ON \((\sigma, \tau)\)– LEFT JORDAN IDEALS AND GENERALIZED DERIVATIONS

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Abstract
Let \(R\) be a prime ring with characteristic not 2 and \(\sigma, \tau, \alpha, \beta, \lambda, \mu, \gamma\) automorphisms of \(R\). Let \(h : R \rightarrow R\) be a nonzero right (resp. left) generalized \((\alpha, \beta)\)– derivation associated with \((\alpha, \beta)\)– derivation \(d\) (resp. \(d_1\)). Let \(W, V\) be nonzero left \((\sigma, \tau)\)-Jordan ideals of \(R\). The main object in this paper is to study the situations. (1) \(h(W) = 0\), (2) \([b, W]_{\lambda, \mu} = 0\) or \([W, b]_{\lambda, \mu} = 0\), \((3)\) \([b, W]_{\lambda, \mu} = 0\) or \([W, b]_{\lambda, \mu} = 0\), \((4)\) \([b, W]_{\lambda, \mu} = 0\) or \([W, b]_{\lambda, \mu} = 0\), \((5)\) \([b, W]_{\lambda, \mu} = 0\) or \([W, b]_{\lambda, \mu} = 0\), \((6)\) \(bW \subset C_{\lambda, \mu}(V)\) or \(Wb \subset C_{\lambda, \mu}(V)\). (7) \((h(R), b)_{\lambda, \mu}a = 0\) or \(a(h(R), b)_{\lambda, \mu} = 0\).

1 Introduction

Let \(R\) be a ring and \(\sigma, \tau\) two mappings of \(R\). For each \(r, s \in R\) we set \([r, s] = rs - sr, (r, s) = rs + sr, (r, s)_{\sigma, \tau} = r\sigma(s) - \tau(s)r\) and \((r, s)_{\sigma, \tau} = r\sigma(s) + \tau(s)r\). Let \(U\) be an additive subgroup of \(R\). If \((U, R) \subset U\) then \(U\) is called a Jordan ideal of \(R\). The definition of \((\sigma, \tau)\)–Jordan ideal of \(R\) is introduced in [6] as follows: (i) \(U\) is called a right \((\sigma, \tau)\)–Jordan ideal of \(R\) if \((U, R)_{\sigma, \tau} \subset U\), (ii) \(U\) is called a left \((\sigma, \tau)\)–Jordan ideal if \((R, U)_{\sigma, \tau} \subset U\). (iii) \(U\) is called a \((\sigma, \tau)\)–Jordan ideal if \(U\) is both right and left \((\sigma, \tau)\)–Jordan ideal of \(R\). Every Jordan ideal of \(R\) is a \((1, 1)\)–Jordan ideal of \(R\), where \(1 : R \rightarrow R\) is an identity map. The following example is given in [6]. If \(R = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x \text{ and } y \text{ are integers}\}\), \(U = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x \text{ is integer}\}\), \(\sigma\)\((\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\) and \(\tau\)\((\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix}\) then \(U\) is \((\sigma, \tau)\)–right Jordan ideal but not a Jordan ideal of \(R\).

A derivation \(d\) is an additive mapping on \(R\) which satisfies \(d(rs) = d(r)s + rd(s), \forall r, s \in R\). The notion of generalized derivation was introduced by

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Brešar [2] as follows. An additive mapping $h : R \rightarrow R$ will be called a generalized derivation if there exists a derivation $d$ of $R$ such that $h(xy) = h(x)y + xd(y)$, $\forall x, y \in R$.

An additive mapping $d : R \rightarrow R$ is said to be a $(\sigma, \tau)$-derivation if $d(rs) = d(r)\sigma(s) + \tau(r)d(s)$ for all $r, s \in R$. Every derivation $d : R \rightarrow R$ is a $(1,1)$-derivation. Chang [3] gave the following definition. Let $R$ be a ring, $\sigma$ and $\tau$ automorphisms of $R$ and $d : R \rightarrow R$ a $(\sigma, \tau)$-derivation. An additive mapping $h : R \rightarrow R$ is said to be a left generalized $(\sigma, \tau)$-derivation of $R$ associated with derivation $d$ if $h(xy) = h(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$ and $h$ is said to be a left generalized $(\sigma, \tau)$-derivation of $R$ associated with $d$ if $h(xy) = d(x)\sigma(y) + \tau(x)h(y)$, for all $x, y \in R$. $h$ is said to be a generalized $(\sigma, \tau)$-derivation of $R$ associated with $d$ if it is both a left and right generalized $(\sigma, \tau)$-derivation of $R$ associated with $d$. Besides every $(\sigma, \tau)$-derivation $d : R \rightarrow R$ is a generalized $(\sigma, \tau)$-derivation associated with $d$ and every derivation $d : R \rightarrow R$ is a generalized $(1,1)$-derivation associated with $d$.

A generalized $(1,1)$-derivation is simply called a generalized derivation. It is clear that the definition of generalized derivation which is given in [2] is a right generalized derivation associated with derivation $d$ according to Chang’s definition.

The mapping $h(r) = (a, r)_{\sigma, \tau}, \forall r \in R$ is a left-generalized $(\sigma, \tau)$-derivation with $(\sigma, \tau)$-derivation $d_1(r) = [a, r]_{\sigma, \tau}, \forall r \in R$ and right-generalized $(\sigma, \tau)$-derivation with $(\sigma, \tau)$-derivation $d_1(r) = -[a, r]_{\sigma, \tau}, \forall r \in R$. Every $(\sigma, \tau)$-derivation $d : R \rightarrow R$ is a generalized $(\sigma, \tau)$-derivation with $d$.

