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ON $(\sigma, \tau)-$ LEFT JORDAN IDEALS AND GENERALIZED DERIVIATIONS

E. Güven

Dept. of Mathematics, Faculty of Art and Sciences
Kocaeli University, Kocaeli - TURKEY
e-mail: evrim@kocaeli.edu.tr

Abstract

Let R be a prime ring with characteristic not 2 and $\sigma, \tau, \alpha, \beta, \lambda, \mu, \gamma$ automorphisms of R. Let $h : R \longrightarrow R$ be a nonzero *right (resp. left)-generalized* $(\alpha, \beta)-$ derivation associated with $(\alpha, \beta)-$ derivation d (resp. d_1). Let W, V be nonzero left $(\sigma, \tau)-$Jordan ideals of R. The main object in this paper is to study the situations. (1) $h(W) = 0$, (2) $[b, W]_{\lambda,\mu} = 0$ or $[W, b]_{\lambda,\mu} = 0$, (3) $(b, W)_{\lambda,\mu} = 0$ or $(W, b)_{\lambda,\mu} = 0$, (4) $b[W, a]_{\lambda,\mu} = 0$ or $[W, a]_{\lambda,\mu} b = 0$ (5) $b(W, a)_{\lambda,\mu} = 0$ or $(W, a)_{\lambda,\mu} b = 0$, (6) $bW \subset C_{\lambda,\mu}(V)$ or $Wb \subset C_{\lambda,\mu}(V)$. (7) $(h(R), b)_{\lambda,\mu} a = 0$ or $a(h(R), b)_{\lambda,\mu} = 0$.

1 Introduction

Let R be a ring and σ, τ two mappings of R. For each $r, s \in R$ we set $[r, s] = rs-sr$, $(r, s) = rs+sr$, $[r, s]_{\sigma,\tau} = r\sigma(s)-\tau(s)r$ and $(r, s)_{\sigma,\tau} = r\sigma(s)+\tau(s)r$. Let U be an additive subgroup of R. If $(U, R) \subset U$ then U is called a Jordan ideal of R. The definition of $(\sigma, \tau)-$Jordan ideal of R is introduced in [6] as follows: (i) U is called a right $(\sigma, \tau)-$Jordan ideal of R if $(U, R)_{\sigma,\tau} \subset U$, (ii) U is called a left $(\sigma, \tau)-$Jordan ideal if $(R, U)_{\sigma,\tau} \subset U$. (iii) U is called a $(\sigma, \tau)-$Jordan ideal if U is both right and left $(\sigma, \tau)-$Jordan ideal of R. Every Jordan ideal of R is a $(1, 1)-$Jordan ideal of R, where $1 : R \to R$ is an identity map. The following example is given in [6]. If $R = \{ \left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) \upharpoonright x$ and y are integers$\}$, $U = \{ \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) \upharpoonright x$ is integer$\}$, $\sigma\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\tau\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & -y \\ 0 & 0 \end{smallmatrix}\right)$ then U is $(\sigma, \tau)-$right Jordan ideal but not a Jordan ideal of R.

A derivation d is an additive mapping on R which satisfies $d(rs) = d(r)s + rd(s), \forall r, s \in R$. The notion of generalized derivation was introduced by

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Brešar [2] as follows. An additive mapping $h : R \rightarrow R$ will be called a generalized derivation if there exists a derivation d of R such that $h(xy) = h(x)y + xd(y), \forall x, y \in R$.

An additive mapping $d : R \rightarrow R$ is said to be a $(\sigma,\tau)-$derivation if $d(rs) = d(r)\sigma(s) + \tau(r)d(s)$ for all $r, s \in R$. Every derivation $d : R \rightarrow R$ is a $(1,1)-$derivation. Chang [3] gave the following definition. Let R be a ring, σ and τ automorphisms of R and $d : R \rightarrow R$ a $(\sigma,\tau)-$derivation. An additive mapping $h : R \rightarrow R$ is said to be a right generalized $(\sigma,\tau)-$derivation of R associated with d if $h(xy) = h(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$ and h is said to be a left generalized $(\sigma,\tau)-$derivation of R associated with d if $h(xy) = d(x)\sigma(y) + \tau(x)h(y)$, for all $x, y \in R$. h is said to be a generalized $(\sigma,\tau)-$derivation of R associated with d if it is both a left and right generalized $(\sigma,\tau)-$derivation of R associated with d. Besides every $(\sigma,\tau)-$derivation $d : R \rightarrow R$ is a generalized $(\sigma,\tau)-$derivation associated with d and every derivation $d : R \rightarrow R$ is a generalized $(1,1)-$derivation associated with d. A generalized $(1,1)-$derivation is simply called a generalized derivation. It is clear that the definition of generalized derivation which is given in [2] is a right generalized derivation associated with derivation d according to Chang's definition.

The mapping $h(r) = (a,r)_{\sigma,\tau}, \forall r \in R$ is a left-generalized $(\sigma,\tau)-$ derivation with $(\sigma,\tau)-$derivation $d_1(r) = [a,r]_{\sigma,\tau}, \forall r \in R$ and right-generalized $(\sigma,\tau)-$derivation with $(\sigma,\tau)-$derivation $d(r) = -[a,r]_{\sigma,\tau}, \forall r \in R$. Every $(\sigma,\tau)-$derivation $d : R \rightarrow R$ is a generalized $(\sigma,\tau)-$derivation with d.

