ is also short-solid. This can be verified directly by checking that $V_3$ satisfies $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ for every identity $a \approx v$ in our basis $\Sigma$ for $V_3$ and every $\sigma$ in Short. (In fact, it suffices to test only those $\sigma$ in $\text{Mon} \cup \text{ShE}$, as we saw in Section 4.) This proves the following result.

Lemma 5.1 The largest type (3) short-solid variety $V_3$ is also the largest Mon-solid variety of type (3).

However, it is not the case that Mon-solidity and Short-solidity are equivalent, even for type (3). As an example, we consider the subvariety $V$ of $V_3$ defined by the four identities of $V_3$ plus three additional identities:

\[
\begin{align*}
    f(f(x, y, z), a, b) &\approx f(f(x, z, y), a, b), & f(a, f(y, x, z), b) &\approx f(a, f(z, x, y), b), \\
    f(a, b, f(y, z, x)) &\approx f(a, b, f(z, y, x)).
\end{align*}
\]

Then it is straightforward to check that this variety is Mon-solid but not short-solid; in particular, the hypersubstitution using $f(x_1, x_1, x_2)$ gives an identity which is not regular (uses different variables on each side), whereas all identities in $V$ must be regular.

Nor is it the case that the variety $V_n$ is Short-solid, for $n \geq 4$. For $\sigma \in \text{ShE}$, the identities obtained by applying $\hat{\sigma}$ to $\text{SA}_n^p$ need not be identities of $V_n$. For example, for $n = 4$, applying the hypersubstitution $\sigma_t$ for $t = f(x_1, x_2, x_3, x_1)$ gives the identity
set of identities holds in \( W \). So let the relation be a set of representatives of the equivalence classes of elements of \( \text{ShE} \) under the relation \( \mathcal{R} \) on \( \text{Short} \) (or \( \text{Hyp}(n) \), since these are the same). Let \( \Sigma \) be the set of identities \( \{SA_1^n, \ldots, SA_n^n\} \). Let \( S(\Sigma) \) be the set

\[
\{\hat{\sigma}[u] \approx \hat{\sigma}[v] : u \approx v \in \Sigma \text{ and } \sigma \in S\}.
\]

Let \( PS(\Sigma) \) be the set

\[
\{\hat{\rho}[s] \approx \hat{\rho}[t] : s \approx t \in S(\Sigma) \text{ and } \rho \in \text{Perm}\}.
\]

Finally, let \( \Sigma^* = \{\text{Id} \} \cup \Sigma \cup S(\Sigma) \cup PS(\Sigma) \).

**Theorem 5.2** Let \( W_n = \text{Mod}(\Sigma^*) \). Then \( W_n \) is the largest short-solid superassociative variety of type \((n)\).

**Proof** We note that by definition we have \( RA \leq W_n \leq V_n \). We will show first that \( W_n \) is perm-solid, and then by Lemma 3.3 it will be at least \( \text{Mon} \)-solid. So let \( u \approx v \) be any identity in \( \Sigma^* \) and let \( \sigma \) be any hypersubstitution from \( \text{Perm} \). Clearly if \( u \approx v \) is the normalized idempotent identity \( \text{Id} \), then \( \hat{\sigma}[u] \approx \hat{\sigma}[v] \) is a consequence of \( \text{Id} \) and holds in \( W_n \). If \( u \approx v \) is in \( \Sigma \), we know from Lemma 3.1 that \( \hat{\sigma}[u] \approx \hat{\sigma}[v] \) holds in \( V_n \) and hence in \( W_n \). If \( u \approx v \) is in \( S(\Sigma) \), then \( \hat{\sigma}[u] \approx \hat{\sigma}[v] \) is an identity in \( PS(\Sigma) \subseteq \Sigma^* \), and hence holds in \( W_n \). Finally, if \( u \approx v \) is in \( PS(\Sigma) \), then it has the form \( \hat{\rho}[s] \approx \hat{\rho}[t] \) for some \( \rho \) in \( \text{Perm} \) and some \( s \approx t \) in \( S(\Sigma) \). But then \( \hat{\sigma}[u] \approx \hat{\sigma}[v] \) has the form \( (\sigma \circ_h \rho)^{-1}[s] \approx (\sigma \circ_h \rho)^{-1}[t] \). But \( \text{Perm} \) is a monoid, so we have \( \sigma \circ_h \rho \) in \( \text{Perm} \) and \( s \approx t \) in \( S(\Sigma) \), and our identity is also in \( PS(\Sigma) \). We have verified that \( W_n \) is \( \text{Perm} \)-solid, and thus by Lemma 3.3 it is \( \text{Mon} \)-solid.