Throughout the paper, $R$ will be a prime ring with center $Z$, characteristic not 2 and $\sigma, \tau, \alpha, \beta, \lambda, \mu, \gamma$ automorphisms of $R$. We set $C_{\sigma, \tau}(R) = \{c \in R \mid c\sigma(r) = \tau(r)c, \forall r \in R\}$, and shall use the following relations frequently:

$[rs, t]_{\sigma, \tau} = r[s, t]_{\sigma, \tau} + [r, \tau(t)]s = r[s, \sigma(t)] + [r, t]_{\sigma, \tau}s$

$[r, st]_{\sigma, \tau} = \tau(s)[r, t]_{\sigma, \tau} + [r, s]_{\sigma, \tau}\sigma(t)$

$(r, s)_{\sigma, \tau} = \tau(s)(r, t)_{\sigma, \tau} - [r, \tau(t)]s = r[s, \sigma(t)] + (r, t)_{\sigma, \tau}s$.

$C_{\lambda, \mu}(I)$

$0 = [ba, x]_{\lambda, \mu} = b[a, \lambda(x)] + [b, x]_{\lambda, \mu}a = b[a, \lambda(x)]$

2 Results

Lemma 1. [1, Lemma 3] Let $U$ be a nonzero ideal of $R$ and $d : R \rightarrow R$ a $(\sigma, \tau)$-derivation. If $a \in R$ such that $ad(U) = 0$ (or $d(U)a = 0$) then $a = 0$ or $d = 0$.

Lemma 2. Let $I$ be a nonzero ideal of $R$ and $a, b \in R$.

(i) If $b, ba \in C_{\lambda, \mu}(I)$ or $(b, ab \in C_{\lambda, \mu}(I))$ then $b = 0$ or $a \in Z$.

(ii) If $b\gamma(I, a)_{\alpha, \beta} = 0$ or $\gamma(I, a)_{\alpha, \beta}b = 0$ then $b = 0$ or $a \in Z$.

Proof. (i) If $b, ba \in C_{\lambda, \mu}(I)$ then we have, for all $x \in I$

$0 = [ba, x]_{\lambda, \mu} = b[a, \lambda(x)] + [b, x]_{\lambda, \mu}a = b[a, \lambda(x)]$
Corollary 1. [7, Lemma 4] Let $b$ and $ab$ be in the center of a prime ring $R$. If $b$ is not zero, then $a$ is in $Z$, the center of $R$.

Theorem 1. Let $I, J$ be nonzero ideals of $R$ and $a, b \in R$. Let $W$ be a left $(\sigma, \tau)$—Jordan ideal of $R$.

(i) If $b\gamma(I, a)_{\alpha, \beta} \in C_{\lambda, \mu}(J)$ or $\gamma(I, a)_{\alpha, \beta} b \in C_{\lambda, \mu}(J)$ then $b = 0$ or $a \in Z$.

(ii) If $b\gamma(W) \in C_{\lambda, \mu}(J)$ or $\gamma(W) b \in C_{\lambda, \mu}(J)$ then $b = 0$ or $W \in Z$.

Proof. (i) If $b\gamma(I, a)_{\alpha, \beta} \in C_{\lambda, \mu}(J)$ then we have, for all $x \in I$

\[ C_{\lambda, \mu}(J) \ni b\gamma(xa(a), a)_{\alpha, \beta} = b\gamma(x)\gamma[\alpha(a), \alpha(a)] + b\gamma(x, a)_{\alpha, \beta} \gamma\alpha(a) = b\gamma(x, a)_{\alpha, \beta} \gamma\alpha(a). \]

Then

\[ b\gamma(I, a)_{\alpha, \beta} \gamma\alpha(a) \in C_{\lambda, \mu}(J). \]  \hspace{1cm} (2.1)

If we use Lemma 2 (i) in (2.1) then we get

\[ b\gamma(I, a)_{\alpha, \beta} = 0 \text{ or } a \in Z. \]

If $b\gamma(I, a)_{\alpha, \beta} = 0$ then we have $b = 0$ or $a \in Z$ by Lemma 2 (ii).

If $\gamma(I, a)_{\alpha, \beta} b \in C_{\lambda, \mu}(J)$ then we have, for all $x \in I$

\[ C_{\lambda, \mu}(J) \ni \gamma(b(\alpha)x(a), a)_{\alpha, \beta} b = \gamma(b(\alpha)\gamma(x, a)_{\alpha, \beta} b - \gamma(\beta(a), \beta(a))\gamma(x) b = \gamma(\beta(a)\gamma(x, a)_{\alpha, \beta} b. \]

That is

\[ \gamma(\beta(a)\gamma(I, a)_{\alpha, \beta} b \in C_{\lambda, \mu}(J). \]  \hspace{1cm} (2.2)
If we use Lemma 2 (i) then (2.2) gives that
\[ \gamma(I, a)_{\alpha, \beta}b = 0 \text{ or } a \in Z. \]

If \( \gamma(I, a)_{\alpha, \beta}b = 0 \) then using Lemma 2 (ii) we obtain the required result.

(ii) If \( b_\gamma(W) \subset C_{\lambda, \mu}(J) \text{ or } \gamma(W)b \subset C_{\lambda, \mu}(J) \) then we have \( b_\gamma(R, W)_{\sigma, \tau} \subset C_{\lambda, \mu}(J) \text{ or } \gamma(R, W)_{\sigma, \tau}b \subset C_{\lambda, \mu}(J) \). Thus \( b = 0 \) or \( W \subset Z \) by (i).

