A NOTE ON STRONGLY IFP SUBMODULES
AND MODULES

Bunthita Chattae and N. V. Sanh*

Department of Mathematics
Faculty of Science, Mahidol University
Center of Excellence in Mathematics
Bangkok 10400, Thailand

e-mail: nguyen.san@mahidol.ac.th

Abstract

In this note we rename the structure of strongly IFP submodules and make some corrections on a paper of some authors in our group that was appeared recently.

1 Introduction

Throughout this paper, all rings are associative rings with identity and all modules are unitary right $R$-modules. Let $R$ be a ring and $M$ a right $R$-module. Denote $S = \text{End}_R(M)$, the endomorphism ring of the module $M$. A submodule $X$ of $M$ is called a fully invariant submodule of $M$, if $f(X) \subseteq X$ for any $f \in S$. Especially, a right ideal of $R$ is a fully invariant submodule of $R_R$ if it is a two-sided ideal of $R$. The class of all fully invariant submodules of $M$ is non-empty and closed under intersections and sums. A right $R$-module $M$ is called a self-generator if it generates all its submodules. Following [10], a fully invariant proper submodule $X$ of $M$ is called a prime submodule of $M$ if for any ideal $I$ of $S = \text{End}_R(M)$, and any fully invariant submodule $U$ of $M$, $I(U) \subseteq X$ implies that either $I(M) \subseteq X$ or $U \subseteq X$. A fully invariant submodule $X$ of $M$ is called a strongly prime submodule of $M$ if for any $\varphi \in S = \text{End}_R(M)$ and $m \in M$, $\varphi(m) \in X$ implies that either $\varphi(M) \subseteq X$ or $m \in X$. The basic Theorem 2.1 in [10] shows that the class of prime submodules of a given module has some properties similar to that of prime ideals in an associative ring. Following this

*Corresponding author

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theorem, a fully invariant proper submodule $X$ of $M$ is prime if and only if for any $\varphi \in S$ and $m \in M$, $\varphi Sm \subset X$ implies that $\varphi(M) \subset X$ or $m \in X$. Using this property, one can see that every strongly prime submodule is prime.

Following [18, Definition 2.1], a submodule $X$ of a right $R$-module $M$ is said to have insertion factor property (briefly, an IFP-submodule) if for any endomorphism $\varphi$ of $M$ and any element $m \in M$, $\varphi Sm \subset X$ implies that $\varphi(M) \subset X$ or $m \in X$. Using this property, one can see that every strongly prime submodule is prime.

Following [18, Definition 2.1], a submodule $X$ of a right $R$-module $M$ is said to have insertion factor property (briefly, an IFP-submodule) if for any endomorphism $\varphi$ of $M$ and any element $m \in M$, $\varphi Sm \subset X$ implies that $\varphi(M) \subset X$ or $m \in X$. Using this property, one can see that every strongly prime submodule is prime.

A right ideal $I$ of $R$ is an IFP-right ideal if it is an IFP submodule of $R_R$, that is for any $a, b \in R$, if $ab \in I$, then $aRb \subset I$. A right $R$-module $M$ is called an IFP-module if $0$ is an IFP submodule of $R$.

A ring $R$ is IFP if $0$ is an IFP ideal. A right $R$-module $M$ is called a semiprime module if $0$ is a semiprime submodule of $M$.

Proposition 1.1. [1, Proposition 2.3] Let $M$ be a right $R$-module which is a self-generator and $X$, a fully invariant submodule of $M$. Then $X$ is a semiprime submodule if and only if whenever $f \in S$ with $fSf(M) \subset X$, then $f(M) \subset X$.

2 Strongly IFP-submodules and modules.

Definition 2.1. A fully invariant proper submodule $X$ of $M$ is called strongly IFP if for any $\psi \in S$ and $m \in M$, $\psi^2(m) \in X$ implies $\psi Sm \subset X$. A right $R$-module $M$ is called a strongly IFP-module if $0$ is a strongly IFP submodule of $M$.