Throughout the paper, R will be a prime ring with center Z, characteristic not 2 and $\sigma, \tau, \alpha, \beta, \lambda, \mu, \gamma$ automorphisms of R. We set $C_{\sigma,\tau}(R) = \{c \in R \mid c\sigma(r) = \tau(r)c, \forall\, r \in R\}$, and shall use the following relations frequently:

$[rs,t]_{\sigma,\tau} = r[s,t]_{\sigma,\tau} + [r,\tau(t)]s = r[s,\sigma(t)] + [r,t]_{\sigma,\tau}s$

$[r,st]_{\sigma,\tau} = \tau(s)[r,t]_{\sigma,\tau} + [r,s]_{\sigma,\tau}\sigma(t)$

$(rs,t)_{\sigma,\tau} = r(s,t)_{\sigma,\tau} - [r,\tau(t)]s = r[s,\sigma(t)] + (r,t)_{\sigma,\tau}s.$

$(r,st)_{\sigma,\tau} = \tau(s)(r,t)_{\sigma,\tau} + [r,s]_{\sigma,\tau}\sigma(t) = -\tau(s)[r,t]_{\sigma,\tau} + (r,s)_{\sigma,\tau}\sigma(t)$

2 Results

Lemma 1. *[1, Lemma 3] Let U be a nonzero ideal of R and $d : R \longrightarrow R$ a $(\sigma,\tau)-$derivation. If $a \in R$ such that $ad(U) = 0$ (or $d(U)a = 0$) then $a = 0$ or $d = 0$.*

Lemma 2. *Let I be a nonzero ideal of R and $a, b \in R$.*

(i) If $b, ba \in C_{\lambda,\mu}(I)$ or $(b, ab \in C_{\lambda,\mu}(I))$ then $b = 0$ or $a \in Z$.
(ii) If $b\gamma(I,a)_{\alpha,\beta} = 0$ or $\gamma(I,a)_{\alpha,\beta}b = 0$ then $b = 0$ or $a \in Z$.

Proof. (i) If $b, ba \in C_{\lambda,\mu}(I)$ then we have, for all $x \in I$

$$0 = [ba,x]_{\lambda,\mu} = b[a,\lambda(x)] + [b,x]_{\lambda,\mu}a = b[a,\lambda(x)]$$

and so $b[a,\lambda(x)]=0,\forall x\in I$. Replacing x by $xr, r\in R$ then we get $b\lambda(I)[a,R]=0$. Since $\lambda(I)$ is a nonzero ideal of R, then we have $b=0$ or $a\in Z$.

If $b, ab\in C_{\lambda,\mu}(I)$ then considering as above and using the relation

$$0=[ab,x]_{\lambda,\mu}=a[b,x]_{\lambda,\mu}+[a,\mu(x)]b=[a,\mu(x)]b,\forall x\in I$$

we obtain the result.

(ii) If $b\gamma(I,a)_{\alpha,\beta}=0$ then, for all $x\in I, r\in R$

$$0=b\gamma(xr,a)_{\alpha,\beta}=b\gamma(x)\gamma[r,\alpha(a)]+b\gamma(x,a)_{\alpha,\beta}\gamma(r)=b\gamma(x)\gamma[r,\alpha(a)].$$

That is $b\gamma(I)\gamma[R,\alpha(a)]=0$. Since $\gamma(I)$ is a nonzero ideal, then we obtain that $b=0$ or $a\in Z$ by the last relation

If $\gamma(I,a)_{\alpha,\beta}b=0$ then, for all $x\in I, r\in R$

$$0=\gamma(rx,a)_{\alpha,\beta}b=\gamma(r)\gamma(x,a)_{\alpha,\beta}b-\gamma[r,\beta(a)]\gamma(x)b=-\gamma[r,\beta(a)]\gamma(x)b$$

which gives $\gamma[R,\beta(a)]\gamma(I)b=0$. Considering as above we get the required result. □

Corollary 1. *[7, Lemma 4] Let b and ab be in the center of a prime ring R. If b is not zero, then a is in Z, the center of R.*

Theorem 1. *Let I, J be nonzero ideals of R and $a,b\in R$. Let W be a left $(\sigma,\tau)-$Jordan ideal of R.*

(i) If $b\gamma(I,a)_{\alpha,\beta}\subset C_{\lambda,\mu}(J)$ or $\gamma(I,a)_{\alpha,\beta}b\subset C_{\lambda,\mu}(J)$ then $b=0$ or $a\in Z$.
(ii) If $b\gamma(W)\subset C_{\lambda,\mu}(J)$ or $\gamma(W)b\subset C_{\lambda,\mu}(J)$ then $b=0$ or $W\subset Z$.

Proof. (i) If $b\gamma(I,a)_{\alpha,\beta}\subset C_{\lambda,\mu}(J)$ then we have, for all $x\in I$

$$C_{\lambda,\mu}(J)\ni b\gamma(x\alpha(a),a)_{\alpha,\beta}=b\gamma(x)\gamma[\alpha(a),\alpha(a)]+b\gamma(x,a)_{\alpha,\beta}\gamma\alpha(a)=b\gamma(x,a)_{\alpha,\beta}\gamma\alpha(a).$$

Then

$$b\gamma(I,a)_{\alpha,\beta}\gamma\alpha(a)\subset C_{\lambda,\mu}(J). \tag{2.1}$$

If we use Lemma2 (i) in (2.1) then we get

$$b\gamma(I,a)_{\alpha,\beta}=0 \text{ or } a\in Z.$$

If $b\gamma(I,a)_{\alpha,\beta}=0$ then we have $b=0$ or $a\in Z$ by Lemma2 (ii).

If $\gamma(I,a)_{\alpha,\beta}b\subset C_{\lambda,\mu}(J)$ then we have, for all $x\in I$

$$C_{\lambda,\mu}(J)\ni\gamma(\beta(a)x,a)_{\alpha,\beta}b=\gamma\beta(a)\gamma(x,a)_{\alpha,\beta}b-\gamma[\beta(a),\beta(a)]\gamma(x)b=\gamma\beta(a)\gamma(x,a)_{\alpha,\beta}b.$$

That is

$$\gamma\beta(a)\gamma(I,a)_{\alpha,\beta}b\subset C_{\lambda,\mu}(J). \tag{2.2}$$

If we use Lemma2 (i) then (2.2) gives that

$$\gamma(I,a)_{\alpha,\beta}b=0 \text{ or } a\in Z.$$

If $\gamma(I,a)_{\alpha,\beta}b=0$ then using Lemma2 (ii) we obtain the required result.