To complete our proof, we have to show that for any \( u \approx v \) in \( \Sigma^* \) and any \( \sigma \in \text{Short} \cup \text{Oth} \), the identity \( \hat{\sigma}[u] \approx \hat{\sigma}[v] \) also holds in \( W_n \). In fact, by Corollary 4.5, we need only consider \( \sigma \in S \). We consider four cases, depending on the four possibilities for \( u \approx v \) in \( \Sigma^* \):

1. Case 1: If \( u \approx v \) is the identity \( \text{Id} \), the claim is true.
2. Case 2: If \( u \approx v \) is in \( \Sigma \), the identity \( \hat{\sigma}[u] \approx \hat{\sigma}[v] \) is in \( S(\Sigma) \) and by definition holds in \( W_n \).
3. Case 3: If \( u \approx v \) is in \( S(\Sigma) \) then the identity \( u \approx v \) has the form \( \hat{\alpha}[s] \approx \hat{\alpha}[t] \) for some identity \( s \approx t \) in \( \Sigma \) and some \( \alpha \) in \( S \). Thus \( \hat{\sigma}[u] = \hat{\sigma}[\hat{\alpha}[s]] = (\sigma \circ_h \alpha)^{-1}[s] \), and similarly for \( \hat{\sigma}[v] \). If we consider the hypersubstitution \( \sigma \circ_h \alpha \), which for convenience we shall call \( \gamma \), and the corresponding identity \( \hat{\gamma}[s] \approx \hat{\gamma}[t] \). If \( \gamma \) is in \( \text{Mon} \), then the \( \text{Mon} \)-solidity of \( W_n \) combined with the fact that \( s \approx t \) is in
\( \Sigma \) means that our identity holds in \( W_n \). Otherwise, if \( \gamma \) is in \( ShE \cup Oth \), we know that \( \gamma \) is \( R \)-equivalent to some element \( \beta \) in \( S \). Since the variety \( W_n \) is \( Perm \)-solid, we can apply Lemma 4.3 to say that \( \hat{\gamma}[s] \approx \hat{\gamma}[t] \) holds in \( W_n \) if \( \beta[s] \approx \beta[t] \) does. But this last identity does hold in \( W_n \), since it is one of the identities in \( S(\Sigma) \).

Case 4: If \( u \approx v \) is in \( PS(\Sigma) \) then the identity \( u \approx v \) has the form \( \hat{\alpha}[\hat{\kappa}[s]] \approx \hat{\alpha}[\hat{\kappa}[t]] \) for some identity \( s \approx t \) in \( \Sigma \) and some \( \alpha \) in \( Perm \) and \( \kappa \) in \( S \). Thus \( \hat{\sigma}[u] = \hat{\sigma}[\hat{\alpha}[\hat{\kappa}[s]]] = (\sigma \circ_h \alpha \circ_h \kappa)[s] \), and similarly for \( \hat{\sigma}[v] \). So we consider the hypersubstitution \( \sigma \circ_h \alpha \circ_h \kappa \), which for convenience we shall call \( \delta \), and the corresponding identity \( \delta[s] \approx \delta[t] \). If \( \delta \) is in \( Mon \), we are again done by \( Mon \)-solidity of \( W_n \). If \( \delta \) is not in \( Mon \), then as in Case 3 it is \( R \)-related to some element \( \beta \) in \( S \), and we argue exactly as in case 3.

The set \( \Sigma^* \) of identities used as a basis for our variety \( W_n \) is finite, but quite large. In fact it is possible to find a smaller basis, as we illustrate with \( n = 3 \) and \( n = 4 \). We consider first the set \( S(\Sigma) \). It turns out that many of the identities in this set are already in \( \Sigma \), and can be omitted. We have the following observations, which are straightforward to check.