**Corollary 2.** [4, Theorem 2.2] Let \( W \) be a left \((\sigma, \tau)\)-Jordan ideal of \( R \). If \( aW = 0 \) (or \( Wa = 0 \)) and \( a \in R \), then \( a = 0 \) or \( W \subset Z \).

**Lemma 3.** Let \( h : R \rightarrow R \) be a nonzero right generalized \((\alpha, \beta)\)-derivation associated with a nonzero \((\alpha, \beta)\)-derivation \( d : R \rightarrow R \) and \( I \) a nonzero ideal of \( R \). If \( a \) is a noncentral element of \( R \) such that \( h\lambda(I, a)_{\sigma, \tau} = 0 \) or \( (h\lambda(I), a)_{\alpha, \beta} = 0 \) then \( d\lambda(a) = 0 \).

**Proof.** If \( h\lambda(I, a)_{\sigma, \tau} = 0 \) then we get, for all \( x \in I \)
\[
0 = h\lambda(xa(a), a)_{\sigma, \tau} = h\{\lambda(x)\lambda(a, a) + \lambda(x, a)_{\sigma, \tau}a\}
= h\{\lambda(x)_{\sigma, \tau}a\lambda(a)\}
= \beta\lambda(x, a)_{\sigma, \tau}d\lambda(a).
\]
That is
\[
\beta\lambda(I, a)_{\sigma, \tau}d\lambda(a) = 0. \tag{2.3}
\]
Since \( a \notin Z \) then (2.3) gives that \( d\lambda(a) = 0 \) by Lemma 2 (ii).

If \( (h\lambda(I), a)_{\alpha, \beta} = 0 \) then we have, for all \( x \in I \)
\[
0 = (h\lambda(xa(a)), a)_{\alpha, \beta} = (h\lambda(x)\alpha\lambda(a) + \beta\lambda(x)d\lambda(a), a)_{\alpha, \beta}
= h\lambda(x)_{\alpha, \beta}a\lambda(a) + (h\lambda(x), a)_{\alpha, \beta}\alpha\lambda(a)
+ \beta\lambda(x)(d\lambda(a), a)_{\alpha, \beta}
= h\lambda(x)_{\alpha, \beta}d\lambda(a)
\]
and so
\[
\beta\lambda(x)(k) = 0, \forall x \in I \tag{2.4}
\]
where \( k = d\lambda(a) \). Replacing \( x \) by \( rx, r \in R \) in (2.4) and using (2.4) we get, for all \( r \in R, x \in I \)
\[
0 = \beta\lambda(r)\beta\lambda(x)(k) - [\beta\lambda(r), \beta\lambda(a)]k = -[\beta\lambda(r), \beta\lambda(a)]\beta\lambda(x)k
\]
so \( [R, \beta\lambda(a)]\beta\lambda(I)k = 0 \). Since \( \beta\lambda(I) \) is a nonzero ideal of \( R \) and \( a \notin Z \), then we obtain that \( d\lambda(a) = 0 \) by the last relation. □
Lemma 4. Let $I$ be a nonzero ideal of $R$ and $h : R \rightarrow R$ a nonzero left-generalized $(\alpha, \beta)$—derivation associated with a nonzero $(\alpha, \beta)$—derivation $d_1 : R \rightarrow R$. If $a$ is a noncentral element of $R$ such that $h\lambda(I, a)_{\sigma, \tau} = 0$ or $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$ then $d_1\lambda\tau(a) = 0$.

Proof. If $(h\lambda(I), a)_{\sigma, \tau} = 0$ then we have, for all $x \in I$

\[
0 = h\lambda(\tau(a)x, a)_{\sigma, \tau} = h\{\lambda\tau(a)\lambda(x, a)_{\sigma, \tau} - \lambda[\tau(a), \tau(a)]\lambda(x)\} = h\{\lambda\tau(a)\lambda(x, a)_{\sigma, \tau}\}
\]

\[
= d_1\lambda\tau(a)\alpha\lambda(x, a)_{\sigma, \tau} + \beta\lambda\tau(a)h\lambda(x, a)_{\sigma, \tau} = d_1\lambda\tau(a)\alpha\lambda(x, a)_{\sigma, \tau}.
\]

That is

\[
d_1\lambda\tau(a)\alpha\lambda(I, a)_{\sigma, \tau} = 0. \tag{2.5}
\]

Since $a$ is noncentral, using 2 (ii) and (2.5) we get $d_1\lambda\tau(a) = 0$.

Similarly, if $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$ then we get, for all $x \in I$

\[
0 = (h\lambda(\tau(a)x, a)_{\alpha\lambda\sigma, \beta\lambda\tau} = (d_1\lambda\tau(a)\alpha\lambda(x) + \beta\lambda\tau(a)h\lambda(x, a)_{\alpha\lambda\sigma, \beta\lambda\tau})
\]

\[
= d_1\lambda\tau(a)[\alpha\lambda(x, \alpha\lambda\sigma(a))] + (d_1\lambda\tau(a), a)_{\alpha\lambda\sigma, \beta\lambda\tau}\alpha\lambda(x)
\]

\[
+ \beta\lambda\tau(a)(h\lambda(x, a)_{\alpha\lambda\sigma, \beta\lambda\tau} - [\beta\lambda\tau(a), \beta\lambda\tau(a)]h\lambda(x)
\]

\[
= d_1\lambda\tau(a)[\alpha\lambda(x, \alpha\lambda\sigma(a))] + (d_1\lambda\tau(a), a)_{\alpha\lambda\sigma, \beta\lambda\tau}\alpha\lambda(x).
\]