(ii) If $b\gamma(W)\subset C_{\lambda,\mu}(J)$ or $\gamma(W)b\subset C_{\lambda,\mu}(J)$ then we have $b\gamma(R,W)_{\sigma,\tau}\subset C_{\lambda,\mu}(J)$ or $\gamma(R,W)_{\sigma,\tau}b\subset C_{\lambda,\mu}(J)$. Thus $b=0$ or $W\subset Z$ by (i). □

Corollary 2. *[4, Theorem 2.2] Let W be a left $(\sigma,\tau)-$Jordan ideal of R. If $aW=0$ (or $Wa=0$) and $a\in R$, then $a=0$ or $W\subset Z$.*

Lemma 3. *Let $h:R\longrightarrow R$ be a nonzero right generalized $(\alpha,\beta)-$ derivation associated with a nonzero $(\alpha,\beta)-$ derivation $d:R\longrightarrow R$ and I a nonzero ideal of R. If a is a noncentral element of R such that $h\lambda(I,a)_{\sigma,\tau}=0$ or $(h\lambda(I),a)_{\alpha\lambda\sigma,\beta\lambda\tau}=0$ then $d\lambda\sigma(a)=0$.*

Proof. If $h\lambda(I,a)_{\sigma,\tau}=0$ then we get, for all $x\in I$

$$\begin{aligned}0&=h\lambda(x\sigma(a),a)_{\sigma,\tau}=h\{\lambda(x)\lambda[\sigma(a),\sigma(a)]+\lambda(x,a)_{\sigma,\tau}\lambda\sigma(a)\}\\&=h\{\lambda(x,a)_{\sigma,\tau}\lambda\sigma(a)\}=h\lambda(x,a)_{\sigma,\tau}\alpha\lambda\sigma(a)+\beta\lambda(x,a)_{\sigma,\tau}d\lambda\sigma(a)\\&=\beta\lambda(x,a)_{\sigma,\tau}d\lambda\sigma(a).\end{aligned}$$

That is

$$\beta\lambda(I,a)_{\sigma,\tau}d\lambda\sigma(a)=0. \tag{2.3}$$

Since $a\notin Z$ then (2.3) gives that $d\lambda\sigma(a)=0$ by Lemma2 (ii) .

If $(h\lambda(I),a)_{\alpha\lambda\sigma,\beta\lambda\tau}=0$ then we have, for all $x\in I$

$$\begin{aligned}0&=(h\lambda(x\sigma(a)),a)_{\alpha\lambda\sigma,\beta\lambda\tau}=(h\lambda(x)\alpha\lambda\sigma(a)+\beta\lambda(x)d\lambda\sigma(a),a)_{\alpha\lambda\sigma,\beta\lambda\tau}\\&=h\lambda(x)[\alpha\lambda\sigma(a),\alpha\lambda\sigma(a)]+(h\lambda(x),a)_{\alpha\lambda\sigma,\beta\lambda\tau}\alpha\lambda\sigma(a)\\&+\beta\lambda(x)(d\lambda\sigma(a),a)_{\alpha\lambda\sigma,\beta\lambda\tau}-[\beta\lambda(x),\beta\lambda\tau(a)]d\lambda\sigma(a)\\&=\beta\lambda(x)(d\lambda\sigma(a),a)_{\alpha\lambda\sigma,\beta\lambda\tau}-[\beta\lambda(x),\beta\lambda\tau(a)]d\lambda\sigma(a)\end{aligned}$$

and so

$$\beta\lambda(x)(k,a)_{\alpha\lambda\sigma,\beta\lambda\tau}-[\beta\lambda(x),\beta\lambda\tau(a)]k=0,\forall x\in I \tag{2.4}$$

where $k=d\lambda\sigma(a)$. Replacing x by $rx, r\in R$ in (2.4) and using (2.4) we get, for all $r\in R, x\in I$

$$\begin{aligned}0&=\beta\lambda(r)\beta\lambda(x)(k,a)_{\alpha\lambda\sigma,\beta\lambda\tau}-\beta\lambda(r)[\beta\lambda(x),\beta\lambda\tau(a)]k-[\beta\lambda(r),\beta\lambda\tau(a)]\beta\lambda(x)k\\&=-[\beta\lambda(r),\beta\lambda\tau(a)]\beta\lambda(x)k\end{aligned}$$

so $[R,\beta\lambda\tau(a)]\beta\lambda(I)k=0$. Since $\beta\lambda(I)$ is a nonzero ideal of R and $a\notin Z$, then we obtain that $d\lambda\sigma(a)=0$ by the last relation. □

Lemma 4. *Let I be a nonzero ideal of R and $h : R \longrightarrow R$ a nonzero left-generalized $(\alpha,\beta)-$ derivation associated with a nonzero $(\alpha,\beta)-$ derivation $d_1 : R \longrightarrow R$. If a is a noncentral element of R such that $h\lambda(I,a)_{\sigma,\tau} = 0$ or $(h\lambda(I),a)_{\alpha\lambda\sigma,\beta\lambda\tau} = 0$ then $d_1\lambda\tau(a) = 0$.*

Proof. If $h\lambda(I,a)_{\sigma,\tau} = 0$ then we have, for all $x \in I$

$$0 = h\lambda(\tau(a)x,a)_{\sigma,\tau} = h\{\lambda\tau(a)\lambda(x,a)_{\sigma,\tau} - \lambda[\tau(a),\tau(a)]\lambda(x)\} = h\{\lambda\tau(a)\lambda(x,a)_{\sigma,\tau}\}$$
$$= d_1\lambda\tau(a)\alpha\lambda(x,a)_{\sigma,\tau} + \beta\lambda\tau(a)h\lambda(x,a)_{\sigma,\tau} = d_1\lambda\tau(a)\alpha\lambda(x,a)_{\sigma,\tau}.$$

That is

$$d_1\lambda\tau(a)\alpha\lambda(I,a)_{\sigma,\tau} = 0. \tag{2.5}$$

Since a is noncentral, using 2 (ii) and (2.5) we get $d_1\lambda\tau(a) = 0$.