Lemma 5.3 Let \( 1 \leq i \leq n \), and let \( \sigma_i \) be in \( S \). We write the identity \( SA^n_i \) as \( u_i \approx v_i \).

i) If the variable \( x_i \) does not occur in the content of the term \( t \), then \( \hat{\sigma}_i[u_i] \approx \hat{\sigma}_i[v_i] \) is a consequence of the identity \( NId \).

ii) If the variable \( x_i \) occurs exactly once in the term \( t \), then since \( \sigma_i \) is an idempotent the \( i \)-th entry in \( t \) must be \( x_i \). In this case, \( \hat{\sigma}_i[u_i] \approx \hat{\sigma}_i[v_i] \) is a consequence (by identification of variables) of \( SA^n_i \) itself, and need not be included in our set \( S(\Sigma) \).

iii) If \( x_i \) occurs \( n-1 \) times in \( t \), with one other variable \( x_j \) occurring exactly once in position \( j \), then \( \hat{\sigma}_i[u_i] \approx \hat{\sigma}_i[v_i] \) is a consequence (by identification of variables) of \( SA^n_j \).

Example 5.4 : Let us apply these observations to the case \( n = 3 \). It is easy to show that the 18 elements of \( Oth \) (see Example 2.5) reduce to three \( R \)-classes. We choose one \( ShE \) element from each class, to set

\[ S = \{ \sigma_f(x_1,x_2,x_1), \sigma_f(x_1,x_2,x_2), \sigma_f(x_1,x_1,x_1) \} \]

In this case Lemma 5.3 show us that all 9 identities in \( S(\Sigma) \) are already consequences of \( \Sigma \), and need not be included. This means that \( PS(\Sigma) \) is also unnecessary. This reduces our basis \( \Sigma^* \) from Theorem 5.2 to the set \( \Sigma \), and bears out our remark at the beginning of this section that for \( n = 3 \), the largest short-solid superassociative variety is in fact the same as the largest \( Mon \)-solid superassociative variety, \( V_3 \).

Example 5.5 For \( n = 4 \), we have four identities in \( \Sigma \) and 13 \( R \) classes on \( ShE \cup Oth \), giving a set \( S \) of size 52. We use the following terms \( t \) for the representatives \( \sigma_i \) for \( S \):
Short-Solid Superassociative Type (n) Varieties

\[ f(x_1, x_2, x_3, x_1), \quad f(x_1, x_2, x_3, x_2), \quad f(x_1, x_2, x_3, x_3), \quad f(x_1, x_2, x_3, x_4), \]
\[ f(x_1, x_2, x_3, x_1), \quad f(x_1, x_2, x_3, x_2), \quad f(x_1, x_2, x_3, x_3), \quad f(x_1, x_2, x_3, x_4). \]

By Lemma 5.3 we can eliminate 40 of our 52 identities in \( S(\Sigma) \). We can also eliminate 6 more, by the following fact: if \( x_i \) occurs between 2 and \( n - 2 \) times in \( t \), and we replace one other variable \( x_j \) by another variable \( x_k \) which also occurs, to make a new term \( t \), then the identity \( \hat{\sigma}_i[u_i] \approx \hat{\sigma}_i[v_i] \) is a consequence of \( \hat{\sigma}_i[u_i] \approx \hat{\sigma}_i[v_i] \). For instance, the identity obtained from applying \( f(x_1, x_2, x_2, x_1) \) to the identity \( SA_1^3 \) is a consequence of that obtained from \( f(x_1, x_2, x_3, x_1) \). This leaves us with a reduced version of \( S \) containing only 6 new identities. We can then verify that the set \( PS(\Sigma) \) contains no new identities, and need not be used. So for our basis we can use \( NId \cup \Sigma \cup T \), where \( T \) contains the following 6 identities:

\[ \hat{\sigma}_f(x_1, x_2, x_3, x_1)[u_1] \approx \hat{\sigma}_f(x_1, x_2, x_3, x_2)[v_1], \quad \hat{\sigma}_f(x_1, x_2, x_3, x_3)[u_1] \approx \hat{\sigma}_f(x_1, x_2, x_3, x_4)[v_1]. \]

This gives a basis of size 11 for the variety \( W_4 \) in this case.

As a corollary of our previous results, we obtain the following characterization of all short-solid superassociative varieties, analogous to Theorem 2.5 and Lemma 3.3.

**Theorem 5.6** Let \( S \) be a set of representatives of the \( R \)-classes on the set \( ShE \). Let \( V \) be a type \((n)\) superassociative variety. Then \( V \) is short-solid if and only if \( V \) satisfies the following conditions:

1. \( RA \leq V \leq W_n \),
2. \( V \) is perm-solid, and
3. the identities of \( V \) are closed under every \( \sigma \) in \( S \).

Results analogous to Lemmas 3.4, 3.6 and 3.7 can also be proved for the superassociative case, using the fact that the varieties \( Med \) and \( Idem \) are solid.

**References**