That is

\[
k[\alpha\lambda(x, \alpha\lambda\sigma(a))] + (k, a)_{\alpha\lambda\sigma, \beta\lambda\tau}\alpha\lambda(x) = 0, \forall x \in I \tag{2.6}
\]

where $k = d_1\lambda\tau(a)$. Replacing $x$ by $xr, r \in R$ in (2.6) we get, for all $x \in I, r \in R$

\[
0 = k\lambda(\alpha\lambda(r), \alpha\lambda\sigma(a)] + [k\alpha\lambda(x, \alpha\lambda\sigma(a)])\alpha\lambda(r) + (k, a)_{\alpha\lambda\sigma, \beta\lambda\tau}\alpha\lambda(x)\alpha\lambda(r)
\]

\[
= k\lambda(\alpha\lambda(r, \alpha\lambda\sigma(a)])
\]

so $k\lambda(I)[R, \alpha\lambda\sigma(a)] = 0$. The last relation gives that $d_1\lambda\tau(a) = 0$ or $[R, \alpha\lambda\sigma(a)] = 0$. Since $a \notin Z$ then we have $d_1\lambda\tau(a) = 0$. \qed

Theorem 2. Let $h : R \rightarrow R$ be a nonzero right-generalized $(\alpha, \beta)$—derivation associated with $(\alpha, \beta)$—derivation $d : R \rightarrow R$ and left-generalized $(\alpha, \beta)$—derivation associated with $(\alpha, \beta)$—derivation $d_1 : R \rightarrow R$. Let $a$ be a noncentral element of $R$ and $I$ a nonzero ideal of $R$. Then $h\lambda(I, a)_{\sigma, \tau} = 0$ if and only if $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$.

Proof. If $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$ or $h\lambda(I, a)_{\sigma, \tau} = 0$ then $d\lambda\sigma(a) = 0$ and $d_1\lambda\tau(a) = 0$ by Lemma 3 and Lemma 4. Using these results we get, for all $x \in I$

\[
h\lambda(x, a)_{\sigma, \tau} = 0 \iff h\lambda(xa\sigma + \tau(x)) = 0
\]

\[
\iff h\lambda(x)\alpha\lambda\sigma(a) + \beta\lambda(x)d\lambda\sigma(a) + d_1\lambda\tau(a)\alpha\lambda(x) + \beta\lambda\tau(a)h\lambda(x) = 0
\]

\[
\iff h\lambda(x)\alpha\lambda\sigma(a) + \beta\lambda\tau(a)h\lambda(x) = 0
\]

\[
\iff (h\lambda(x), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0.
\]
That is \((h\lambda(I), a)_{\alpha \lambda \sigma, \beta \lambda \tau} = 0\) if and only if \(h\lambda(I, a)_{\sigma, \tau} = 0\).

\[\text{Corollary 3. \cite[Theorem 7]{5}}\] Let \(R\) be a prime ring of characteristic different from two, \(d : R \rightarrow R\) a nonzero derivation and \(a \in R\). Then \((d(R), a) = 0\) if and only if \(d(R, a) = 0\).

**Theorem 3.** Let \(d : R \rightarrow R\) be a nonzero \((\alpha, \beta)\)-derivation and \(I\) a nonzero ideal of \(R\). If \(b \in R\) such that \(d\lambda(I, b)_{\sigma, \tau} = 0\) then \(\sigma(b) - \tau(b) \in Z\).

**Proof.** If \(b \in Z\) then we have \(\sigma(b) - \tau(b) \in Z\). Hence let \(b \notin Z\). Since \(d\) is an \((\alpha, \beta)\)-derivation then \(d\) is a right (and left) generalized \((\alpha, \beta)\)-derivation associated with \(d\).

If \(d\lambda(I, b)_{\sigma, \tau} = 0\) then \((d\lambda(I), b)_{\alpha \lambda \sigma, \beta \lambda \tau} = 0\) by Theorem 2. Using this relation, we get, for all \(x, y \in I\)

\[0 = (d\lambda(x, y, b)_{\sigma, \tau}, b)_{\alpha \lambda \sigma, \beta \lambda \tau} = (d\lambda(x)\alpha \lambda(y, b)_{\sigma, \tau} + \beta \lambda(x)d\lambda(y, b)_{\sigma, \tau}, b)_{\alpha \lambda \sigma, \beta \lambda \tau} = (d\lambda(x)\alpha \lambda(y, b)_{\sigma, \tau}, b)_{\alpha \lambda \sigma, \beta \lambda \tau} + (d\lambda(x), b)_{\alpha \lambda \sigma, \beta \lambda \tau} \alpha \lambda(y, b)_{\sigma, \tau} = d\lambda(x)\alpha \lambda(y, b)_{\sigma, \tau}, \alpha \lambda(b)\]

which gives that \(d\lambda(I)[\alpha \lambda(y, b)_{\sigma, \tau}, \alpha \lambda(b)] = 0\), \(\forall y \in I\). Since \(\lambda(I)\) is a nonzero ideal of \(R\) and \(d \neq 0\) then using Lemma 1 we obtain that

\[(y, b)_{\sigma, \tau}, \sigma(b)] = 0, \forall y \in I. \quad (2.7)\]

Replacing \(y\) by \(\tau(b)y\) in (2.7) we have, for all \(y \in I\)

\[0 = [(\tau(b)y, b)_{\sigma, \tau}, \sigma(b)] = [\tau(b){{y}}_{\sigma, \tau} - [\tau(b), \tau(b)]y, \sigma(b)] = [\tau(b)(y, b)_{\sigma, \tau}, \sigma(b)] = \tau(b)[(y, b)_{\sigma, \tau}, \sigma(b)] + [\tau(b), \sigma(b)](y, b)_{\sigma, \tau} = [\tau(b), \sigma(b)](y, b)_{\sigma, \tau}\]