Similarly, if $(h\lambda(I),a)_{\alpha\lambda\sigma,\beta\lambda\tau} = 0$ then we get, for all $x \in I$

$$\begin{aligned}0 &= (h\lambda(\tau(a)x),a)_{\alpha\lambda\sigma,\beta\lambda\tau} = (d_1\lambda\tau(a)\alpha\lambda(x) + \beta\lambda\tau(a)h\lambda(x),a)_{\alpha\lambda\sigma,\beta\lambda\tau}\\ &= d_1\lambda\tau(a)[\alpha\lambda(x),\alpha\lambda\sigma(a)] + (d_1\lambda\tau(a),a)_{\alpha\lambda\sigma,\beta\lambda\tau}\alpha\lambda(x)\\ &+ \beta\lambda\tau(a)(h\lambda(x),a)_{\alpha\lambda\sigma,\beta\lambda\tau} - [\beta\lambda\tau(a),\beta\lambda\tau(a)]h\lambda(x)\\ &= d_1\lambda\tau(a)[\alpha\lambda(x),\alpha\lambda\sigma(a)] + (d_1\lambda\tau(a),a)_{\alpha\lambda\sigma,\beta\lambda\tau}\alpha\lambda(x).\end{aligned}$$

That is

$$k[\alpha\lambda(x),\alpha\lambda\sigma(a)] + (k,a)_{\alpha\lambda\sigma,\beta\lambda\tau}\alpha\lambda(x) = 0, \forall x \in I \tag{2.6}$$

where $k = d_1\lambda\tau(a)$. Replacing x by $xr, r \in R$ in (2.6) we get, for all $x \in I, r \in R$

$$0 = k\alpha\lambda(x)[\alpha\lambda(r),\alpha\lambda\sigma(a)] + k[\alpha\lambda(x),\alpha\lambda\sigma(a)]\alpha\lambda(r) + (k,a)_{\alpha\lambda\sigma,\beta\lambda\tau}\alpha\lambda(x)\alpha\lambda(r)$$
$$= k\alpha\lambda(x)[\alpha\lambda(r),\alpha\lambda\sigma(a)]$$

so $k\alpha\lambda(I)[R,\alpha\lambda\sigma(a)] = 0$. The last relation gives that $d_1\lambda\tau(a) = 0$ or $[R,\alpha\lambda\sigma(a)] = 0$. Since $a \notin Z$ then we have $d_1\lambda\tau(a) = 0$. □

Theorem 2. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha,\beta)-$ derivation associated with $(\alpha,\beta)-$ derivation $d : R \longrightarrow R$ and left-generalized $(\alpha,\beta)-$ derivation associated with $(\alpha,\beta)-$ derivation $d_1 : R \longrightarrow R$. Let a be a noncentral element of R and I a nonzero ideal of R. Then $h\lambda(I,a)_{\sigma,\tau} = 0$ if and only if $(h\lambda(I),a)_{\alpha\lambda\sigma,\beta\lambda\tau} = 0$.*

Proof. If $(h\lambda(I),a)_{\alpha\lambda\sigma,\beta\lambda\tau} = 0$ or $h\lambda(I,a)_{\sigma,\tau} = 0$ then $d\lambda\sigma(a) = 0$ and $d_1\lambda\tau(a) = 0$ by Lemma 3 and Lemma 4. Using these results we get, for all $x \in I$

$$\begin{aligned}h\lambda(x,a)_{\sigma,\tau} = 0 &\Leftrightarrow h\lambda(x\sigma(a) + \tau(a)x) = 0\\ &\Leftrightarrow h\lambda(x)\alpha\lambda\sigma(a) + \beta\lambda(x)d\lambda\sigma(a) + d_1\lambda\tau(a)\alpha\lambda(x) + \beta\lambda\tau(a)h\lambda(x) = 0\\ &\Leftrightarrow h\lambda(x)\alpha\lambda\sigma(a) + \beta\lambda\tau(a)h\lambda(x) = 0\\ &\Leftrightarrow (h\lambda(x),a)_{\alpha\lambda\sigma,\beta\lambda\tau} = 0.\end{aligned}$$

That is $(h\lambda(I),a)_{\alpha\lambda\sigma,\beta\lambda\tau}=0$ if and only if $h\lambda(I,a)_{\sigma,\tau}=0$. □

Corollary 3. *[5, Theorem 7] Let R be a prime ring of characteristic different from two, $d:R\longrightarrow R$ a nonzero derivation and $a\in R$. Then $(d(R),a)=0$ if and only if $d(R,a)=0$*

Theorem 3. *Let $d:R\longrightarrow R$ be a nonzero $(\alpha,\beta)-$ derivation and I a nonzero ideal of R. If $b\in R$ such that $d\lambda(I,b)_{\sigma,\tau}=0$ then $\sigma(b)-\tau(b)\in Z$.*

Proof. If $b\in Z$ then we have $\sigma(b)-\tau(b)\in Z$. Hence let $b\notin Z$. Since d is an $(\alpha,\beta)-$ derivation then d is a right(and left)-generalized $(\alpha,\beta)-$ derivation associated with d.

