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A NOTE ON STRONGLY IFP SUBMODULES AND MODULES

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Abstract

In this note we rename the structure of strongly IFP submodules and make some corrections on a paper of some authors in our group that was appeared recently.

1 Introduction

Throughout this paper, all rings are associative rings with identity and all modules are unitary right R-modules. Let R be a ring and M a right R-module. Denote $S = \text{End}_R(M)$, the endomorphism ring of the module M. A submodule X of M is called a fully invariant submodule of M, if $f(X) \subset X$ for any $f \in S$. Especially, a right ideal of R is a fully invariant submodule of R_R if it is a two-sided ideal of R. The class of all fully invariant submodules of M is non-empty and closed under intersections and sums. A right R-module M is called a self-generator if it generates all its submodules. Following [10], a fully invariant proper submodule X of M is called a *prime submodule* of M if for any ideal I of $S = \text{End}_R(M)$, and any fully invariant submodule U of M, $I(U) \subset X$ implies that either $I(M) \subset X$ or $U \subset X$. A fully invariant submodule X of M is called a *strongly prime submodule* of M if for any $\varphi \in S = \text{End}_R(M)$ and $m \in M$, $\varphi(m) \in X$ implies that either $\varphi(M) \subset X$ or $m \in X$. The basic Theorem 2.1 in [10] shows that the class of prime submodules of a given module has some properties similar to that of prime ideals in an associative ring. Following this

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theorem, a fully invariant proper submodule X of M is prime if and only if for any $\varphi \in S$ and $m \in M$, $\varphi Sm \subset X$ implies that $\varphi(M) \subset X$ or $m \in X$. Using this property, one can see that *every strongly prime submodule is prime.*

Following [18, Definition 2.1], a submodule X of a right R-module M is said to have *insertion factor property* (briefly, an IFP-submodule) if for any endomorphism φ of M and any element $m \in M$, if $\varphi(m) \in X$, then $\varphi Sm \subset X$. A right ideal I of R is an *IFP-right ideal* if it is an IFP-submodule of R_R, that is for any $a, b \in R$, if $ab \in I$, then $aRb \subset I$. A right R-module M is called an *IFP-module* if 0 is an IFP-submodule of M. A ring R is IFP if 0 is an IFP-ideal.

A fully invariant submodule X of a right R-module M is called a *semiprime submodule* if it is an intersection of prime submodules of M. A right R-module M is called a *semiprime module* if 0 is a semiprime submodule of M. Thus, the ring R is a *semiprime ring* if R_R is semiprime. By symmetry, the ring R is a semiprime ring if R_R is a semiprime left R-module.

Proposition 1.1. *[1, Proposition 2.3] Let M be a right R-module which is a self-generator and X, a fully invariant submodule of M. Then X is a semiprime submodule if and only if whenever $f \in S$ with $fSf(M) \subset X$, then $f(M) \subset X$.*

2 Strongly IFP-submodules and modules.

Definition 2.1. A fully invariant proper submodule X of M is called *strongly IFP* if for any $\psi \in S$ and $m \in M$, $\psi^2(m) \in X$ implies $\psi Sm \subset X$. A right R-module M is called a *strongly IFP-module* if 0 is a strongly IFP-submodule of M.

In [8], authors had a confusion in applying Proposition 1.1. In this result, we need the condition of self-generator and because of this, we could not call it *completely semiprime.* Moreover, authors did not define completely prime submodules. By the Proposition 2.3 below, we call such a submodule *strongly IFP.*

Remark 2.2. If M is a self-generator, then every strongly IFP-submodule is semiprime.

Proof. The proof can be found in [8, Remark 2.2]. □

Proposition 2.3. *Let X be a strongly IFP submodule of M, and $S = \mathrm{End}(M_R)$. Then,*

1. *X is an IFP-submodule of M,*
2. *if $\varphi, \psi \in S$ and $m \in M$ such that $\varphi\psi(m) \in X$, then $\psi\varphi(m) \in X$.*

Proof. The proof can be found in [8] and we give here for the sake completeness.

(1.) Let $\varphi \in S$ and $m \in M$ such that $\varphi(m) \in X$. Since X is fully invariant, we get $\varphi^2(m) \in X$. By definition of strongly IFP submodules, we get $\varphi Sm \subset X$, proving that X is IFP.

(2.) Take any $\varphi, \psi \in S$, $m \in M$ with $\varphi\psi(m) \in X$. Since X is fully invariant, we get $(\psi\varphi\psi)^2(m) \in X$. By definition 2.1, we get $(\psi\varphi\psi)Sm \subset X$. Hence, $\psi\varphi\psi\varphi(m) \in X$ or $(\psi\varphi)^2(m) \in X$. Since X is strongly IFP, $\psi\varphi Sm \subset X$. This shows that $\psi\varphi(m) \in X$, proving our claim. □

The following Proposition is a correction of [8, 2.10]. The condition that being finitely generated is needed.

Proposition 2.4. *Let M be a right R-module and $S = \mathrm{End}(M_R)$.*

(1) If X is a strongly IFP submodule of M, then I_X is a strongly IFP ideal of S.

(2) Let P be a strongly IFP-ideal of S. If M is finitely generated and a self-generator, then $X = P(M)$ is a strongly IFP submodule of M and $I_X = P$.

Proof. (1). Let $\varphi^2\psi \in I_X$. Then $\varphi^2\psi(M) \subset X$. This means for any $m \in M$ we have $\varphi^2\psi(m) \in X$. Since X is strongly IFP, we get $\varphi S\psi(m) \subset X$. It follows that $\varphi S\psi(M) \subset X$, showing that $\varphi S\psi \subset I_X$.

(2). Let P be a strongly IFP ideal of S and put $X = P(M)$. Since M is finitely generated, by [20, 18.4], we get $I_X = P$. Let $\varphi^2(m) \in X$ with $\varphi \in S$ and $m \in M$. Since M is a self-generator, $mR = \sum_{i \in I} \psi_i(M)$, where $\psi_i \in S$ for some set I. It follows that $\varphi^2\psi_i(M) \subset X$. Thus $\varphi^2\psi_i \in I_X = P$. By assumption, $\varphi S\psi_i \subset P$. Hence $\varphi S(mR) \subset X$, and therefore $\varphi Sm \subset X$, proving that X is a strongly IFP submodule of M. □

Proposition 2.5. *Let X be a fully invariant submodule of a right R-module M. X is strongly prime if and only if it is prime and strongly IFP.*

Proof. From [3], X is strongly prime if and only if it is prime and IFP. By Proposition 2.3, the result follows. □

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