That is \([\tau(b), \sigma(b)](I, b)_{\sigma, \tau} = 0\). Since \(b \notin Z\), then using Lemma 2 (ii) we get

\[\tau(b), \sigma(b) = 0\] and so

\[\sigma(b)\tau(b) = \tau(b)\sigma(b). \quad (2.8)\]

If we consider (2.7) and (2.8) we have, for all \(y \in I\)

\[0 = [(y, b)_{\sigma, \tau}, \sigma(b)] = y\sigma(b)\sigma(b) + \tau(b)y\sigma(b) - \sigma(b)y\sigma(b) - \sigma(b)\tau(b)y = y\sigma(b)\sigma(b) + \tau(b)y\sigma(b) - \sigma(b)y\sigma(b) - \tau(b)\sigma(b)y = [y, \sigma(b)]\sigma(b) + \tau(b)[y, \sigma(b)] = ([y, \sigma(b)], b)_{\sigma, \tau}\]

That is

\[([y, \sigma(b)], b)_{\sigma, \tau} = 0, \forall y \in I. \quad (2.9)\]
Replacing $y$ by $yz$, $z \in I$ in (2.9) we get, for all $y, z \in I$

$$0 = (y[z, \sigma(b)] + [y, \sigma(b)]z, b)_{\sigma, \tau} = (y[z, \sigma(b)], b)_{\sigma, \tau} + ([y, \sigma(b)]z, b)_{\sigma, \tau}$$

$$= y([z, \sigma(b)], b)_{\sigma, \tau} - [y, \tau(b)][z, \sigma(b)] + [y, \sigma(b)][z, \sigma(b)] + ([y, \sigma(b)], b)_{\sigma, \tau}z$$

$$= -[y, \tau(b)][z, \sigma(b)] + [y, \sigma(b)][z, \sigma(b)] = [y, \sigma(b) - \tau(b)][z, \sigma(b)]$$

which gives that

$$[y, \sigma(b) - \tau(b)][I, \sigma(b)] = 0, \forall y \in I. \hspace{1cm} (2.10)$$

Replacing $y$ by $ry$, $r \in R$ in (2.10) we have $[R, \sigma(b) - \tau(b)]I[1, \sigma(b)] = 0$

Since $b$ is noncentral then we obtain that $\sigma(b) - \tau(b) \in Z(R)$ by the last relation.

\[ \square \]

**Corollary 4.** Let $d : R \longrightarrow R$ be a nonzero $(\alpha, \beta)$-derivation and $W$ a nonzero left $(\sigma, \tau)$-Jordan ideal of $R$. If $d\gamma(W) = 0$ then $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

**Proof.** If $d\gamma(W) = 0$ then $d\gamma(R, v)_{\sigma, \tau} = 0, \forall v \in W$. This gives that $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Theorem3. \[ \square \]

**Theorem 4.** Let $W$ be a nonzero left $(\sigma, \tau)$-Jordan ideal of $R$. Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha, \beta)$-derivation associated with a nonzero $(\alpha, \beta)$-derivation $d_1 : R \longrightarrow R$ of $R$ and $b \in R$.

(i) If $h\lambda(W) = 0$ then $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

(ii) If $[W, b]_{\lambda, \mu} = 0$ then $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

(iii) If $[b, W]_{\lambda, \mu} = 0$ then $b \in C_{\lambda, \mu}(R)$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

**Proof.** (i) If $h\lambda(W) = 0$ then $h\lambda(R, v)_{\sigma, \tau} = 0, \forall v \in W$. This means that, for any $v \in W$,

$$v \in Z \text{ or } d_1 \lambda \tau(v) = 0$$

by Lemma4. Let $K = \{v \in W \mid v \in Z\}$ and $L = \{v \in W \mid d_1 \lambda \tau(v) = 0\}$. Then $K$ and $L$ are two additive subgroups of $W$ such that $W = K \cup L$. Since a group cannot be the union of proper subgroups, according to Brauer’s Trick either $W = K$ or $W = L$. That is

$$W \subset Z \text{ or } d_1 \lambda \tau(W) = 0.$$ 

It is clear that, if $W \subset Z$ then $\sigma(v) - \tau(v) \in Z, \forall v \in W$. On the other hand, if $d_1 \lambda \tau(W) = 0$ then we have $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Corollary4.

(ii) The mapping $g(r) = [r, b]_{\lambda, \mu}, \forall r \in R$ is a left-generalized derivation associated with derivation $d_2(r) = [r, \mu(b)], \forall r \in R$. If $g = 0$ then $d_2 = 0$ and so $b \in Z$ is obtained.

Let $d_2 \neq 0$. If $[W, b]_{\lambda, \mu} = 0$ then $g(W) = 0$. Using (i) we have $\sigma(v) - \tau(v) \in Z, \forall v \in W$. Finally, we obtain that $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$. 


associated with a nonzero \( d_{\alpha} \), \( \forall r \in R \) is a \((\lambda, \mu)\)-derivation and so left-generalized derivation associated with \( d_{\lambda} \). If \( d_{\lambda} = 0 \) then \( b \in C_{\lambda, \mu}(R) \). Let \( d_{\lambda} \neq 0 \).