If $d\lambda(I,b)_{\sigma,\tau}=0$ then $(d\lambda(I),b)_{\alpha\lambda\sigma,\beta\lambda\tau}=0$ by Theorem2. Using this relation, we get, for all $x,y\in I$

$$\begin{aligned}0&=(d\lambda(x(y,b)_{\sigma,\tau}),b)_{\alpha\lambda\sigma,\beta\lambda\tau}\\&=(d\lambda(x)\alpha\lambda(y,b)_{\sigma,\tau}+\beta\lambda(x)d\lambda(y,b)_{\sigma,\tau},b)_{\alpha\lambda\sigma,\beta\lambda\tau}=(d\lambda(x)\alpha\lambda(y,b)_{\sigma,\tau},b)_{\alpha\lambda\sigma,\beta\lambda\tau}\\&=d\lambda(x)[\alpha\lambda(y,b)_{\sigma,\tau},\alpha\lambda\sigma(b)]+(d\lambda(x),b)_{\alpha\lambda\sigma,\beta\lambda\tau}\alpha\lambda(y,b)_{\sigma,\tau}\\&=d\lambda(x)[\alpha\lambda(y,b)_{\sigma,\tau},\alpha\lambda\sigma(b)]\end{aligned}$$

which gives that $d\lambda(I)[\alpha\lambda(y,b)_{\sigma,\tau},\alpha\lambda\sigma(b)]=0,\forall y\in I$. Since $\lambda(I)$ is a nonzero ideal of R and $d\neq 0$ then using Lemma1 we obtain that

$$[(y,b)_{\sigma,\tau},\sigma(b)]=0,\forall y\in I. \tag{2.7}$$

Replacing y by $\tau(b)y$ in (2.7) we have, for all $y\in I$

$$\begin{aligned}0&=[(\tau(b)y,b)_{\sigma,\tau},\sigma(b)]=[\tau(b)(y,b)_{\sigma,\tau}-[\tau(b),\tau(b)]y,\sigma(b)]\\&=[\tau(b)(y,b)_{\sigma,\tau},\sigma(b)]=\tau(b)[(y,b)_{\sigma,\tau},\sigma(b)]+[\tau(b),\sigma(b)](y,b)_{\sigma,\tau}\\&=[\tau(b),\sigma(b)](y,b)_{\sigma,\tau}.\end{aligned}$$

That is $[\tau(b),\sigma(b)](I,b)_{\sigma,\tau}=0$. Since $b\notin Z$, then using Lemma2 (ii) we get $[\tau(b),\sigma(b)]=0$ and so

$$\sigma(b)\tau(b)=\tau(b)\sigma(b). \tag{2.8}$$

If we consider (2.7) and (2.8) we have, for all $y\in I$

$$\begin{aligned}0&=[(y,b)_{\sigma,\tau},\sigma(b)]=y\sigma(b)\sigma(b)+\tau(b)y\sigma(b)-\sigma(b)y\sigma(b)-\sigma(b)\tau(b)y\\&=y\sigma(b)\sigma(b)+\tau(b)y\sigma(b)-\sigma(b)y\sigma(b)-\tau(b)\sigma(b)y\\&=[y,\sigma(b)]\sigma(b)+\tau(b)[y,\sigma(b)]=([y,\sigma(b)],b)_{\sigma,\tau}.\end{aligned}$$

That is

$$([y,\sigma(b)],b)_{\sigma,\tau}=0,\forall y\in I. \tag{2.9}$$

Replacing y by yz, $z\in I$ in (2.9) we get, for all $y,z\in I$

$$\begin{aligned}0&=(y[z,\sigma(b)]+[y,\sigma(b)]z,b)_{\sigma,\tau}=(y[z,\sigma(b)],b)_{\sigma,\tau}+([y,\sigma(b)]z,b)_{\sigma,\tau}\\&=y([z,\sigma(b)],b)_{\sigma,\tau}-[y,\tau(b)][z,\sigma(b)]+[y,\sigma(b)][z,\sigma(b)]+([y,\sigma(b)],b)_{\sigma,\tau}z\\&=-[y,\tau(b)][z,\sigma(b)]+[y,\sigma(b)][z,\sigma(b)]=[y,\sigma(b)-\tau(b)][z,\sigma(b)]\end{aligned}$$

which gives that

$$[y,\sigma(b)-\tau(b)][I,\sigma(b)]=0,\forall y\in I. \tag{2.10}$$

Replacing y by ry, $r\in R$ in (2.10) we have $[R,\sigma(b)-\tau(b)]I[I,\sigma(b)]=0$ Since b is noncentral then we obtain that $\sigma(b)-\tau(b)\in Z(R)$ by the last relation. □

Corollary 4. *Let $d:R\longrightarrow R$ be a nonzero $(\alpha,\beta)-$ derivation and W a nonzero left $(\sigma,\tau)-$Jordan ideal of R. If $d\gamma(W)=0$ then $\sigma(v)-\tau(v)\in Z,\forall v\in W$.*

Proof. If $d\gamma(W)=0$ then $d\gamma(R,v)_{\sigma,\tau}=0,\forall v\in W$. This gives that $\sigma(v)-\tau(v)\in Z,\forall v\in W$ by Theorem3. □

Theorem 4. *Let W be a nonzero left $(\sigma,\tau)-$Jordan ideal of R. Let $h:R\longrightarrow R$ be a nonzero left-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$ derivation $d_1:R\longrightarrow R$ of R and $b\in R$.*

(i) If $h\lambda(W)=0$ then $\sigma(v)-\tau(v)\in Z,\forall v\in W$.
(ii) If $[W,b]_{\lambda,\mu}=0$ then $b\in Z$ or $\sigma(v)-\tau(v)\in Z,\forall v\in W$.
(iii) If $[b,W]_{\lambda,\mu}=0$ then $b\in C_{\lambda,\mu}(R)$ or $\sigma(v)-\tau(v)\in Z,\forall v\in W$.