In [8], authors had a confusion in applying Proposition 1.1. In this result, we need the condition of self-generator and because of this, we could not call it completely semiprime. Moreover, authors did not define completely prime submodules. By the Proposition 2.3 below, we call such a submodule strongly IFP.

Remark 2.2. If $M$ is a self-generator, then every strongly IFP-submodule is semiprime.

Proof. The proof can be found in [8, Remark 2.2].

Proposition 2.3. Let $X$ be a strongly IFP submodule of $M$, and $S = \text{End}(M_R)$. Then,

1. $X$ is an IFP-submodule of $M$,

2. if $\varphi, \psi \in S$ and $m \in M$ such that $\varphi\psi(m) \in X$, then $\psi\varphi(m) \in X$. 
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Proof. The proof can be found in [8] and we give here for the sake completeness.

(1.) Let \( \varphi \in S \) and \( m \in M \) such that \( \varphi(m) \in X \). Since \( X \) is fully invariant, we get \( \varphi^2(m) \in X \). By definition of strongly IFP submodules, we get \( \varphi Sm \subset X \), proving that \( X \) is IFP.

(2.) Take any \( \varphi, \psi \in S \), \( m \in M \) with \( \varphi \psi(m) \in X \). Since \( X \) is fully invariant, we get \( (\psi \varphi \psi)^2(m) \in X \). By definition 2.1, we get \( (\psi \varphi \psi)^2 Sm \subset X \). Hence, \( \psi \varphi \varphi(m) \in X \) or \( (\psi \varphi)^2(m) \in X \). Since \( X \) is strongly IFP, \( \psi \varphi Sm \subset X \). This shows that \( \psi \varphi(m) \in X \), proving our claim. \( \square \)

The following Proposition is a correction of [8, 2.10]. The condition that being finitely generated is needed.

Proposition 2.4. Let \( M \) be a right \( R \)-module and \( S = \text{End}(M_R) \).

(1) If \( X \) is a strongly IFP submodule of \( M \), then \( I_X \) is a strongly IFP ideal of \( S \).

(2) Let \( P \) be a strongly IFP-ideal of \( S \). If \( M \) is finitely generated and a self-generator, then \( X = P(M) \) is a strongly IFP submodule of \( M \) and \( I_X = P \).

Proof. (1). Let \( \varphi^2 \psi \in I_X \). Then \( \varphi^2 \psi(M) \subset X \). This means for any \( m \in M \) we have \( \varphi^2 \psi(m) \in X \). Since \( X \) is strongly IFP, we get \( \varphi S \psi(m) \subset X \). It follows that \( \varphi S \psi(M) \subset X \), showing that \( \varphi S \psi \subset I_X \).

(2). Let \( P \) be a strongly IFP ideal of \( S \) and put \( X = P(M) \). Since \( M \) is finitely generated, by [20, 18.4], we get \( I_X = P \). Let \( \varphi^2(m) \in X \) with \( \varphi \in S \) and \( m \in M \). Since \( M \) is a self-generator, \( mR = \sum_{i \in I} \psi_i(M) \), where \( \psi_i \in S \) for some set \( I \). It follows that \( \varphi^2 \psi_i(M) \subset X \). Thus \( \varphi^2 \psi_i \in I_X = P \). By assumption, \( \varphi S \psi_i \subset P \). Hence \( \varphi S(mR) \subset X \), and therefore \( \varphi Sm \subset X \), proving that \( X \) is a strongly IFP submodule of \( M \). \( \square \)

Proposition 2.5. Let \( X \) be a fully invariant submodule of a right \( R \)-module \( M \). \( X \) is strongly prime if and only if it is prime and strongly IFP.

Proof. From [3], \( X \) is strongly prime if and only if it is prime and IFP. By Proposition 2.3, the result follows. \( \square \)

References