If \([b, W]_{\lambda, \mu} = 0 \) then \( d_{\lambda}(W) = 0 \). This means that \( \sigma(v) - \tau(v) \in Z, \forall v \in W \) by Corollary 4. Finally, we obtain that \( b \in C_{\lambda, \mu}(R) \) or \( \sigma(v) - \tau(v) \in Z, \forall v \in W \) for any cases. \( \square \)

**Theorem 5.** Let \( W \) be a nonzero left \((\sigma, \tau)\)-Jordan ideal of \( R \) and \( b \in R \).

(i) If \([b, W]_{\lambda, \mu} = 0 \) then \( b \in C_{\lambda, \mu}(R) \) or \( \sigma(v) - \tau(v) \in Z, \forall v \in W \).

(ii) If \((W, b)_{\lambda, \mu} = 0 \) then \( b \in Z \) or \( \sigma(v) - \tau(v) \in Z, \forall v \in W \).

**Proof.** (i) The mapping \( h(r) = (b, r)_{\lambda, \mu}, \forall r \in R \) is a left-generalized \((\lambda, \mu)\)-derivation associated with \((\lambda, \mu)\)- derivation \( d(r) = [b, r]_{\lambda, \mu}, \forall r \in R \). If \( h = 0 \) then \( d = 0 \) and so \( b \in C_{\lambda, \mu}(R) \) is obtained. Let \( h \neq 0 \). If \((W, b)_{\lambda, \mu} = 0 \) then we have \( h(W) = 0 \). Using Theorem 4 (i) we obtain that \( \sigma(v) - \tau(v) \in Z, \forall v \in W \). Finally we obtain that \( b \in C_{\lambda, \mu} \) or \( \sigma(v) - \tau(v) \in Z, \forall v \in W \) for any cases.

(ii) Similarly, the mapping \( g(r) = (r, b)_{\lambda, \mu}, \forall r \in R \) is a left-generalized derivation associated with derivation \( d_1(r) = -[r, \mu(b)], \forall r \in R \). If \( g = 0 \) then \( d_1 = 0 \) and so \( b \in Z \) is obtained. Let \( d_1 \neq 0 \). If \((W, b)_{\lambda, \mu} = 0 \) then \( g(W) = 0 \). This gives that \( \sigma(v) - \tau(v) \in Z, \forall v \in W \) by Theorem 4 (i). Finally, we obtain that \( b \in Z \) or \( \sigma(v) - \tau(v) \in Z, \forall v \in W \) for any cases. \( \square \)

**Lemma 5.** Let \( h : R \rightarrow R \) be a nonzero right-generalized \((\alpha, \beta)\)-derivation associated with a nonzero \((\alpha, \beta)\)-derivation \( d : R \rightarrow R \). If \( a, b \in R \) such that \( a[h(R), b]_{\lambda, \mu} = 0 \) then \( a[a, \mu(b)] = 0 \) or \( d\alpha^{-1}\lambda(b) \).

**Proof.** If \( a[h(R), b]_{\lambda, \mu} = 0 \) then we get, for all \( r \in R \)

\[
0 = a[h(r\alpha^{-1}\lambda(b)), b]_{\lambda, \mu} = a[h(r)\lambda(b) + \beta(r)d\alpha^{-1}\lambda(b), b]_{\lambda, \mu} = a[h(r)\lambda(b), b]_{\lambda, \mu} + a[\beta(r)d\alpha^{-1}\lambda(b), b]_{\lambda, \mu} + a[\beta(r), \mu(b)]d\alpha^{-1}\lambda(b) = a\beta(r)d\alpha^{-1}\lambda(b)_{\lambda, \mu} + a[\beta(r), \mu(b)]d\alpha^{-1}\lambda(b).
\]

That is

\[
a\beta(r)[k, b]_{\lambda, \mu} + a[\beta(r), \mu(b)]k = 0, \forall r \in R.
\]

(2.11)

where \( k = d\alpha^{-1}\lambda(b) \). Replacing \( r \) by \( \beta^{-1}(a)r \) in (2.11) and using (2.11) we have, for all \( r \in R \)

\[
0 = aa\beta(r)[k, b]_{\lambda, \mu} + a[a\beta(r), \mu(b)]k = aa\beta(r)[k, b]_{\lambda, \mu} + aa[\beta(r), \mu(b)]k + a[a, \mu(b)]\beta(r)k = a[a, \mu(b)]\beta(r)k
\]

which gives \( a[a, \mu(b)]Rk = 0 \). Using that primeness of \( R \) we get \( a[a, \mu(b)] = 0 \) or \( d\alpha^{-1}\lambda(b) = 0 \). \( \square \)
Lemma 6. Let \( h : R \rightarrow R \) be a nonzero left-generalized \((\alpha, \beta)\)-derivation associated with a nonzero \((\alpha, \beta)\)-derivation \( d_1 : R \rightarrow R \). If \( a, b \in R \) such that \( [h(R), b]_{\lambda, \mu}a = 0 \) then \( d_1 \beta^{-1} \mu(b) = 0 \) or \( [a, \lambda(b)]a = 0 \).

**Proof.** If \( [h(R), b]_{\lambda, \mu}a = 0 \) then we get, for all \( r \in R \)

\[
0 = [h(\beta^{-1} \mu(b)r), b]_{\lambda, \mu}a = [d_1 \beta^{-1} \mu(b)\alpha(r) + \mu(b)h(r), b]_{\lambda, \mu}a \\
= d_1 \beta^{-1} \mu(b)[\alpha(r), \lambda(b)]a + [d_1 \beta^{-1} \mu(b), b]_{\lambda, \mu}\alpha(r)a \\
+ \mu(b)[h(r), b]_{\lambda, \mu}a + [\mu(b), \mu(b)]h(r)a \\
= d_1 \beta^{-1} \mu(b)[\alpha(r), \lambda(b)]a + [d_1 \beta^{-1} \mu(b), b]_{\lambda, \mu}\alpha(r)a
\]