Proof. (i) If $h\lambda(W)=0$ then $h\lambda(R,v)_{\sigma,\tau}=0,\forall v\in W$. This means that, for any $v\in W$,

$$v\in Z \text{ or } d_1\lambda\tau(v)=0$$

by Lemma4. Let $K=\{v\in W\mid v\in Z\}$ and $L=\{v\in W\mid d_1\lambda\tau(v)=0\}$. Then K and L are two additive subgroups of W such that $W=K\cup L$. Since a group cannot be the union of proper subgroups, according to Brauer's Trick either $W=K$ or $W=L$. That is

$$W\subset Z \text{ or } d_1\lambda\tau(W)=0.$$

It is clear that, if $W\subset Z$ then $\sigma(v)-\tau(v)\in Z,\forall v\in W$. On the other hand, if $d_1\lambda\tau(W)=0$ then we have $\sigma(v)-\tau(v)\in Z,\forall v\in W$ by Corollary4.

(ii) The mapping $g(r)=[r,b]_{\lambda,\mu},\forall r\in R$ is a left-generalized derivation associated with derivation $d_2(r)=[r,\mu(b)],\forall r\in R$. If $g=0$ then $d_2=0$ and so $b\in Z$ is obtained.

Let $d_2\neq 0$. If $[W,b]_{\lambda,\mu}=0$ then $g(W)=0$. Using (i) we have $\sigma(v)-\tau(v)\in Z,\forall v\in W$. Finally, we obtain that $b\in Z$ or $\sigma(v)-\tau(v)\in Z,\forall v\in W$.

(iii) The mapping $d_3(r) = [b, r]_{\lambda,\mu}, \forall r \in R$ is a $(\lambda, \mu)-$derivation and so left-generalized derivation associated with d_3. If $d_3 = 0$ then $b \in C_{\lambda,\mu}(R)$. Let $d_3 \neq 0$.

If $[b, W]_{\lambda,\mu} = 0$ then $d_3(W) = 0$. This means that $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Corollary4. Finally, we obtain that $b \in C_{\lambda,\mu}(R)$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases. □

Theorem 5. *Let W be a nonzero left $(\sigma, \tau)-$Jordan ideal of R and $b \in R$.*

(i) If $(b, W)_{\lambda,\mu} = 0$ then $b \in C_{\lambda,\mu}(R)$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.
(ii) If $(W, b)_{\lambda,\mu} = 0$ then $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

Proof. (i) The mapping $h(r) = (b, r)_{\lambda,\mu}, \forall r \in R$ is a left-generalized $(\lambda, \mu)-$derivation associated with $(\lambda, \mu)-$ derivation $d(r) = [b, r]_{\lambda,\mu}, \forall r \in R$. If $h = 0$ then $d = 0$ and so $b \in C_{\lambda,\mu}(R)$ is obtained. Let $d \neq 0$. If $(b, W)_{\lambda,\mu} = 0$ then we have $h(W) = 0$. Using Theorem4 (i) we obtain that $\sigma(v) - \tau(v) \in Z, \forall v \in W$. Finally we obtain that $b \in C_{\lambda,\mu}$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases.

(ii) Similarly, the mapping $g(r) = (r, b)_{\lambda,\mu}, \forall r \in R$ is a left-generalized derivation associated with derivation $d_1(r) = -[r, \mu(b)], \forall r \in R$. If $g = 0$ then $d_1 = 0$ and so $b \in Z$ is obtained. Let $d_1 \neq 0$. If $(W, b)_{\lambda,\mu} = 0$ then $g(W) = 0$. This gives that $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Theorem4 (i). Finally, we obtain that $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases. □

Lemma 5. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha, \beta)-$derivation associated with a nonzero $(\alpha, \beta)-$derivation $d : R \longrightarrow R$. If $a, b \in R$ such that $a[h(R), b]_{\lambda,\mu} = 0$ then $a[a, \mu(b)] = 0$ or $d\alpha^{-1}\lambda(b)$.*

Proof. If $a[h(R), b]_{\lambda,\mu} = 0$ then we get, for all $r \in R$

$$\begin{aligned}0 &= a[h(r\alpha^{-1}\lambda(b)), b]_{\lambda,\mu} = a[h(r)\lambda(b) + \beta(r)d\alpha^{-1}\lambda(b), b]_{\lambda,\mu} \\ &= ah(r)[\lambda(b), \lambda(b)] + a[h(r), b]_{\lambda,\mu}\lambda(b) + a\beta(r)[d\alpha^{-1}\lambda(b), b]_{\lambda,\mu} \\ &+ a[\beta(r), \mu(b)]d\alpha^{-1}\lambda(b) \\ &= a\beta(r)[d\alpha^{-1}\lambda(b), b]_{\lambda,\mu} + a[\beta(r), \mu(b)]d\alpha^{-1}\lambda(b).\end{aligned}$$

That is

$$a\beta(r)[k, b]_{\lambda,\mu} + a[\beta(r), \mu(b)]k = 0, \forall r \in R. \tag{2.11}$$

where $k = d\alpha^{-1}\lambda(b)$. Replacing r by $\beta^{-1}(a)r$ in (2.11) and using (2.11) we have, for all $r \in R$

$$\begin{aligned}0 &= aa\beta(r)[k, b]_{\lambda,\mu} + a[a\beta(r), \mu(b)]k \\ &= aa\beta(r)[k, b]_{\lambda,\mu} + aa[\beta(r), \mu(b)]k + a[a, \mu(b)]\beta(r)k \\ &= a[a, \mu(b)]\beta(r)k\end{aligned}$$

which gives $a[a, \mu(b)]Rk = 0$. Using that primeness of R we get $a[a, \mu(b)] = 0$ or $d\alpha^{-1}\lambda(b) = 0$. □

Lemma 6. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation $d_1 : R \longrightarrow R$. If $a,b \in R$ such that $[h(R),b]_{\lambda,\mu}a = 0$ then $d_1\beta^{-1}\mu(b) = 0$ or $[a,\lambda(b)]a = 0$.*

Proof. If $[h(R),b]_{\lambda,\mu}a = 0$ then we get, for all $r \in R$

$$\begin{aligned}0 &= [h(\beta^{-1}\mu(b)r),b]_{\lambda,\mu}a = [d_1\beta^{-1}\mu(b)\alpha(r) + \mu(b)h(r),b]_{\lambda,\mu}a\\ &= d_1\beta^{-1}\mu(b)[\alpha(r),\lambda(b)]a + [d_1\beta^{-1}\mu(b),b]_{\lambda,\mu}\alpha(r)a\\ &+ \mu(b)[h(r),b]_{\lambda,\mu}a + [\mu(b),\mu(b)]h(r)a\\ &= d_1\beta^{-1}\mu(b)[\alpha(r),\lambda(b)]a + [d_1\beta^{-1}\mu(b),b]_{\lambda,\mu}\alpha(r)a\end{aligned}$$

which gives that

$$k[\alpha(r),\lambda(b)]a + [k,b]_{\lambda,\mu}\alpha(r)a = 0, \forall r \in R \tag{2.12}$$

where $k = d_1\beta^{-1}\mu(b)$. Replacing r by $r\alpha^{-1}(a)$ in (2.12) and using (2.12) we have, for all $r \in R$

$$\begin{aligned}0 &= k[\alpha(r)a,\lambda(b)]a + [k,b]_{\lambda,\mu}\alpha(r)aa\\ &= k\alpha(r)[a,\lambda(b)]a + k[\alpha(r),\lambda(b)]aa + [k,b]_{\lambda,\mu}\alpha(r)aa\\ &= k\alpha(r)[a,\lambda(b)]a.\end{aligned}$$

That is $kR[a,\lambda(b)]a = 0$. Since R is a prime ring, then the last relation gives that $d_1\beta^{-1}\mu(b) = 0$ or $[a,\lambda(b)]a = 0$. □

Theorem 6. *Let W be a nonzero left $(\sigma,\tau)-$Jordan ideal of R and $a,b \in R$.*

(i) If $a[W,b]_{\lambda,\mu} = 0$ then $a[a,\mu(b)] = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.
(ii) If $[W,b]_{\lambda,\mu}a = 0$ then $[a,\lambda(b)]a = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

Proof. Let us consider a nonzero element v of W. The mapping defined by $h(r) = (r,v)_{\sigma,\tau}, \forall r \in R$ is a left-generalized derivation associated with derivation $d_1(r) = -[r,\tau(v)], \forall r \in R$ and right-generalized derivation associated with derivation $d(r) = [r,\sigma(v)], \forall r \in R$. If $h = 0$ then $d = 0 = d_1$ and so $v \in Z$ is obtained. Let $d \neq 0$ and $d_1 \neq 0$.

(i) If $a[W,b]_{\lambda,\mu} = 0$ then we have $a[(R,v)_{\sigma,\tau},b]_{\lambda,\mu} = 0$ and so $a[h(R),b]_{\lambda,\mu} = 0$. Since h is a right-generalized derivation associated with d, then using Lemma 5 we get $d\lambda(b) = 0$ or $a[a,\mu(b)] = 0$. That is

$$[\lambda(b),\sigma(v)] = 0 \text{ or } a[a,\mu(b)] = 0.$$

On the other hand, if $v = 0$ then $[\lambda(b),\sigma(v)] = 0$. Hence, considering the same argument for all $v \in W$ we have

$$[\sigma^{-1}\lambda(b),W] = 0 \text{ or } a[a,\mu(b)] = 0.$$

If $[\sigma^{-1}\lambda(b), W] = 0$ then $b \in Z$ or $\sigma(v)-\tau(v) \in Z, \forall v \in W$ by Theorem4(iii). Finally we obtained that $a[a, \mu(b)] = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases.

(ii) If $[W, b]_{\lambda,\mu}a = 0$ then we have $[(R, v)_{\sigma,\tau}, b]_{\lambda,\mu}a = 0$ and so $[h(R), b]_{\lambda,\mu}a = 0$. Since h is a left-generalized derivation associated with d_1 then, using Lemma 6, we get $d_1\mu(b) = 0$ or $[a, \lambda(b)]a = 0$. That is $[\mu(b), \tau(v)] = 0$ or $[a, \lambda(b)]a = 0$. Considering as in the proof of (i) we get

$$[\tau^{-1}\mu(b), W] = 0 \text{ or } [a, \lambda(b)]a = 0.$$

If $[\tau^{-1}\mu(b), W] = 0$ then we have $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Theorem4(iii). If $b \in Z$ then $[a, \lambda(b)]a = 0$.

Finally, we obtained that $[a, \lambda(b)]a = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases. □

Lemma 7. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha, \beta)-$derivation associated with a nonzero $(\alpha, \beta)-$derivation $d : R \longrightarrow R$ and $a, b \in R$. If $a(h(R), b)_{\lambda,\mu} = 0$ then $d\alpha^{-1}\lambda(b) = 0$ or $a[a, \mu(b)] = 0$.*