which gives that

\[
k[\alpha(r), \lambda(b)]a + [k, b]_{\lambda, \mu}\alpha(r)a = 0, \forall r \in R \tag{2.12}
\]

where \( k = d_1 \beta^{-1} \mu(b) \). Replacing \( r \) by \( r\alpha^{-1}(a) \) in (2.12) and using (2.12) we have, for all \( r \in R \)

\[
0 = k[\alpha(r)\alpha(r), \lambda(b)]a + [k, b]_{\lambda, \mu}\alpha(r)\alpha(r)a \\
= k\alpha(r)[\alpha(r), \lambda(b)]a + k[\alpha(r), \lambda(b)]\alpha(r)aa + [k, b]_{\lambda, \mu}\alpha(r)\alpha(r)a \\
= k\alpha(r)[\alpha(r), \lambda(b)]a.
\]

That is \( kR[\alpha, \lambda(b)]a = 0 \). Since \( R \) is a prime ring, then the last relation gives that \( d_1 \beta^{-1} \mu(b) = 0 \) or \( [a, \lambda(b)]a = 0 \). \( \square \)

**Theorem 6.** Let \( W \) be a nonzero left \((\sigma, \tau)\)-Jordan ideal of \( R \) and \( a, b \in R \).

(i) If \( a[W, b]_{\lambda, \mu} = 0 \) then \( a[a, \mu(b)] = 0 \) or \( \sigma(v) - \tau(v) \in Z, \forall v \in W \).

(ii) If \( [W, b]_{\lambda, \mu}a = 0 \) then \( [a, \lambda(b)]a = 0 \) or \( \sigma(v) - \tau(v) \in Z, \forall v \in W \).

**Proof.** Let us consider a nonzero element \( v \) of \( W \). The mapping defined by \( h(r) = (r, v)_{\sigma, \tau}, \forall r \in R \) is a left-generalized derivation associated with derivation \( d_1 (r) = -[r, \sigma(v)], \forall r \in R \) and right-generalized derivation associated with derivation \( d(r) = [r, \sigma(v)], \forall r \in R \). If \( h = 0 \) then \( d = 0 = d_1 \) and so \( v \in Z \) is obtained. Let \( d \neq 0 \) and \( d_1 \neq 0 \).

(i) If \( a[W, b]_{\lambda, \mu} = 0 \) then we have \( a[(R, v)_{\sigma, \tau}, b]_{\lambda, \mu} = 0 \) and so \( a[h(R), b]_{\lambda, \mu} = 0 \). Since \( h \) is a right-generalized derivation associated with \( d \), then using Lemma 5 we get \( d\lambda(b) = 0 \) or \( a[a, \mu(b)] = 0 \). That is

\[
[\lambda(b), \sigma(v)] = 0 \text{ or } a[a, \mu(b)] = 0.
\]

On the other hand, if \( v = 0 \) then \( [\lambda(b), \sigma(v)] = 0 \). Hence, considering the same argument for all \( v \in W \) we have

\[
[\sigma^{-1} \lambda(b), W] = 0 \text{ or } a[a, \mu(b)] = 0.
\]
If $\sigma^{-1}(\lambda(b), W) = 0$ then $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Theorem 4(iii).
Finally we obtained that $a[\alpha, \mu(b)] = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases.

(ii) If $[W, b]_{\lambda, \mu} = 0$ then we have $[(R, v)\sigma, \tau, b]_{\lambda, \mu} = 0$ and so $[h(R), b]_{\lambda, \mu} = 0$.
Since $h$ is a left-generalized derivation associated with $d_1$ then, using Lemma 6, we get $d_1\mu(b) = 0$ or $[\alpha, \lambda(b)]a = 0$. That is $[\mu(b), \tau(v)] = 0$ or $[\alpha, \lambda(b)]a = 0$.
Considering as in the proof of (i) we get

$$[\tau^{-1}\mu(b), W] = 0 \text{ or } [\alpha, \lambda(b)]a = 0.$$ 

If $[\sigma^{-1}\mu(b), W] = 0$ then we have $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Theorem 4(iii).
Finally, we obtained that $[\alpha, \lambda(b)]a = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases.
\[ \square \]

**Lemma 7.** Let $h : R \to R$ be a nonzero right-generalized $(\alpha, \beta)$-derivation associated with a nonzero $(\alpha, \beta)$-derivation $d : R \to R$ and $a, b \in R$.

If $a(h(R), b)_{\lambda, \mu} = 0$ then $d\sigma^{-1}\lambda(b) = 0$ or $a[\alpha, \mu(b)] = 0$.

**Proof.** If $a(h(R), b)_{\lambda, \mu} = 0$ then we get, for all $r \in R$

$$0 = a(h(\sigma^{-1}\lambda(b)), b)_{\lambda, \mu} = a(h(r)\lambda(b) + \beta(r)d\sigma^{-1}\lambda(b), b)_{\lambda, \mu}$$
$$= ah(r)[\lambda(b), \lambda(b)] + a(h(r), b)_{\lambda, \mu}\lambda(b) + a\beta(r)(d\sigma^{-1}\lambda(b), b)_{\lambda, \mu}$$
$$- a[\beta(r), \mu(b)]d\sigma^{-1}\lambda(b)$$

which gives that

$$a\beta(r)(k, b)_{\lambda, \mu} - a[\beta(r), \mu(b)]k = 0, \forall r \in R. \quad (2.13)$$

where $k = d\sigma^{-1}\lambda(b)$. Replacing $r$ by $\beta^{-1}(a)r$ in (2.13) we have, for all $r \in R$

$$0 = aa\beta(r)(k, b)_{\lambda, \mu} - a[a\beta(r), \mu(b)]k$$
$$= aa\beta(r)(k, b)_{\lambda, \mu} - aa[\beta(r), \mu(b)]k - a[a, \mu(b)]\beta(r)k$$
$$= -a[a, \mu(b)]\beta(r)k.$$