Proof. If $a(h(R), b)_{\lambda,\mu} = 0$ then we get, for all $r \in R$

$$\begin{aligned} 0 &= a(h(r\alpha^{-1}\lambda(b)), b)_{\lambda,\mu} = a(h(r)\lambda(b) + \beta(r)d\alpha^{-1}\lambda(b), b)_{\lambda,\mu} \\ &= ah(r)[\lambda(b), \lambda(b)] + a(h(r), b)_{\lambda,\mu}\lambda(b) + a\beta(r)(d\alpha^{-1}\lambda(b), b)_{\lambda,\mu} \\ &- a[\beta(r), \mu(b)]d\alpha^{-1}\lambda(b) \end{aligned}$$

which gives that

$$a\beta(r)(k, b)_{\lambda,\mu} - a[\beta(r), \mu(b)]k = 0, \forall r \in R. \tag{2.13}$$

where $k = d\alpha^{-1}\lambda(b)$. Replacing r by $\beta^{-1}(a)r$ in (2.13) we have, for all $r \in R$

$$\begin{aligned} 0 &= aa\beta(r)(k, b)_{\lambda,\mu} - a[a\beta(r), \mu(b)]k \\ &= aa\beta(r)(k, b)_{\lambda,\mu} - aa[\beta(r), \mu(b)]k - a[a, \mu(b)]\beta(r)k \\ &= -a[a, \mu(b)]\beta(r)k. \end{aligned}$$

That is $a[a, \mu(b)]Rd\alpha^{-1}\lambda(b) = 0$. Since R is a prime ring, we obtain that $d\alpha^{-1}\lambda(b) = 0$ or $a[a, \mu(b)] = 0$. □

Lemma 8. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha, \beta)-$derivation associated with a nonzero $(\alpha, \beta)-$derivation $d_1 : R \longrightarrow R$ and $a, b \in R$. If $(h(R), b)_{\lambda,\mu}a = 0$ then $d_1\beta^{-1}\mu(b) = 0$ or $[a, \lambda(b)]a = 0$.*

Proof. If $(h(R),b)_{\lambda,\mu}a=0$ then we get, for all $r\in R$

$$\begin{aligned}0&=(h(\beta^{-1}\mu(b)r),b)_{\lambda,\mu}a=(d_1\beta^{-1}\mu(b)\alpha(r)+\mu(b)h(r),b)_{\lambda,\mu}a\\&=d_1\beta^{-1}\mu(b)[\alpha(r),\lambda(b)]a+(d_1\beta^{-1}\mu(b),b)_{\lambda,\mu}\alpha(r)a\\&+\mu(b)(h(r),b)_{\lambda,\mu}a-[\mu(b),\mu(b)]h(r)a\\&=d_1\beta^{-1}\mu(b)[\alpha(r),\lambda(b)]a+(d_1\beta^{-1}\mu(b),b)_{\lambda,\mu}\alpha(r)a\end{aligned}$$

which gives that

$$k[\alpha(r),\lambda(b)]a+(k,b)_{\lambda,\mu}\alpha(r)a=0,\forall r\in R. \tag{2.14}$$

where $k=d_1\beta^{-1}\mu(b)$. If we take $r\alpha^{-1}(a)$ instead of r in (2.14), we get, for all $r\in R$

$$\begin{aligned}0&=k[\alpha(r)a,\lambda(b)]a+(k,b)_{\lambda,\mu}\alpha(r)aa\\&=k\alpha(r)[a,\lambda(b)]a+k[\alpha(r),\lambda(b)]aa+(k,b)_{\lambda,\mu}\alpha(r)aa\\&=k\alpha(r)[a,\lambda(b)]a.\end{aligned}$$

That is $kR[a,\lambda(b)]a=0$. This gives that $d_1\beta^{-1}\mu(b)=0$ or $[a,\lambda(b)]a=0$ in prime rings. □

Theorem 7. *Let W be a nonzero left $(\sigma,\tau)-$Jordan ideal of R and $a,b\in R$.*

(i) If $a(W,b)_{\lambda,\mu}=0$ then $a[a,\mu(b)]=0$ or $\sigma(v)-\tau(v)\in Z,\forall v\in V$.
(ii) If $(W,b)_{\lambda,\mu}a=0$ then $[a,\lambda(b)]a=0$ or $\sigma(v)-\tau(v)\in Z,\forall v\in V$.

Proof. Let us consider a nonzero element v of W. The mapping defined by $h(r)=(r,v)_{\sigma,\tau},\forall r\in R$ is a left-generalized derivation associated with derivation $d_1(r)=-[r,\tau(v)],\forall r\in R$ and right-generalized derivation associated with derivation $d(r)=[r,\sigma(v)],\forall r\in R$. If $h=0$ then $d=0=d_1$ and so $v\in Z$ is obtained. Let $d\neq 0$ and $d_1\neq 0$.

(i) If $a(W,b)_{\lambda,\mu}=0$ then we have $a((R,v)_{\sigma,\tau},b)_{\lambda,\mu}=0$. That is $a(h(R),b)_{\lambda,\mu}=0$. Since h is a right-generalized derivation with d then using Lemma 7 we obtain that $d\lambda(b)=0$ or $a[a,\mu(b)]=0$. That is

$$[\lambda(b),\sigma(v)]=0 \text{ or } a[a,\mu(b)]=0.$$

On the other hand, if $v=0$ then $[\lambda(b),\sigma(v)]=0$. Hence, considering the same argument for all $v\in W$ we have

$$[b,W]=0 \text{ or } a[a,\mu(b)]=0.$$

If $[b,W]=0$ then we have $b\in Z$ or $\sigma(v)-\tau(v)\in Z,\forall v\in W$ by Theorem4 (iii). If $b\in Z$ then $a[a,\mu(b)]=0$.

Finally, we obtained that $a[a,\mu(b)]=0$ or $\sigma(v)-\tau(v)\in Z,\forall v\in V$ for any cases.

(ii) If $(W,b)_{\lambda,\mu}a=0$ then $(h(R),b)_{\lambda,\mu}a$ is obtained. Since h is a left-generalized derivation associated with d_1 then using Lemma 8 and considering as in the proof of (i) we get $[a,\lambda(b)]a=0$ or $\sigma(v)-\tau(v)\in Z,\forall v\in V$. □

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