That is $a[a, \mu(b)]Rda^{-1}\lambda(b) = 0$. Since $R$ is a prime ring, we obtain that

$$d\sigma^{-1}\lambda(b) = 0 \text{ or } a[\alpha, \mu(b)] = 0. \quad \square$$

**Lemma 8.** Let $h : R \to R$ be a nonzero left-generalized $(\alpha, \beta)$-derivation associated with a nonzero $(\alpha, \beta)$-derivation $d_1 : R \to R$ and $a, b \in R$.

If $(h(R), b)_{\lambda, \mu} = 0$ then $d_1\beta^{-1}\mu(b) = 0$ or $[\alpha, \lambda(b)]a = 0$. 


Proof. If \((h(R), b)_{\lambda,\mu}a = 0\) then we get, for all \(r \in R\)
\[
0 = (h(\beta^{-1}\mu(b)r), b)_{\lambda,\mu}a = (d_1\beta^{-1}\mu(b)\alpha(r) + \mu(b)h(r), b)_{\lambda,\mu}a
\]
\[
= d_1\beta^{-1}\mu(b)[\alpha(r), \lambda(b)]a + (d_1\beta^{-1}\mu(b), b)_{\lambda,\mu}\alpha(r)a
\]
\[
+ \mu(b)(h(r), b)_{\lambda,\mu}a - [\mu(b), \mu(b)]h(r)a
\]
\[
= d_1\beta^{-1}\mu(b)[\alpha(r), \lambda(b)]a + (d_1\beta^{-1}\mu(b), b)_{\lambda,\mu}\alpha(r)a
\]
which gives that
\[
k[\alpha(r), \lambda(b)]a + (k, b)_{\lambda,\mu}\alpha(r)a = 0, \forall r \in R. \tag{2.14}
\]
where \(k = d_1\beta^{-1}\mu(b)\). If we take \(r\alpha^{-1}(a)\) instead of \(r\) in (2.14), we get, for all \(r \in R\)
\[
0 = k[\alpha(r)a, \lambda(b)]a + (k, b)_{\lambda,\mu}\alpha(r)aa
\]
\[
= k\alpha(r)[a, \lambda(b)]a + k[\alpha(r), \lambda(b)]aa + (k, b)_{\lambda,\mu}\alpha(r)aa
\]
\[
= k\alpha(r)[a, \lambda(b)]a.
\]
That is \(k R[a, \lambda(b)]a = 0\). This gives that \(d_1\beta^{-1}\mu(b) = 0\) or \([a, \lambda(b)]a = 0\) in prime rings.

Theorem 7. Let \(W\) be a nonzero left \((\sigma, \tau)\)–Jordan ideal of \(R\) and \(a, b \in R\).

(i) If \(a(W, b)_{\lambda,\mu} = 0\) then \([a, \mu(b)] = 0\) or \(\sigma(v) - \tau(v) \in Z\), \(\forall v \in V\).

(ii) If \((W, b)_{\lambda,\mu}a = 0\) then \([a, \lambda(b)]a = 0\) or \(\sigma(v) - \tau(v) \in Z\), \(\forall v \in V\).

Proof. Let us consider a nonzero element \(v \in W\). The mapping defined by
\(h(r) = (r, v)_{\sigma,\tau}, \forall r \in R\) is a left-generalized derivation associated with derivation \(d_1(r) = -[r, \tau(v)], \forall r \in R\) and right-generalized derivation associated with derivation \(d(r) = [r, \sigma(v)], \forall r \in R\). If \(h = 0\) then \(d = 0 = d_1\) and so \(v \in Z\) is obtained. Let \(d \neq 0\) and \(d_1 \neq 0\).

(i) If \(a(W, b)_{\lambda,\mu} = 0\) then we have \(a((R, v)_{\sigma,\tau}, b)_{\lambda,\mu} = 0\). That is \(a(h(R), b)_{\lambda,\mu} = 0\). Since \(h\) is a right-generalized derivation with \(d\) then using Lemma 7 we obtain that \(d \lambda(b) = 0\) or \(a[a, \mu(b)] = 0\). That is
\[
[\lambda(b), \sigma(v)] = 0 \text{ or } a[a, \mu(b)] = 0.
\]

On the other hand, if \(v = 0\) then \([\lambda(b), \sigma(v)] = 0\). Hence, considering the same argument for all \(v \in W\) we have
\[
[b, W] = 0 \text{ or } a[a, \mu(b)] = 0.
\]
If \([b, W] = 0\) then we have \(b \in Z\) or \(\sigma(v) - \tau(v) \in Z\), \(\forall v \in W\) by Theorem 4 (iii). If \(b \in Z\) then \(a[a, \mu(b)] = 0\).

Finally, we obtained that \(a[a, \mu(b)] = 0\) or \(\sigma(v) - \tau(v) \in Z\), \(\forall v \in V\) for any cases.

(ii) If \((W, b)_{\lambda,\mu}a = 0\) then \((h(R), b)_{\lambda,\mu}a\) is obtained. Since \(h\) is a left-generalized derivation associated with \(d_1\) then using Lemma 8 and considering as in the proof of (i) we get \([a, \lambda(b)]a = 0\) or \(\sigma(v) - \tau(v) \in Z\), \(\forall v \in V\).
References


